# Topology Proceedings


**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

**Topology Proceedings**
**Volume 24, Summer 1999, 421–502**

# TOPOLOGICAL GROUPS: WHERE TO FROM HERE?

Vladimir Pestov

**Abstract**

This is an account of one man's view of the current perspective of theory of topological groups. We survey some recent developments which are, from our viewpoint, indicative of the future directions, concentrating on actions of topological groups on compacta, embeddings of topological groups, free topological groups, and 'massive' groups (such as groups of homeomorphisms of compacta and groups of isometries of various metric spaces).

## 1. Introduction

### 1.1. Motivation

For a randomly selected mathematician outside of the field of general topology — or, to be more precise, 'general topological algebra' — the words 'topological group' most probably sound synonymous with 'locally compact group.' Indeed, the depth, beauty and importance of theory of locally compact groups, in particular representation theory, abstract harmonic analysis, duality theory and structure theory, are overwhelming, while the richness of links with other areas of mathematics, physics, chemistry, computing and other sciences is hard to match. A natural question to ask is therefore:

---

*Mathematics Subject Classification*: 22A05, 22A10, 54H20
*Key words*: Embeddings of topological groups, free topological groups, 'massive' topological groups, topological transformation groups.

• *Is there life beyond local compactness?*

In the present audience the question sounds hollow rhetoric as the answer is known to be in the affirmative to just about everyone. Therefore, I will sharpen it up:

• *Does there exist a potential for a theory of* (*some classes of*) *non-locally compact groups of a comparable depth to that of theory of locally compact groups?*

The answer to this question is much less obvious, as probably only very few people would rush into betting on a definite 'yes' or 'no.' There is a not uncommonly held opinion (which some of us would dispute) that the most recent result in the theory of topological groups that really mattered for the rest of mathematics was the solution to Hilbert's Fifth Problem [72]; can we presently discern where the next result about topological groups to have an equally resounding impact could emerge from? As a matter of fact, every professional working in the field is guided by a vision of his/her own, and the totality of views coming from different researchers would span a remarkably diverse range of opinions. This paper offers such an opinion belonging to the present author. Accordingly, the article does not attempt to be in any way comprehensive and compete with such substantial surveys on topological groups as [19, 109], or else [5, 7].

One obvious approach to exploring the general question stated above is to try and directly extend concepts and results from the locally compact case to more general classes of groups. In this way, one surrounds the class of locally compact groups with a larger 'halo' formed by topological groups that inherit some or other features of locally compact groups. One of such species of topological groups thriving in the penumbral shadows of local compactness is the class of nuclear (abelian) groups advertised by Banaszczyk [11], whose *raison d'être* is essentially testing the limits of Pontryagin–van Kampen duality, as well as of the entire body of closely related structural results such as Glicksberg's theorem [12]. Another such species is the class of pseudocompact groups which is very popular nowadays largely due to the consistent efforts of Wis Comfort over the past few decades [18, 17]. There can hardly be a better recipe for achieving a deeper insight into the nature of (locally) compact groups than walking a few steps away and having a good look from the outside! As a matter of fact, this approach can lead to fruitful insights into the structure and properties of topological groups of very general nature. One would expect essential further progress achieved in this direction in the future. Nevertheless, one meets certain limitations on this way, since most of tools making the theory of LC groups a success are intrinsic to local compactness – like the left-invariant locally finite finitely additive Borel measure, positive on non-void open sets [2].

This makes an alternative approach unavoidable: to isolate new classes of topological groups of importance on their own right with a view towards understanding their structure and properties. Such large ('esentially non-locally compact') topological groups originate in many different contexts: in set-theoretic and smooth topology, ergodic theory, representation theory, functional analysis, and topological dynamics, to name a few. Below we will exhibit examples of such groups and show that some of their properties have no analogue in the locally compact case.

## 1.2. What is Included in the Article

The paper is loosely 'coordinatized' by the following four notions.

*1. Embeddings.*

The following question has stimulated investigations in the theory of topological groups over many years. Let $\mathcal{P}$ be a nonempty class of topological groups (possibly consisting just of a single group). When is a given topological group $G$ isomorphic to a subgroup of a group from $\mathcal{P}$?

2. *Actions.*

The situation where a topological group $G$ acts continuously on a topological space $X$ emerges very often in disparate contexts throughout mathematics. If $X$ is compact, then the triple formed by $X$, $G$, and the action of $G$ on $X$ forms, formally speaking, the object of study of abstract topological dynamics. However, the accents in topological dynamics are put on different concepts — having more to do with the structure and properties of orbits — from those we are interested in. Our emphasis will be also different from theory of $G$-spaces, where the main attention is paid to the phase space $X$. Therefore, we prefer to talk simply of actions.

3. *'Massive,' or 'large,' groups.*

It is impossible to give a formal definition of this concept, yet 'large' groups are easy to recognize. Examples include the group of all self-homeomorphisms of a (sufficiently homogeneous) compact or locally compact space equipped with the compact-open topology (or related topologies), the full unitary group of an infinite-dimensional Hilbert space, typically with the strong operator topology, the group of measure-preserving transformations of a Lebesgue measure space with the weak topology, and so forth. The role played by such groups in mathematics is fundamental, yet there is no coherent unifying theory in existence for the time being.

4. *Free topological groups.*

Free topological groups, introduced by Markov in 1941 along with their closest counterparts such as free abelian topological groups and free locally convex spaces, served as an inspiration for the concept of a universal arrow to a functor introduced by Pierre Samuel. Free topological groups remain a very useful source of examples and building blocks in general topological group theory. Apart from that, these objects have never enjoyed much popularity and are often perceived as exotic. However, such objects, properly disguised, often resurface in other areas of mathematics, meaning that the accumulated expertise of topological group theorists is very probably applicable to problems that at the first sight have nothing to do with topological algebra.

Each of the above four concepts will feature below in three different roles: as a tool, as an object of study, and as a link between 'general topological algebra' and other disciplines of mathematics. Notice that all of the above are closely intertwined and their role in our presentation is that of a coordinate system rather than a linear index.

### 1.3. What is Left Out

It is hardly surprising that the vast majority of research directions pertinent to 'large' topological groups is left untouched by the present paper. As the first such omission, we want to mention descriptive set theory, or more exactly descriptive theory of group actions, where non-locally compact Polish (that is, completely metrizable) topological groups appear very naturally. A topological algebraist must certainly keep one eye on further developments in the area, which provides both a motivation and a guidance for the future general theory of non-locally compact topological groups. The present author does not feel qualified to touch upon this subject, but fortunately there are excellent references available, e.g. [13, 51].

Another omission, but this time quite purposeful, is theory of infinite-dimensional Lie groups. Indeed, many concrete large topological groups support a natural structure of ($C^\infty$ or sometimes even analytic) Lie groups modelled over locally convex spaces of infinite dimension. While such a theory for Banach–Lie groups goes back to the 30's and is well established [16, 47], there are several competing versions of infinite-dimensional Lie theory beyond the Banach–Lie case [53, 70, 57] (see also [76], which was referee's suggestion). Already for Lie groups modelled over nuclear Fréchet spaces the theory meets some difficulties of fundamental nature, the most upsetting of which is the apparent need to explicitely require the existence of exponential map: it is still an open problem whether or not a $C^\infty$ Fréchet manifold equipped with smooth group operations admits an exponential map from the tangent Lie algebra at the identity (*loco citato*).

However, the reason why infinite-dimensional groups are not featured in this essay is of a different kind: for all we know, they cannot be considered as mere *topological groups,* but rather as *topological groups with additional structure* — unlike in the finite dimensional case, where the classical Montgomery–Zippin theorem allows one to identify Lie groups with locally Euclidean topological groups! No analogue of the Montromery–Zippin theorem is known even for Banach–Lie groups. Moreover, such an analogue is impossible to state in terms of topology alone, in view of the following result by Keesling [52]: every separable metrizable topological group embeds as a topological subgroup into a topological group homeomorphic to the separable Hilbert space $l_2$. (Now recall that for every Banach–Lie group the restrictions of the two natural uniformities to a suitable neighbourhood of identity coincide, and this property, while inherited by topological subgroups, is typically absent in Polish topological groups.)

Finally, it is beyond reasonable doubt that topological *semigroups* will play an ever increasing role in theory of 'large' topological groups. Semigroups do appear in this survey on a few occasions, but present author does not feel sufficiently competent in the area to offer topological semigroups the prominent place they richly deserve. A couple of references on the subject are [14, 19].

**Acknowledgements**
The author expresses his gratitude to the Organizers of the 14-th Summer Conference on General Topology and its Applications for selecting him as an Invited Speaker and thus offering an opportunity to put in order his thoughts about topological groups. This article is a (somewhat extended) write-up of the actual lecture.

Thanks also go to Vladimir Uspenskij, Michael Megrelishvili, and the anonymous referee for their constructive criticisms of the initial version of this paper (dated 29.10.99), and to Alekos Kechris for useful remarks.

Some of the research the article is based upon was supported by the Victoria University of Wellington Research Development Fund.

## 2. Actions and Representations

### 2.1. Some Basics

We will be only considering Hausdorff topological groups, Tychonoff topological spaces, and separated uniform spaces. A topological group $G$ *acts* on a topological space $X$ if there is a continuous mapping $\tau\colon G\times X\to X$, called an *action,* where the image of a pair $(g,x)\in G\times X$ is usually denoted either by $\tau_g x$ or simply by $g\cdot x$, having the properties that $g\cdot(h\cdot x)=(gh)\cdot x$ and $e_G\cdot x=x$ for every $g,h\in G$ and $x\in X$. Here and in the sequel, $e=e_G$ denotes the identity element of the acting group. The entire triple $\mathfrak{X}=(X,G,\tau)$ is called an (abstract) *topological dynamical system,* while $X$ together with the action $\tau$ is referred to as a *$G$-space.* The triple $\mathfrak{X}$ is also known under the name a *topological transformation group.*

If the space $X$ is compact, a continuous action of a topological group $G$ on $X$ can be identified with a topological group homomorphism $G\to \mathrm{Homeo}_c X$, where the subscript '$c$' denotes the compact-open topology on the homeomorphism group. Namely, to every element $g\in G$ one associates the mapping

$$X\ni x\mapsto g\cdot x\in X,$$

which is a self-homeomorphism of $X$ (with the inverse $x\mapsto g^{-1}\cdot x$). Of course, this mapping makes sense for a non-compact space $X$ as well, and is called a *motion*, or a *translation*. However, only in the case of compact $X$ one can always prove that the homomorphism $G \to \mathrm{Homeo}_c X$ is continuous. Conversely, every continuous homomorphism from a topological group $G$ to the full homeomorphism group $\mathrm{Homeo}_c X$ of a compact space $X$ determines in a unique way a continuous action of $G$ on $X$. For general topological spaces $X$ the group of self-homeomorphisms $\mathrm{Homeo}\, X$ no longer supports an apparent group topology making such a convenient identification possible, though there are important exceptions.

Let us single out two particularly significant classes of actions. An action is called *effective* if every element $g \in G$, different from the identity, acts in a non-trivial fashion on *some* element of the phase space $X$, that is, $\forall g \neq e$, $\exists x$, $g \cdot x \neq x$. An action is called *free* if every element $g$ different from the identity acts in a non-trivial way on *every* element of $X$, that is, $\forall g \neq e$, $\forall x$, $g \cdot x \neq x$.

**Examples 2.1.1.** 1. Every topological group acts on itself freely by left translations through $g \cdot x = gx$, $g, x \in G$.
2. The action of a topopological group $G$ on itself by conjugations,

$$g \cdot x = g^{-1} x g, \quad g, x \in G,$$

is effective if and only if the centre of $G$ is trivial.
3. The canonical action of the group $\mathrm{Homeo}_c \mathbb{S}^1$ on the circle $\mathbb{S}^1$ is effective but not free.

The topological space $X$ (the *phase space*) can also support additional structures of various sort. If $X = (X, \rho)$ is a metric space, then an action of a topological group $G$ on $X$ is said to be *isometric*, or an action *by isometries,* if every motion $x \mapsto g \cdot x$, $g \in G$, is an isometry of $X$ onto itself. In this case, the continuity of the action is equivalent to the continuity of the associated homomorphism from $G$ to the group of isometries $\mathrm{Iso}\,(X)$ equipped with the pointwise topology (that is, one inherited from $X^X$, or else from $C_p(X,X)$). Moreover, the group $\mathrm{Iso}\,(X)$ with the pointwise topology forms a Hausdorff topological group. Both statements are very easy to verify directly.

If $X$ is a Banach space – more precisely, a complete normed linear space – and $G$ acts on $X$ continuously, then the action $\tau$ is called a *continuous representation of $G$ in $E$* if every motion $E \ni x \mapsto g \cdot x \in E$, $g \in G$, is a linear operator. Effective actions are known in this context as *faithful representations*. Every such action has zero element as the fixed point and therefore cannot be free. One says that the action $\tau$ on a Banach space $E$ is *by isometries* if every homeomorphism of the form $x \mapsto g \cdot x$, $g \in G$, is a linear isometric transformation of $X$. In such a case, the action of $G$ on $X$ is often referred to as a representation of $G$ in $X$ *by isometries.* For such actions instead of continuous representations one normally speaks of *strongly continuous* representations. To explain this peculiarity in terminology, recall that the *strong operator topology* on the space $\mathcal{L}(E,E)$ of all bounded linear operators on a locally convex space $E$ is the topology of pointwise convergence, that is, one induced from the standard embedding $\mathcal{L}(E,E) \hookrightarrow E^E$, where the latter space carries the Tychonoff product topology. The restriction of the strong operator topology to the group $\mathrm{Iso}\,(E)_s$ therefore coincides with the topology of pointwise convergence and, as we noted above, an isometric representation of $G$ on $E$ is continuous if and only if the associated homomorphism $G \to \mathrm{Iso}\,(X)$ is continuous with respect to the strong operator topology on the latter group.

Finally, if $X = \mathcal{H}$ is a (complex) Hilbert space, then a representation $\tau$ of a group $G$ in $\mathcal{H}$ is called *unitary* if every motion $\tau_g \colon \mathcal{H} \ni x \mapsto g{\cdot}x \in \mathcal{H}$ is a unitary operator: $(\tau_g x, y) = (x, \tau_{g^{-1}} y)$ for all $x, y \in \mathcal{H}$. It is a particular case of a representation by isometries, and in fact namely strongly continuous unitary representations of topological groups are of overwhelming importance. The collection of all unitary operators on a Hilbert space $\mathcal{H}$ forms a group, called the *full unitary group* of $\mathcal{H}$ and denoted by $U(\mathcal{H})$. The subscript 's' will denote the strong operator topology, and the topological group $U(\mathcal{H})_s$ (where $\dim \mathcal{H} = \infty$) is one of the most important 'massive' groups. If $\mathcal{H}$ has finite dimension $n$, then $U(\mathcal{H}) = U(n)$ is the group of $n \times n$ unitary matrices with the natural compact topology.

### 2.2. Teleman's Theorem

The following fundamental result asserts, roughly speaking, that effective actions of topological groups are sufficiently common.

**Theorem 2.2.1** (Silviu Teleman, 1957, [105])**.** *Every Hausdorff topological group $G$ acts effectively*
*($*$) on a Banach space by isometries;*
*($**$) on a compact space.*

Before proceeding to the proof, we will say a few words about uniformities on topological groups. The difference between the two standard uniform structures on a topological group turns out to be surprisingly important and, in particular, provides the clue to some of the subtlest results in theory of topological groups and their actions known to date, which is why taking special care of terminology and notation is important. If $G$ is a topological group, then the *right equicontinuous* (in short, *right e.c.*) uniform structure on $G$, which we will denote by $\mathcal{U}_{\Rsh}(G)$, has basic entourages of the diagonal of the form

$$V_{\Rsh} := \{(g,h) \in G \times G \colon gh^{-1} \in V\},$$

as $V$ runs over a neighbourhood basis of $G$. The uniformity $\mathcal{U}_{\Rsh}(G)$ is right equicontinuous in the sense that the family of all right translations $R_g$, $g \in G$, where $R_g(x) := xg$, is uniformly equicontinuous as a family of mappings from the uniform space $(G, \mathcal{U}_{\Rsh}(G))$ to itself. [The basic entourages $V_{\Rsh}$ are invariant with respect to the action of $G$ on $G \times G$ on the right:

$$(h,f) \cdot g := (hg, fg).]$$

The right equicontinuous uniformity is called *right uniformity* by some authors and *left uniformity* by others (consult our paper [88] for references to the both kinds of usage). Since the word *right equicontinuous* is not only unambiguous but fully descriptive, we will adopt it in this article. In a similar way, one can consider the *left equicontinuous* uniformity (suggested notation: $\mathcal{U}_{\urcorner}(G)$), determined by the basic neighbourhoods of the form

$$V_{\urcorner} := \{(g,h) \in G \times G \colon g^{-1}h \in V\}.$$

Denote by $C^b_{\ulcorner}(G)$ the vector space formed by all bounded (real or complex-valued) functions on $G$ that are uniformly continuous with respect to the right e.c. uniform structure:

$$\forall \epsilon > 0, \exists V \ni e, xy^{-1} \in V \Rightarrow |f(x) - f(y)| < \epsilon.$$

If equipped with the supremum norm,

$$\|f\| := \sup_{x \in G} |f(x)|,$$

$C^b_{\ulcorner}(G)$ becomes a Banach space.

In a similar way, one can define the Banach space $C^b_{\urcorner}(G)$ of all bounded functions that are uniformly continuous with respect to the left e.c. structure on $G$.

**Remark 2.2.2.** If the two uniformities on a group $G$ coincide, then one says that $G$ is a *SIN group* — from *S*mall *I*nvariant *N*eighbourhoods — or else a *balanced group*. The SIN property of a topological group $G$ is equivalent to the existence of a neighbourhood basis at $e_G$ formed by invariant sets $V$, that is, such that $gVg^{-1} = V$ for all $g \in G$. For example, every compact and every abelian topological group are SIN. Another prominent example of a SIN group is the full unitary group $U(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ equipped with the *uniform* (not strong!) operator topology, that is, the topology induced by the natural embedding $U(\mathcal{H}) \subset C^b(B, \mathcal{H})$, where $B$ is the closed unit ball in $\mathcal{H}$ and the space of bounded continuous functions is equipped with the supremum norm.

Clearly, for every SIN group $G$ the two function spaces are identical:

$$C^b_{\ulcorner}(G) = C^b_{\urcorner}(G).$$

Rather astonishingly, it remains unknown to date if the converse is true! A similar question can be asked about spaces of functions that are not necessarily bounded: suppose $C_{\ulcorner}(G) = C_{\urcorner}(G)$, is then $G$ necessarily a SIN group? This question was first asked by Itzkowitz [48], and a positive answer has been since obtained for a number of particular (and quite disparate) classes of topological groups. (See, for instance, [66] and references contained in the paper.) It remains also unknown if the two versions of Itzkowitz's question (bounded and unbounded ones) are equivalent between themselves. A comprehensive survey of what is known of the problem to date, written by Itzkowitz himself, appears in this Volume of Topology Proceedings.

Now — back to the proof of Teleman's result, which is remarkably straightforward.

($*$) The group $G$ acts on the Banach space $C^b_{\ulcorner}(G)$ by left translations:

$$(g, f) \mapsto L_g(f), \quad L_g(f)(x) := f(gx).$$

The action $L\colon G \times E \to E$ is easily verified to be a continuous and effective action and each operator $L_g\colon E \to E$ is a linear isometry. The first claim is established.

**Remark 2.2.3.** Observe that in the above proof the group $G$ acts by *left* translations on the space of functions that are uniformly continuous with respect to the *right* equicontinuous uniform structure! As a particularly instructive exercise, the reader is advised to try and find where the proof breaks down if one considers the (perfectly well-defined) action of $G$ by left translations on the space $C^b_{\urcorner}(G)$.

($**$) The group $G$ acts in a natural way on the dual space of $E = C^b_{\mathfrak{r}}(G)$ as follows: $(g \cdot \phi)(f) := \phi(g^{-1} \cdot f)$, where $\phi\colon E \to \mathbb{K}$ is a linear functional (an element of $E'$), $g \in G$, and $f \in E = C^b_{\mathfrak{r}}(G)$. This action is a representation of $G$ in the dual Banach space by isometries (the so-called *contragredient representation* to the left regular representation $L$) and consequently can be restricted to the unit ball, $B$, of $E'$. It is now an easy exercise to verify that the action of $G$ on $B$ is continuous if $B$ is equipped with the (always compact) weak$^*$ topology (that is, the weakest topology making each evaluation mapping of the form $E' \ni \phi \mapsto \phi(f)$, $f \in E$ continuous).

In fact, one can verify somewhat more: that the continuous monomorphism $G \hookrightarrow \mathrm{Homeo}_c(B_w)$ is an embedding of topological subgroups. Here $B_w$ is the dual ball with the weak$^*$ topology.

**Corollary 2.2.4.** *Every topological group $G$ is topologically isomorphic with a subgroup of the group of homeomorphisms of a suitable compact space equipped with the compact-open topology.*

One can reformulate the above statement by saying that every topological group acts *topologically effectively* on an appropriate compact space.

**Remark 2.2.5.** One of the most interesting and best known open problems related to actions of topological groups on compacta is the *Hilbert–Smith conjecture,* which can be given the following equivalent reformulation: if a zero-dimensional compact group $G$ acts effectively on a finite-dimensional topological manifold $X$, then $G$ is necessarily finite. See the related papers [78, 126, 94, 59].

*A universal group with countable base*

Now we will proceed to a particularly nice application of Teleman's construction. In mid-30's Ulam asked in the Scottish Book (cf. Problem 103 in [60]) if there existed a *universal separable topological group,* that is, a separable group $G$ containing an isomorphic copy of every other separable topological group. As it was discovered rather soon, the negative answer follows from simple set-theoretic considerations: indeed, there are more pairwise non-isomorphic separable groups than there are different subgroups in any single separable group (*ibid.*) In his comments to Ulam's problem, Kallman had suggested several 'corrected' versions of the same question, one of them being as follows: does there exist a universal *second-countable* topological group? Apparently quite independently, the same question was promoted by Arhangel'skiĭ [5, 7]. The answer is contained in the following result, which exemplifies an interplay between embeddings and actions (as well as massive groups such as $\mathrm{Homeo}\,(\mathbb{I}^\omega)$ undoubtedly is).

**Theorem 2.2.6** (Uspenskij, 1985, [111])**.** $\mathrm{Homeo}_c(\mathbb{I}^\omega)$ *is a universal second-countable topological group.*

*Proof.* Let $G$ be a second-countable group. Then the Banach space $C^b_{r}(G)$ conains a $G$-invariant separable Banach subspace $E$ whose elements separate points and closed subsets of $G$. The unit ball $B_w$ of the dual space to $E$ in the weak* topology, being a convex compact subset of the separable Fréchet (= completely metrizable locally convex) space $E'_{w^*}$, is homeomorphic to the Hilbert cube $\mathbb{I}^\omega$ by force of Keller's theorem (cf. [15]). Consequently,

$$G \hookrightarrow \mathrm{Homeo}_c(B_w) \cong \mathrm{Homeo}_c(\mathbb{I}^\omega). \qquad \square$$

**Remark 2.2.7.** A more accurate rendering of the same idea shows that the pair $(\mathrm{Homeo}\,(\mathbb{I}^\omega), \mathbb{I}^\omega)$ forms a *universal second-countable topological transformation group:* every compact $G$-space $X$, where both $G$ and $X$ are second-countable, embeds into $(\mathrm{Homeo}\,(\mathbb{I}^\omega), \mathbb{I}^\omega)$ in a clear sense. This result seems to have been obtained by Megrelishvili independently from Uspenskij, only to be published a decade later in [63].

## 2.3. Urysohn Metric Spaces and their Groups of Isometries

We are going to steer towards another example of a universal group with countable base. A metric space $M$ is a (generalized) *Urysohn space* if for every finite metric space $X$ and every finite subspace $Y$, every isometric embedding $Y \hookrightarrow M$ extends to an isometric embedding of $X$ into $M$. (Cf. [110], [49], or [38], $3.11_+$.) Every separable Urysohn metric space $M$ contains an isometric copy of every other separable metric space, and if $M$ is in addition complete, it is unique up to isometry; we will denote it by $\mathbb{U}$. (A proof of the latter statement consists of shuttling between the two spaces and building up a recursive sequence of extensions using increasing chains of finite subspaces with everywhere dense union chosen in each space.) The groups of the form $\mathrm{Iso}\,(M)_s$ provide yet another series of examples of 'massive' topological groups.

A metric space $X$ is called *$n$-homogeneous*, where $n$ is a natural number, if every isometry between two subspaces of $X$ containing at most $n$ elements each extends to an isometry of $X$ onto itself. If $X$ is $n$-homogeneous for every natural $n$, then it is said to be *$\omega$-homogeneous.* The complete separable Urysohn space $\mathbb{U}$ is $\omega$-homogeneous and moreover enjoys the stronger property: every isometry between two compact subspaces of $X$ extends to an isometry of $X$ onto itself. At the same time, non-separable Urysohn metric spaces need not have this property.

The following construction of a Urysohn metric space extension of a given metric space was suggested by Katětov [49]. Let $X$ be a metric space and let $Y \subseteq X$ be a metric subspace. Let us say, following [49, 119], that a 1-Lipschitz real-valued function $f$ on a $X$ is *supported on,* or else *controlled by,* $Y$, if for every $x \in X$

$$f(x) = \inf\{\rho(x,y) + f(y) \colon y \in Y\}.$$

In other words, $f$ is the largest among all 1-Lipschitz functions on $X$ having the prescribed restriction to $Y$. As an example, every distance function $x \mapsto \rho(x, x_0)$ from a point $x_0$ is controlled by a singleton, $\{x_0\}$. Remark that every 1-Lipschitz function on $Y$ extends in a unique way to a 1-Lipschitz function on all of $X$ controlled by $Y$.

Denote by $X^\dagger$ the collection of all 1-Lipschitz functions $f\colon X \to \mathbb{R}$ that are controlled by finite subspaces of $X$ (depending on the function). If equipped with the supremum metric,

$$d_{X^\dagger}(f,g) := \sup_{x\in X} |f(x) - f(y)|,$$

$X^\dagger$ becomes a metric space of the same weight as $X$. Moreover, $X$ isometrically embeds into $X^\dagger$ in a natural way:

$$X \ni x \mapsto [d_x\colon X \ni y \mapsto d(x,y) \in \mathbb{R}] \in X^\dagger.$$

The embedding $X \hookrightarrow X^\dagger$ has a much stronger property than being just isometric:

(A) whenever $Y$ is a finite metric subspace of $X$ and $Y' = Y \cup \{x^*\}$ is an arbitrary one-point metric extension of $Y$, the metric space $Y'$ is isometric to the space $Y \cup \{f\}$ for some $f \in X^\dagger$ under the identification $x^* \leftrightarrow f$.

Indeed, the distance function $f\colon x \mapsto d_X(x, x^*)$ is in $X^\dagger$, and the metric spaces $Y'$ and $Y \cup \{f\}$ are isometric.

Apply the Katětov extension $X \mapsto X^\dagger$ to an arbitrary metric space $X$ in a recursive fashion $\omega$ times, and denote by $\widetilde{X}$ the completion of the metric space union of all finite iterations:

$$\cup_{n\in\mathbb{N}} X^{\dagger\dagger\dagger\cdots\dagger\ (n \text{ times})}.$$

The metric space $\widetilde{X}$ is generalized Urysohn. In particular, if $X$ is separable, then $\widetilde{X}$ is isometric to a complete separable Urysohn space $\mathbb{U}$.

The following theorem also appears as an exercise in Gromov's book [38], Ch. $3\frac{1}{2}$, published in 1999.

**Theorem 2.3.1** (Uspenskij, 1990, [112])**.** *The topological group* $\mathrm{Iso}\,(\mathbb{U})_s$ *is a universal second-countable group.*

*Proof.* The proof is based on the following remarkable property of the functor $X \mapsto X^\dagger$, first noticed and put to use by Uspenskij: every action of a group $G$ on the space $X$ by isometries extends in a canonical way to an action of $G$ on $X^\dagger$ (by left translations), and if the original action of $G$ on $X$ was continuous, so will be the extended action of $G$ on $X^\dagger$. (To better appreciate the usefulness of Katětov functions, notice that in general the same action of $G$ on the space of *all* 1-Lipschitz functions on $X$ need *not* be continuous!) Now it is rather evident that every continuous action of a topological group $G$ on a metric space $X$ by isometries extends to a continuous action of $G$ on $\widetilde{X}$ be isometries in a canonical sort of way. If the original action on $X$ determined an embedding of topological groups $G \hookrightarrow \mathrm{Iso}\,(X)_s$, then clearly $G$ is a topological subgroup of $\mathrm{Iso}\,(\widetilde{X})_s$ as well.

If $G$ is a separable topological group, then one can start with any separable metric space $X$ whose group of isometries $\mathrm{Iso}\,(X)_s$ contains $G$ as a topological subgroup to obtain the desired conclusion. (For example, $X = E$ as in the proof of Teleman's theorem, or simply $X = G$ itself equipped with a right-invariant metric generating the topology.) The space $\widetilde{X}$ is then separable and isometric to $\mathbb{U}$ as in the statement of the theorem. □

A more complicated argument, due to the same author, establishes the following result.

**Theorem 2.3.2** (Uspenskij, 1998, [119]). *Every topological group $G$ embeds as a topological subgroup into the isometry group of a suitable generalized Urysohn space $M$ that is $\omega$-homogeneous and has the same weight as $G$.*

The proof resembles that of Theorem 2.3.1. However, in order to achieve $\omega$-homogeneity of the union space, one has to alternate between the Katětov metric extension $X^\dagger$ and the 'equivariant homogenization' extension, $H(\cdot)$. This extension, forming the nontrivial technical core of the proof, is described in the following result.

**Theorem 2.3.3** (Uspenskij [119])**.** *Every metric space $X$ embeds, as a metric subspace, into an $\omega$-homogeneous metric space $H(X)$ of the same weight as $X$ in such a way that there is a continuous group homomorphism $e\colon \mathrm{Iso}\,(X) \to \mathrm{Iso}\,(H(X))$ with the property that for each $g \in \mathrm{Iso}\,(X)$, $e(g)|_X = g$.*

However, since there is no apparent reason for non-separable complete Urysohn spaces to be unique up to isometry, the above result cannot be used in order to answer the following open question:

**Question 2.3.4.** *Let $\tau > \aleph_0$ be a cardinal. Does there exist a universal topological group of weight $\tau$?*

**Remark 2.3.5.** A recent result by Shkarin [99] states that there exists a universal *abelian* second-countable topological group.

Returning to the groups of isometries of generalized Urysohn spaces, we want to mention the following result, whose proof is based on the lower compactification of the groups $\mathrm{Iso}\,(U)$. (Cf. subsection 2.5.) Recall that a topological group $G$ is called *minimal* if it admits no strictly coarser Hausdorff group topology, and *topologically simple* if $G$ contains no closed normal subgroups other than $G$ itself and $\{e_G\}$.

**Theorem 2.3.6** (Uspenskij, [119])**.** *Let $U$ be an $\omega$-homogeneous generalized Urysohn metric space. Then the group $\mathrm{Iso}\,(U)$ is minimal and topologically simple.*

**Corollary 2.3.7** (*loco citato*)**.** *Every topological group $G$ embeds, as a topological subgroup, into a minimal topologically simple group.*

*Application to embeddings*

The last two results by Uspenskij provide a very interesting and quite unexpected insight into a general problem about embeddability of topological groups into groups with various additional properties.

The following question used to be quite popular among the members of the school of topological algebra headed by V. Alexander Arhangel'skiĭ at Moscow University. (Cf. [5, 7].) Suppose $\mathcal{G}$ is a certain class of topological groups. Under which conditions is a given topological group $G$ isomorphic with a topological subgroup of the direct product of a subfamily of $\mathcal{G}$? In particular, is *every* topological group $G$ isomorphic with such a subgroup? Using the notation adopted in theory of varieties of topological groups [73], the latter question can be restated as follows: given a class $\mathcal{G}$ of topological groups, when is it true that $SP(\mathcal{G})$ contains all (Hausdorff) topological groups? (Here the letters $S$ and $P$ indicate the transition to a topological subgroup and the direct product, respectively.)

We believe that the following result (conjectured a few years ago by the present author in [84]) appears here for the first time.

**Corollary 2.3.8.** *Let $\mathcal{G}$ be a class of topological groups such that every topological group is isomorphic with a topological subgroup of the direct product of a family of groups from $\mathcal{G}$. Then every topological group is isomorphic with a topological subgroup of a suitable group from $\mathcal{G}$.*

*Put otherwise, if $SP(\mathcal{G})$ is the class of all topological groups, then already $S(\mathcal{G})$ is the class of all topological groups.*

*Proof.* Let $G$ be an arbitrary topological group. Without loss in generality, assume that $G \neq \{e_G\}$. Using Uspenskij's Corollary 2.3.7, embed $G$ into a minimal, topologically simple group $H$. Let now $H$ be embedded, as a topological subgroup, into $\prod_{\alpha\in A} G_\alpha$, where $G_\alpha \in \mathcal{G}$. For at least one $\alpha \in A$, the image of $H$ under the $\alpha$-th coordinate projection $\pi_\alpha$ is non-trivial. Because of topological simplicity of $H$, the kernel of $\pi_\alpha|_H$ is $\{e_H\}$, and thus the restriction $\pi_\alpha|_H$ is a (continuous) group monomorphism. Because of minimality of $H$, the latter monomorphism is in fact a topological isomorphism. Consequently, $\pi_\alpha|_G$is a topological isomorphism of $G$ with a topological subgroup of $G_\alpha$. □

**Corollary 2.3.9.** *If $\mathcal{G}$ is a class of topological groups closed under formation of topological subgroups and having the property that every topological group is isomorphic with a topological subgroup of the direct product of a family of groups from $\mathcal{G}$, then $\mathcal{G}$ is the class of all topological groups.*

The above corollaries from Uspenskij's results provide a rather efficient tool for handling questions of the type mentioned at the beginning of the subsection.

## 2.4. Representations in Reflexive Spaces

Let us now re-examine Teleman's theorem 2.2.1 again. It is evident from the proof that the result has the following stronger form (which we have already repeatedly used above).

**Theorem 2.4.1** (Teleman, 1957). *Every Hausdorff topological group $G$ embeds, as a topological subgroup, into the group* $\mathrm{Iso}\,(E)_s$ *of isometries of a suitable Banach space $E$ equipped with the strong operator topology.*

Can the above result be further strengthened?

The first thing to be observed is that $E$ cannot be replaced with a *Hilbert* space. Indeed, there are known to exist topological groups possessing no nontrivial strongly continuous unitary representations.

**Example 2.4.2.** The first example of such kind seems to belong to Herer and Christensen [43], and the group in question is an abelian topological group. Here we will convey the idea of the construction.

A non-negative real-valued function $\varphi$ defined on elements of a sigma-algebra $\mathcal{A}$ of subsets of a set $X$ is called a *submeasure* if it is countably subadditive (that is, always $\varphi(\cup_{i\in\mathbb{N}}A_i) \leq \sum_{i\in\mathbb{N}}\varphi(A_i)$), monotone (that is, $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$), and satisfies $\varphi(\emptyset) = 0$. A submeasure $\varphi$ is called *pathological* if it is not identically zero and there exists no sigma-additive measure on $(X, \mathcal{A})$ all of whose null sets are null sets with respect to $\varphi$. It can be shown that pathological submeasures exist on every non-atomic sigma-algebra of sets. In particular, there exists such a submeasure, $\varphi$, on the sigma-algebra of Borel subsets of the Cantor set $X = \{0,1\}^\omega$, and moreover one can assume the 'regularity' condition: $\varphi(V) > 0$ for every nonempty open and closed subset of $X$. Equip the vector space $C(X)$ with the topology of convergence in submeasure $\varphi$: basic neighbourhoods of zero are of the form

$$V_{\epsilon,\delta} := \{f \in C(X) \colon \varphi(\{x \in X \colon |f(x)| > \delta\}) < \epsilon\}, \epsilon, \delta > 0.$$

It is shown in [43] that the additive topological group of the topological vector space $C(X)$ with the above topology admits no nontrivial strongly continuous unitary representations in Hilbert spaces.

**Example 2.4.3.** Later on, Banaszczyk [10] had shown the existence of abelian Banach–Lie groups without nontrivial strongly continuous unitary representations. They are topological factor-groups of the additive group of the separable Hilbert space $l_2$ by a suitably chosen discrete subgroup $\Gamma$.

Recall that a topological group $G$ together with a smooth structure modelled on a (possibly infinite-dimensional) Banach space is called a *Banach–Lie group*. The above smooth structure is determined by a neighbourhood of identity, $V$, in $G$ and a homeomorphism $\phi$ between $V$ and the open unit ball $B$ in a suitable Banach space $E$ satisfying a certain smoothness condition. To describe it, notice that for each $g \in G$ the formula ${}_g\phi(h) := \phi(g^{-1}h)$ determines a homeomorphism between the neighbourhood $gV$ of $g$ and the ball $B$. The domain of the composition map $\phi \circ ({}_g\phi)^{-1}$ is therefore an open subset of $V$ (possibly empty). The smoothness condition now is this: whenever $g \in G$, the map $\phi \circ ({}_g\phi)^{-1}$ is $C^\infty$ in its domain of definition. In particular, it follows easily that a topological factor group of a Banach-Lie group by a discrete (though not necessarily by an arbitrary closed) subgroup inherits the Banach-Lie structure in a canonical way. A good introduction to the theory of Banach–Lie groups along these lines can be found in Karl Hofmann's lecture notes [46], and some of it made its way into the recently published book [47].

Banach-Lie groups form in a sense the closest class of topological groups (with additional structure) to that of finite-dimensional Lie groups, and such a drastic difference in behaviour between groups from two classes is rather stunning. Indeed, it is worth recalling a well-known and easy to prove fact: every locally compact group admits a faithful strongly continuous unitary representation. Such is the left regular representation of $G$ in the space $L_2(G)$ of all square-integrable complex-valued functions on $G$ with respect to the Haar measure. Locally compact abelian (LCA) groups enjoy a much stronger property: characters (that is, continuous homomorphisms to the circle rotation group $\mathbb{T} = U(1)$) separate points in LCA groups. Notice that the groups in Banaszczyk's example are abelian, and having no nontrivial unitary representations is a much more restrictive property than just having no continuous characters.

The next natural thing to ask is, can one at least assume that the Banach space $E$ in Theorem 2.4.1 is reflexive?

A negative answer was very recently given by Megrelishvili [65] who has shown that the topological group $\mathrm{Homeo}_+(\mathbb{I})$ of all orientation-preserving homeomorphisms of the closed interval equipped with the compact-open topology admits no non-trivial representations in reflexive Banach spaces. The rest of this entire section is loosely grouped around a sketch of the proof of this result.

We will begin with the following criterion. Recall that the weak topology on a topological vector space $E$ is the coarsest topology with respect to which every continuous functional $\phi$ on $E$ remains continuous. The space $E$ equipped with the weak topology will be denoted by $E_w$. Recall also that a (real or complex-valued) bounded function $f$ on a topological group $G$ is called *weakly almost periodic* (*WAP*) [23], if the set of all left translations $\{L_g f : g \in G\}$ of this function is weakly relatively compact in the space $C^b(G)$ of all continuous bounded functions on $G$ with the topology of uniform convergence. In other words, the closure of $\{L_g f : g \in G\}$ in the space $C^b(G)_w$ is compact. It is useful to notice that the concept of weak almost periodicity is partially independent of the topology on the group $G$ in the sense that the same function $f$ remains weakly almost periodic on the group $G$ equipped with the discrete topology.

**Theorem 2.4.4.** (Shtern, 1994, [101] and Megrelishvili, 1998, [64]) *A topological group $G$ embeds into the isometry group of a reflexive Banach space equipped with the strong operator topology if and only if weakly almost periodic functions separate points and closed subsets in $G$.*

The proof will be preceded by a few concepts. Suppose a group $G$ acts on a normed space $E$ by isometries, that is, we are given a homomorphism $\pi\colon G \to \mathrm{Iso}\,(E)$ (a representation of $G$). Fix a vector $\xi \in E$ (usually of norm one) and a bounded linear functional $\phi$ on $E$. The function

$$\tau_{\xi,\phi} : G \ni g \mapsto \phi(g \cdot \xi) \in \mathbb{K}$$

(where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is called the $(\xi, \phi)$-*th matrix coefficient* (of the representation $\pi$). The topology induced on $G$ (or else on the group of isometries $\mathrm{Iso}\,(E)$) by the family of all matrix coefficients is called the *weak operator topology.* If $E$ is reflexive, then the weak operator topology and the strong operator topology on $G$ coincide. This important fact was known for the Hilbert spaces $E$ since long ago, but for general reflexive Banach spaces it was only recently established by Megrelishvili [64]. (Notice that for non-reflexive spaces there are counterexamples distinguishing between the two topologies on groups of isometries, *loco citato.*)

It is easy to notice that if $E$ is a reflexive Banach space, then every matrix coefficient $\tau_{\xi,\phi}$ is a weakly almost periodic function on $G$. Firstly notice that the $G$-orbit of this coefficient consists of all matrix coefficients of the form $\tau_{\xi,g\cdot\phi}$, $g \in G$. Further, the mapping

$$E^* \ni \chi \mapsto \tau_{\xi,\chi} \in C^b(G)$$

is a bounded (= continuous) linear operator with respect to the norm topologies on both spaces (here $\xi \in E$ is fixed), therefore it remains continuous relative to the weak topologies on both spaces in question and, in particular, the restriction

$$B^*_w \ni \chi \mapsto \tau_{\xi,\chi} \in C^b(G)_w$$

is continuous, where both the unit ball in the dual space and the space of continuous functions are equipped with their weak topologies. The weak closure of the $G$-orbit of a unit functional $\phi$ is compact (the reflexivity of a Banach space is equivalent to the weak compactness of the dual unit ball). Consequently, the image of this closure in $C^b(G)_w$ is compact. But this is exactly the weak closure of the $G$-orbit of the matrix coefficient in question.

Hence follows the *necessity* in Theorem 2.4.4: suppose $E$ is a reflexive Banach space and $G \hookrightarrow \mathrm{Iso}\,(E)_s$ is a topological embedding. According to the above result by Megrelishvili, matrix coefficients generate the topology of $G$. It remains to notice that the collection WAP(G) of all weakly almost periodic functions on $G$ (which in fact forms a $C^*$-algebra) is closed under the lattice operations, such as taking maxima of finite families of functions. The rest is just simple routine of general topology.

Now let us outline the proof of *sufficiency* of Th. 2.4.4 as proposed by Megrelishvili [64]. (The proof of Shtern drew upon algebra representation theory.) Make the collection WAP(G) of all weakly almost periodic functions on a topological group $G$ into a Banach space by equipping it with the supremum norm.

Let $f \in \mathrm{WAP(G)}$ be arbitrary, and denote by $E = E_f$ the linear span of the orbit $G \cdot f$ in WAP(G). Denote by $W = W_f$ the convex circled envelope of $G \cdot f$ in $E$. (Recall that a subset $A$ of a vector space is *circled* if $\lambda A \subseteq A$ whenever $\lambda$ is a scalar with $|\lambda| \leq 1$; for real vector spaces circled sets are just symmetric sets.) As a consequence of the Krein–Šmulian Theorem, $W$ is relatively weakly compact (indeed, for every $f \in \mathrm{WAP(G)}$, the orbit $G \cdot f$ is relatively weakly compact).

Now a procedure developed in [21] is applied. For every natural $n$, let $\|\cdot\|_n$ denote the equivalent norm on the space $E$ whose open unit ball is

$$U_n := 2^n W + 2^{-n} B_E,$$

where $B_E$ is the unit ball of $E$ with respect to the induced (supremum) norm. Define

$$\|x\|_* := \left( \sum_{n=1}^{\infty} \|x\|_n^2 \right)^{\frac{1}{2}}$$

for every $x$ from the set $X = X_f$ of elements for which the expression above assumes a finite value. The norm $\|\cdot\|_*$ is invariant under the action of $G$ by left translations (as $X$ is a translation-invariant collection of functions on $G$). The weak compactness of $W$ results in the weak compactness of the unit ball of $\|\cdot\|_*$, which fact implies that $X$ is a reflexive Banach space.

The space $X$ is continuously embedded into $E$ in a canonical way so that on every bounded subset of $X$ the embedding is a homeomorphism with respect to the weak topologies. Since the orbit $G \cdot f$ is bounded in $X$, it follows that the representation of $G$ in $X$ is a strongly continuous representation by isometries.

Form the $l_2$-direct sum of all representations of $G$ obtained in the above way as $f$ runs over the collection of all weakly almost periodic functions on $G$. One obtains a representation of $G$ in a reflexive Banach space $Y = \oplus_{f \in \mathrm{WAP(G)}} X_f$ by isometries which determines a topological group embedding $G \hookrightarrow \mathrm{Iso}\,(Y)_s$.

**Remark 2.4.5.** Of course the above criterion 2.4.4 can be slightly adjusted to suit a variety of different situations. For example, a topological group $G$ admits a separating family of strongly continuous representations in reflexive Banach spaces if and only if WAP functions separate points in $G$. Or else: the existence of a non-trivial representation in a reflexive Banach space is equivalent to the existence of a non-constant WAP function on $G$.

## 2.5. Compactifications of Topological Groups

Now we can proceed to an example of a topological group admitting no non-trivial strongly continuous representations by isometries in reflexive Banach spaces. It is interesting to note that such an example is supplied by one of the most common 'massive' groups.

**Theorem 2.5.1** (Megrelishvili, 1999, [65])**.** *Every weakly almost periodic function on the topological group* $\mathrm{Homeo}_+[0,1]$ *is constant. Consequently, this group admits no non-trivial strongly continuous representations in reflexive Banach spaces.*

To better understand the idea of the proof, it is useful to start with the concept of the *lower uniformity* on a topological group – a concept that has growing significance on its own, especially, it seems, for 'massive' topological groups, and which was first investigated by Roelcke (see e.g. [95]) and since then brought to the limelight through the work of Uspenskij.

Let us begin with an obvious observation: both standard uniformities on a topological group $G$ — namely, the right uniformly equicontinuous uniform structure $\mathcal{U}_{\Rsh}(G)$ and its left counterpart $\mathcal{U}_{\Lsh}(G)$ — are *compatible uniformities*, that is, each of them generates the topology of $G$. As a straightforward consequence, the *upper* uniform structure, which is the supremum of the two,

$$\mathcal{U}_{\vee}(G) := \mathcal{U}_{\Rsh}(G) \vee \mathcal{U}_{\Lsh}(G),$$

is compatible as well. The upper uniform structure is best known in the context of *completeness* of topological groups. Indeed, not every topological group embeds into a topological group that is complete with respect to any (equivalently: both) of the one-sided structures $\mathcal{U}_{\Rsh}(G)$ and $\mathcal{U}_{\Lsh}(G)$ (such groups are called *Weil-complete.*) A counter-example (Dieudonné [22]) is the same group $\mathrm{Homeo}_+[0,1]$. At the same time, every topological group embeds as an everywhere dense subgroup into a group that is complete relative to the upper uniformity. (Which fact of course just means that the concepts of completeness and completion with respect to one-sided uniformity are misfits that must be discarded in favour of two-sided completeness and completion.)

The *lower* uniformity is the infimum of the left and right equicontinuous uniformities:

$$\mathcal{U}_{\wedge}(G) := \mathcal{U}_{\Rsh}(G) \wedge \mathcal{U}_{\Lsh}(G).$$

It is also a compatible uniformity, which fact can be seen from the explicit form of basic entourages,

$$V_{\wedge} := \{(x,y) \in G \times G \colon x \in VyV\},$$

where $V$ runs over neighbourhoods of the identity element $e_G$.

A remarkable observation is that for many 'massive' topological groups the lower uniform structure $\mathcal{U}_{\wedge}(G)$ turns out to be precompact. Uspenskij calls topological groups with this property *Roelcke-precompact.* We side with the author of the review [36] and adopt a more functional terminology, calling such groups *lower precompact,* and their completions with respect to the lower uniformity *lower completions.*

Here are some of the major examples of lower precompact topological groups.

**Example 2.5.2.** The full unitary group $U(\mathcal{H})_s$ of an infinite-dimensional Hilbert space with the strong operator topology. The lower compactification of $U(\mathcal{H})_s$ can be identified with the semigroup of all operators on $\mathcal{H}$ of norm $\leq 1$, equipped with the weak operator topology. [117].

**Example 2.5.3.** Let $X$ be an infinite set. Equip the full group of permutations $S(X)$ with the topology of simple convergence (where $X$ is regarded as a discrete set), that is, the topology induced from the Tychonoff power $X^X$. Then $S(X)$ is lower-precompact. [95]

**Example 2.5.4.** The group $\mathrm{Homeo}_+(I)$ of orientation-preserving homeomorphisms of the closed unit interval with the compact-open topology. Identify each homeomorphism $h\colon I \to I$ with its graph in the square $I \times I$. Then the lower completion of the group $\mathrm{Homeo}_+(I)$ can be identified with the collection of all $C^0$ curves $\gamma$ in $I \times I$ starting at the lower left corner $(0,0)$ and ending at the right upper corner $(1,1)$ and never going either to the left or down — more exactly, if an orientation-preserving parametrization of $\gamma$ is chosen and two values of the parameter satisfy $t_1 \leq t_2$, one necessarily has $\gamma(t_1)_i \leq \gamma(t_2)_i$, $i = 1,2$. (Fig. 1.)

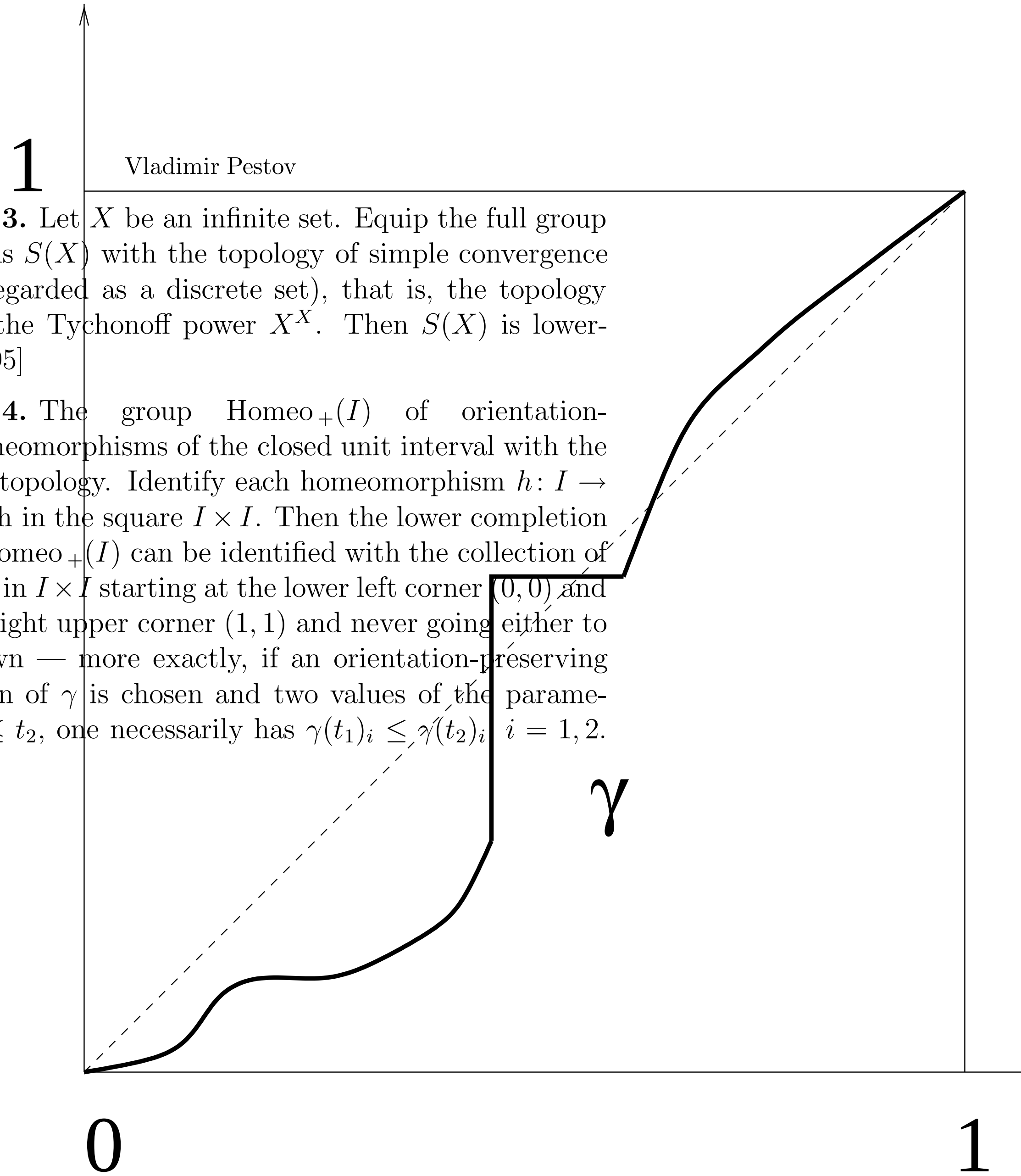


Fig. 1. An example of an element of $\gamma_\wedge(\mathrm{Homeo}_+(\mathbb{I}))$.

The topology on the set of curves is that of uniform convergence. (Uspenskij, [118].)

**Example 2.5.5.** Let $M$ be a complete Urysohn space that is *$\omega$-homogeneous*, that is, every isometry between two finite metric subspaces of $M$ extends to an isometry of $M$ onto itself.

(In the separable case, $\omega$-homogeneity of a complete Urysohn metric space can be taken for granted.) The full group of isometries $\mathrm{Iso}\,(M)_s$ of $M$ equipped with the pointwise topology is lower precompact. (Uspenskij, [119].) In this case, the lower completion of $\mathrm{Iso}\,(M)_s$ can be identified with a set of functions $M \times M \to I$ satisfying a certain set of conditions which we do not reproduce here.

The lower completion of a lower precompact topological group $G$ always supports the structure of a $G$-space, and sometimes that of a semigroup, which structures can be used to establish certain topological-algebraic properties of $G$ itself such as minimality and topological simplicity. We invite the interested reader to consult Uspenskij's papers [117, 119].

**Remark 2.5.6.** There is an interesting yet largely unexplored connection between the above concepts of the lower precompactness and lower completion, on the one side, and modern representation theory of 'massive' groups, on the other. It appears that many major examples of infinite dimensional groups admitting a tractable representation theory in Hilbert spaces and commanding prominence in mathematical physics are lower precompact if equipped with a 'natural' topology.

Namely, in a theory developed over the past decade or two, largely through the efforts of Neretin, G. Ol'shansky and others (the book [77] is the most up-to-date source), to every 'massive' group there is associated a certain compact semigroup called the *mantle* of $G$ and having the same representation theory in Hilbert spaces as $G$. The mantle of a group happens to coincide with the lower completion in the most telling case of the full unitary group. One certainly expects the lower completion of a lower precompact group to be typically 'larger' than the mantle, as the group $\mathrm{Homeo}_+(I)$ examplifies (whose mantle is of course trivial, as one of the consequences of Megrelishvili's result). Yet it may well happen that for many important lower precompact groups the mantle coincides with the lower completion.

Among the topological groups whose mantle has been computed are groups of diffeomorphisms of manifolds, groups associated to Virasoro and other Kac–Moody algebras, various groups of operators in Hilbert spaces (subgroups of unitary groups preserving one structure or other), groups of currents, and groups of automorphisms of measure spaces. It is interesting to examine each of these groups for lower precompactness and to compare the lower completion with the mantle.

What is the relationship between lower precompactness and weak almost periodicity? It turns out that every weakly almost periodic function $f$ on a topological group $G$ is uniformly continuous with respect to both uniformities $\mathcal{U}_{\ulcorner}(G)$, $\mathcal{U}_{\urcorner}(G)$. (For a proof of this fact, see e.g. [97]. Notice that one half of this statement is nearly obvious: the $\mathcal{U}_{\urcorner}(G)$-uniform continuity follows in a straightforward fashion from the compactness of the orbit of $f$ in the *pointwise* topology.) Consequently, the algebra WAP(G) is contained in the algebra $C^b_{\wedge}(G)$ of all bounded lower uniformly continuous functions.

Now recall that every algebra $A$ formed by bounded continuous functions on a topological space $X$ determines a compactification of $X$, which is the completion with respect to the coarsest uniform structure making each function $f \in A$ uniformly continuous. We will denote this compactification by $\gamma_A(X)$ (though in fact Spec $A$ would be more informative and commonly recognizable). The compact space $\gamma_A(X)$ is the maximal ideal space of $A$. Teleman's argument actually tells us that in the case where $X = G$ is a topological group, the algebra $A$ is closed under translations by elements of $G$, and the action of $G$ on $A$ by left translations is continuous (where $A$ is equipped with the topology of uniform convergence on $G$), then the resulting action of $G$ on the compactification $\gamma_A(G)$ is continuous as well, that is, $\gamma_A(G)$ is a compact $G$-space. (Indeed, one only has to notice that in this case $A$ is contained in the algebra $C^b_{\ulcorner}(G)$ as a $G$-invariant subalgebra, and the compactification $\gamma_A(G)$ in question forms a

$G$-invariant compact subset of the unit ball $B^*_w$ in the dual space $C^b_{\mathrm{r}}(G)'$; since the action of $G$ on the ball is continuous, so is the action of $G$ on $\gamma_A(G)$.)

Let us consider some examples of topological group compactifications of this sort.

**Example 2.5.7.** Here is the master concept. A bounded scalar-valued function $f$ on a topological group $G$ is called *almost periodic* if the $G$-orbit of $f$ is relatively compact in $C^b(G)$. The collection of all almost periodic functions is denoted by AP(G), and the corresponding compactification $\gamma_{\mathrm{AP}}(G)$ is known as the *Bohr compactification* of $G$. The Bohr compactification of a topological group is itself a compact group, and it is maximal among all compact groups into which $G$ admits a continuous homomorphism with a dense image. This concept is one of the cornerstones of the classical abstract harmonic analysis [44]. The following two related notions will be used later on. A topological group $G$ is *maximally almost periodic* (*MAP*) if the canonical continuous homomorphism $G \to \gamma_{\mathrm{AP}}(G)$ is a monomorphism, and *minimally almost periodic* (*map*) if $\gamma_{\mathrm{AP}}(G) = \{e\}$. These can be restated in terms of representation theory as follows: $G$ is MAP iff continuous unitary representations in finite-dimensional Hilbert spaces separate points in $G$, and $G$ is map iff it possesses no non-trivial continuous finite-dimensional unitary representations. If $G$ is abelian, then 'finite-dimensional unitary representations' in the above criteria can be replaced with 'characters' (continuous homomorphisms to the circle rotation group $\mathbb{T} = U(1)$).

**Example 2.5.8.** Of singular importance in abstract topological dynamics [9, 124, 88] is the compactification $\gamma_{\mathrm{r}}(G)$ formed with respect to the algebra $A = C^b_{\mathrm{r}}(G)$. This compactification is known as the *greatest ambit* of $G$ and possesses a certain universal property which we will now establish. Let $G$ be a topological group. A compact $G$-space $X$ together with a distinguished point $x^* \in X$ is called an *ambit* if the orbit $G \cdot x^*$

is everywhere dense in $X$. It is clear how to define morphisms between two $G$-ambits $(X, G, \tau_X, x^*)$ and $(Y, G, \tau_Y, y^*)$: such a morphism is a continuous map $\varphi\colon X \to Y$ which is equivariant, that is, commutes with the actions $\tau_X$ and $\tau_Y$ in the sense that, for all $x \in X$ and $g \in G$,

$$\varphi(g \cdot_X x) = g \cdot_Y \varphi(x),$$

and also $\varphi$ preserves the distinguished points:

$$f(x^*) = y^*.$$

One can make the compactification $\gamma_{r}(G)$ into an ambit by marking as the distinguished element $e_G^*$, which is simply the identity element of $G$, or rather its image in the compactification. Since the embedding $G \hookrightarrow \gamma_{r}(G)$ is clearly topological (the bounded $\mathcal{U}_{r}(G)$-uniformly continuous functions separate points and closed subsets in $G$), we can identify $G$ with a topological subspace of the greatest ambit. For every $G$-ambit $(X, x^*)$ there is a unique morphism of $G$-ambits $\varphi\colon \gamma_{r}(G) \to X$. Indeed, for every continuous (real or complex-valued) function $f$ on such a $G$-ambit $X$, the pullback $f_*$ of $F$ to $G$ defined for each $g \in G$ by

$$f_*(g) := f(g \cdot x^*)$$

is $\mathcal{U}_{r}(G)$-uniformly continuous (an easy, direct check). Moreover, the map $f \mapsto f_*$ as above is a homomorphism of $C^*$-algebras from $C(X)$ to $C^b_{r}(G)$ (that is, it is linear, bounded, multiplicative, and – in the complex case – preserves the involution). This map determines a continuous mapping between the corresponding compactifications going in the opposite direction, which is exactly the morphism of $G$-ambits $\gamma_{r}(G) \to X$ we are after. Often the greatest ambit of a topological group $G$ is denoted by $\mathcal{S}(G)$.

While of course there is a canonical morphism of $G$-ambits from $\gamma_{r}(G)$ onto the Bohr compactification $\gamma_{\mathrm{AP}}(G)$, it is one-to-one if and only if $G$ is a precompact group.

**Example 2.5.9.** The compactification $\gamma_{\mathrm{WAP}}(G)$ of a topological group with respect to the algebra WAP(G) of weakly almost periodic functions is known as the *maximal semitopological semigroup compactification of* $G$ and possesses a universal property similar to that of the greatest ambit with respect to all compact semitopological (that is, the operations are *separately* continuous) semigroups containing an image of $G$ as an everywhere dense subsemigroup. Clearly, there is a canonical $G$-equivariant mapping onto $\gamma_{r}(G) \to \gamma_{\mathrm{WAP}}(G)$. Megrelishvili's Theorem 2.5.1 says, equivalently, that the compactification $\gamma_{\mathrm{WAP}}(\mathrm{Homeo}_{+}\mathbb{I})$ is a singleton!

**Example 2.5.10.** If $A = C^b_{\wedge}(G)$ is the algebra of all bounded lower uniformly continuous functions on a topological group $G$, then we will denote the corresponding compactification of $G$ by $\gamma_{\wedge}(G)$. A topological group $G$ is lower precompact if and only if the compactification mapping $G \to \gamma_{\wedge}(G)$ induces on $G$ the lower uniformity. There is a canonical continuous mapping from $\gamma_{\wedge}(G)$ onto the compactification $\gamma_{\mathrm{WAP}}(G)$. Every weakly almost periodic function $f$ uniquely extends from $G$ to $\gamma_{\mathrm{WAP}}(G)$, and therefore clearly factors through the mapping $\gamma_{\wedge}(G) \to \gamma_{\mathrm{WAP}}(G)$. In other words, weakly almost periodic functions on $G$ can be identified with those continuous functions on the lower compactification $\gamma_{\wedge}(G)$ whose orbit under left translations by elements of $G$ is weakly relatively compact in the Banach space $C(\gamma_{\wedge}(G)) \cong C^b_{\wedge}(G)$. For those lower precompact groups whose lower completion admits a transparent geometric interpretation, this observation makes working with weakly almost periodic functions (notoriously evasive objects) somewhat easier. In particular, this applies to the group $\mathrm{Homeo}_{+}(\mathbb{I})$.

**Remark 2.5.11.** A reference to the subject of compactifications of the kind described above is the book [14].

Now we are in a position to convey the flavour of Megrelishvili's proof of Theorem 2.5.1. The proof is remarkably 'graphical.' For each triple of real numbers $a, b, c$ with $0 \leq a \leq c \leq b \leq 1$ denote by $\beta^{a,c,b}$ the element of the lower completion $\gamma_\wedge(\mathrm{Homeo}_+(\mathbb{I}))$ represented by the following curve (Fig. 2).

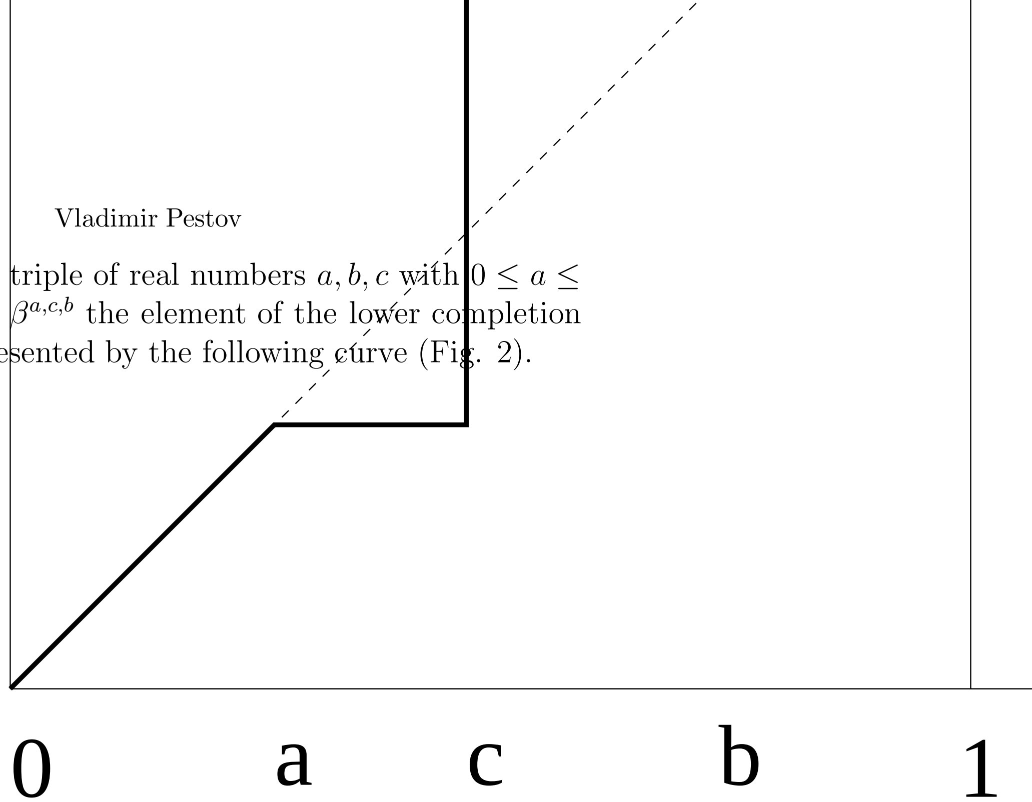


Fig. 2. The curve $\beta^{a,c,b}$.

As we have already remarked, the compactification $\gamma_{\mathrm{WAP}}(\mathrm{Homeo}_+(\mathbb{I}))$ supports a canonical semigroup structure. It turns out that the image of the curve $\beta^{0,1,1}$ in $\gamma_{\mathrm{WAP}}(\mathrm{Homeo}_+(\mathbb{I}))$ forms the zero element. More generally, using the fact that the latter semigroup is semitopological, one can verify that *every* zig-zag curve made up entirely of alternating horizontal and vertical segments, that is, one of the form

$$\beta^{a_0,a_1,a_2} \circ \beta^{a_2,a_3,a_4} \circ \cdots \beta^{a_{n-2},a_{n-1},a_n},$$

where $a_0, a_1, \cdots, a_n$ is a partition of the interval $\mathbb{I}$, goes to zero element under the map

$$\gamma_\wedge(\mathrm{Homeo}_+(\mathbb{I})) \to \gamma_{\mathrm{WAP}}(\mathrm{Homeo}_+(\mathbb{I})).$$

At the same time, as the mesh of the partition tends to zero, the zig-zag curves as above converge to the identity homeomorphism and consequently their images in the semigroup $\gamma_{\mathrm{WAP}}(\mathrm{Homeo}_+(\mathbb{I}))$ must converge to the identity element. But every semigroup whose zero and identity elements coincide with each other is trivial. Q.E.D.

**Remark 2.5.12.** We hope that the above hands-waving argument has not left an impression of Megrelishvili's proof being easy: a precise incarnation of the idea happens to be very involved technically.

**Question 2.5.13** (Megrelishvili; Akin and Glasner)**.** *Does there exist a* monothetic *topological group whose points and closed subsets are not separated by WAP functions?*

As observed by Megrelishvili, this is equivalent to the same question about arbitrary *abelian* topological groups, in view of Theorem 4.1.5 below.

We conclude this Section by recalling a long-standing open problem.

**Problem 2.5.14** (Shtern, [100])**.** *Give an intrinsic description of those topological groups $G$ admitting a separating family of strongly continuous unitary representations in Hilbert spaces (or else embeddable, as topological subgroups, into the unitary groups of Hilbert spaces with the strong operator topology).*

It is worthwhile looking at a well-known description of such topological groups — which however falls short of an acceptable intrinsic criterion — after which the very Theorem 2.4.4 was fashioned. Recall that a complex-valued function $f$ on a group $G$ is called *positive definite* if for every finite collection $g_1, g_2, \cdots, g_n$ of elements of $G$ and every collection of the same size of complex numbers, $\lambda_1, \lambda_2, \cdots, \lambda_n$, one has

$$\sum_{i,j=1}^{n} \lambda_i \overline{\lambda_j} f(g_i^{-1} g_j) \geq 0.$$

The relation between positive definite functions and unitary representations is as follows. If $\pi\colon G \to U(\mathcal{H})$ is a unitary representation of a group, then for every $\xi \in \mathcal{H}$ the function

$$G \ni g \mapsto (g \cdot \xi, \xi) \in \mathbb{C}$$

is positive definite. (Of course, this function is nothing but the matrix coefficient $\tau_{\xi,\hat{\xi}}$, where $\hat{\xi}$ denotes the linear functional represented by $\xi$ in the Hilbert space.) Conversely, given a positive definite function $f$ on $G$, one can construct a unitary representation of $G$ as follows: the Hilbert space $\mathcal{H}$ is the completion of the vector space $\mathrm{lin}\,(\mathrm{G})$ spanned by elements of $G$ as a Hamel basis and equipped with the inner product defined on elements of $G$ by the formula $(g,h) := f(g^{-1}h)$ and extended all over $\mathrm{lin}\,(\mathrm{G})$ by sesquilinearity. (Positive definiteness of $f$ serves to verify the first axiom of the inner product.)

Using the above two constructions, is not difficult to prove that a topological group $G$ embeds into the unitary group $U(\mathcal{H})_s$ of a Hilbert space if and only if continuous positive definite functions on $G$ separate points and elements of some closed subbase for $X$. Even if this description (along with obvious variations) is being extensively used in functional analysis and representation theory, it clearly falls short of what one would accept as an answer to Shtern's question 2.5.14. The theorem of Følner–Cotlar–Ricabarra (Th. 3.5.2 below) expemplifies what is being accepted as a satisfactory answer to questions of the above kind.

It is interesting that the representability in reflexive Banach spaces and unitary representability of topological groups have not been distinguished from each other either.

**Question 2.5.15** (Shtern, [101])**.** *Is it true that a topological group $G$ admits a compete system of strongly continuous representations by isometries in reflexive Banach spaces if and only if $G$ admits a complete system of strongly continuous unitary representations?*

Two obvious candidates for counter-examples are groups constructed by Herer–Christensen (Ex. 2.4.2) and by Banaszczyk (Ex. 2.4.3): how many WAP functions do they possess?

## 3. Free Actions *vs* Fixed Point on Compacta Property

### 3.1. Veech's Theorem

Now that we have examined the question of existence of effective actions on compacta along with a range of broader topics prompted by this question, it is natural to ask: what about the existence of *free* actions?

Rather remarkably, an answer to this question draws a watershed between the locally compact groups and many concrete 'massive' topological groups. By far the most important result in the affirmative direction is the following.

**Theorem 3.1.1** (W. Veech, 1977, [120]). *Every locally compact group $G$ acts freely on a compact space.*

The proof we shall now outline is based on a simplification of Veech's original argument proposed by Pym [91]. First of all, it follows from the 'general nonsense' of the theory of uniform compactifications that if $Y \subseteq (G, \mathcal{U}_{r}(G))$ is a uniformly discrete subspace, then the closure of $Y$ in the greatest ambit $\gamma_{r}(G)$ is canonically homeomorphic to the Stone-Čech compactification $\beta X$ of the space $X$ with the discrete topology. The property of a subset $X$ of a topological group $G$ being $\mathcal{U}_{r}(G)$-uniformly discrete means that for some neighbourhood $U$ of the identity:

$$Ux \cap Uy = \emptyset \text{ for all } x, y \in X,\ x \neq y. \tag{3.1}$$

Let $V$ be an open set with $\mathrm{cl}\,V$ compact and contained in $U$. It follows that the open set $V \cdot \mathrm{cl}\,X$ in $\gamma_{r}(G)$ is canonically homeomorphic to the product $V \times \beta X$.

Given a symmetric compact neighbourhood $U \ni e_G$, one can construct a maximal subset $X$ with the property (3.1). Clearly, the sets $U^2 x$, $x \in X$ form a cover of $G$, and consequently $\mathrm{cl}\,U^2 \cdot \mathrm{cl}\,X$ coincides with all of $\gamma_{r}(G)$. (Notice: here the compactness of $\mathrm{cl}\,U^2$ is used in an essential sort of way, and this is precisely where the argument breaks down for more general topological groups.)

If now $x^* \in \gamma_{\mathfrak{r}}(G)$ is arbitrary, then $x^* = u \cdot x$ for some $u \in \mathrm{cl\,U^2} \subseteq \mathrm{G}$ and $x \in \mathrm{cl\,X}$, and therefore $x^*$ belongs to the closure of the set $uX$ which is a maximal uniformly discrete set with respect to the neighbourhood of identity $uUu^{-1}$.

Let $g \in G$, $g \neq e_G$, and let $x^* \in \gamma_{\mathfrak{r}}(G)$ be arbitrary. As we have seen, there is no loss in generality is assuming that $x^*$ belongs to the closure of some set $X \subseteq G$ which is maximal with respect to the property (3.1), where $U$ is a suitable compact neighbourhood of identity. Of course one can choose $U$ as small as desired, in particular satisfying $g \notin U^2$.

Let $e_G \in V^2 \subset U$. Elementary combinatorial considerations show that $X$ can be partitioned into finitely many pieces $X_1, \ldots, X_k$ in such a way that for each $i = 1, 2, \ldots, k$ the sets $V \cdot X_i$ and $g \cdot X_i$ are disjoint in $G$. From the above description of the topological structure of $U \cdot \mathrm{cl\,X}$ in $\gamma_{\mathfrak{r}}(G)$ it follows that the set $\mathrm{cl\,X_i}$ is disjoint from its translation by $g$. Since $x^* \in \mathrm{cl\,X_{i_0}}$ for some $i_0$, we conclude that $g \cdot x^* \neq x^*$. Q.E.D.

Recall that an action of a group $G$ on a topological space $X$ is called *minimal* if the orbit of every point is everywhere dense in $X$. It is easy to see that if a topological group $G$ admits a free action on a compact space, then $G$ admits a free (and *ipso facto* effective) *minimal* action on a compact space: a direct application of Zorn Lemma to the family of all closed non-empty $G$-subspaces of $X$ leads to the existence of a minimal $G$-subspace, $Y$, and of course the restriction of the action of $G$ to $Y$ is still free. Hence the following corollary, which was, in fact, the *raison d'être* of Veech's result:

**Corollary 3.1.2.** *Every locally compact group admits an effective minimal action on a compact space.*

In particular, every locally compact group admits a fixed point-free action on a compactum.

**Remark 3.1.3.** As pointed out to me by A. Kechris, yet another proof of the Veech Theorem can be found in [1].

**Remark 3.1.4.** It is rather surprising that beyond the class of locally compact groups and some of their most immediate derivatives (such as MAP groups, including free topological groups and additive topological groups of locally convex spaces) we know precious little about topological groups admitting free actions on compacta. One exception is the rather exotic class of *P-groups* [89], that is, those topological groups in which every $G_\delta$-subset is open — hardly a class of great significance! Are there any other visible classes of topological groups admitting free actions on compacta? What we rather have at the moment, is a pageant of counter-examples growing richer by the day, cf. below.

### 3.2. Fixed Point on Compacta Property

If one wishes to go all the way in the opposite direction from the existence of free actions, here is the concept to suit. One says that a topological group $G$ has the *fixed point on compacta property* (*f.p.c.*), or else is *extremely amenable,* if $G$ has a fixed point in every compactum $X$ it acts upon: $g \cdot x^* = x^*$ for some $x^* \in X$ and all $g \in G$.

According to the Veech Theorem, such topological groups can never be locally compact (in particular, discrete). While it is not difficult to contruct even discrete semigroups with the fixed point on compacta property [35], it was at first unclear if topological groups with this property existed at all, as is documented by the relevant question asked in print by T. Mitchell in 1970 [71]. The examples of this kind were hard to come by, and to the best of our knowledge, it was first done by Herer and Christensen [43] (even though they appeared to be unaware of Mitchell's question).

Before discussing such examples, let us make a few fleeting remarks on amenable groups. A topological group $G$ is said to be *amenable* if it possesses an *invariant mean*, that is, a linear real-valued functional $\phi$ on the Banach space $C^b_r(G)$, which is positive (that is, $\phi(f) \geq 0$ whenever $f$ is a non-negative function), of norm one, and invariant under the left action of $G$, that is,

$$\phi(f) = \phi(g \cdot f) \text{ for all } g \in G \text{ and } f \in C^b_{r}(G).$$

The two most immediate classes of amenable groups are given by compact groups, where the invariant mean is just the Haar integral,

$$\phi(f) := \int_G f(x) d\mu(x),$$

and abelian topological groups, for which the proof is somewhat more involved. The problematics of amenability has grown out of the famous Banach–Tarski paradox (which essentially amounts to the non-amenability of the free groups on two generators). A good introduction to amenable groups is the book [125], while the book [37] is a classical reference, and the monograph [81] is the most modern and comprehensive source with greater emphasis put on links with modern analysis.

Notice that every topological group with the fixed point on compacta property is amenable. Indeed, every function $f \in C^b_{r}(G)$ extends to a unique continuous function $\bar{f}$ on the greatest ambit, and by evaluating it at a chosen fixed point $x^* \in \gamma_{r}(G)$ one obtains an invariant mean,

$$\phi(f) := \bar{f}(x^*),$$

which is even multiplicative:

$$\phi(fg) = \phi(f)\phi(g) \text{ for all } f, g.$$

This explains the origin of the name 'extremely amenable group.'

On the other hand, if $G$ is an amenable topological group admitting no nontrivial unitary representations, then $G$ has the fixed point on compacta property, that is, $G$ is extremely amenable. Indeed, suppose that $G$ does not have the f.p.c., that is, admits a nontrivial minimal action on a compact space $X$.

Fix a point $x_0 \in X$, and let $h\colon G \mapsto g \cdot x_0$ be the corresponding orbit map, $h\colon G \to X$. The space $C(X)$ can be made into a pre-Hilbert space through endowing it with the (positive semi-definite) inner product:

$$(f, g) := \phi((h \circ f)(\overline{h \circ g})),$$

where $\phi$ denotes an invariant mean for $G$. The group $G$ acts strongly continuously by isometries on the space $C(X)$, and this representation factors through to the associated Hilbert space $\mathcal{H}$, giving rise to a unitary representation. To prove that the obtained unitary representation is non-trivial, one uses the minimality of $X$. (It is easy to show the existence of a non-zero function $f$ on $X$ whose support is disjoint from the support of a suitable translation of $f$; then the $G$-orbit of the image of $f$ in $\mathcal{H}$ is necessarily non-trivial.) Thus we arrive at the historically first example of a topological group satisfying f.p.c. property.

**Example 3.2.1.** It follows that the above mentioned example 2.4.2 by Herer and Christensen [43] of an abelian topological group without nontrivial unitary representations has the f.p.c. property.

**Example 3.2.2.** The examples 2.4.3 by Banaszczyk also have the f.p.c. property, and thus the Veech Theorem cannot be extended from locally compact groups even to Banach–Lie groups.

In all the fairness, the above two examples look more like genuine, elaborately designed *counter*-examples, and even the name under which topological groups without unitary representations appear in the above quoted papers – *exotic* topological groups – bears a witness to that. However, more recent developments have revealed a highly surprising trend: among 'massive' groups, the fixed point on compacta property is rather common! Here are some further examples known to date.

**Example 3.2.3.** $\boxed{O(\mathcal{H})_s}$ Denote by $O(\mathcal{H})_s$ the group formed by all orthogonal operators on the infinite dimensional separable real Hilbert space $\mathcal{H} \cong l_2$, equipped with the strong operator topology. Gromov and Milman have shown in 1983 [40] that $O(\infty)$ has the fixed point on compacta property. Similarly, the full unitary group $U(\mathcal{H})_s$ of the complex Hilbert space with the strong operator topology has the fixed point on compacta property as well.

**Example 3.2.4.** $\boxed{L_1(X, \mathbb{T})}$ Let $(X, \mu)$ denote a non-atomic Lebesgue probability space. (It is known that every two such spaces are isomorphic, and a standard model of a non-atomic Lebesgue space is the closed unit interval $\mathbb{I}$ equipped with the usual Lebesgue measure; a measure-preserving isomorphism between subsets of full measure of $\mathbb{I}$ and $X$ is called a *parametrization* of $X$.) Denote by $L_1(X, \mathbb{T})$ the group formed by all measurable maps from $X$ to the circle rotation group $\mathbb{T} = U(1)$, where the group operations are defined pointwise. Make $L_1(X, \mathbb{T})$ into a topological group using the $L_1$-distance between measurable functions $f, g \colon X \to \mathbb{T}$:

$$d_1(f, g) := \int_X d(f(x), g(x)) d\mu(x). \tag{3.2}$$

Here $d$ is any metric on the circle group. Eli Glasner [33] (and, independently, Furstenberg and B. Weiss, unpublished) have proved that the topological group $L_1(X, \mathbb{T})$ has the fixed point on compacta property.

**Example 3.2.5.** $\boxed{\mathrm{Homeo}_+(\mathbb{I}) \text{ and } \mathrm{Homeo}_+(\mathbb{R})}$ The present author has proved [88] that the groups of orientation-preserving homeomorphisms of the closed interval, $\mathrm{Homeo}_+(\mathbb{I})$, and of the real line, $\mathrm{Homeo}_+(\mathbb{R})$, equipped with the compact-open topology, have the fixed point on compacta property.

**Example 3.2.6.** $\boxed{\mathrm{Aut}\,(X,\mu)}$ Again, let $X = (X,\mu)$ be a non-atomic Lebesgue measure space. An *automorphism* (or a *measure-preserving transformation*) of such a space is a measurable invertible map $f$ from $X$ to itself such that for every measurable $A \subseteq X$ one has $\mu(f^{-1}(A)) = \mu(A)$. (Then the inverse map $f^{-1}$ is automatically measurable as well.) The collection of all automorphisms of $X$ forms a group, $\mathrm{Aut}\,(X)$. The *strong* topology on the group $\mathrm{Aut}\,(X)_u$ is the Hausdorff group topology whose neighbourhood basis at the identity is formed by all sets of the form

$$N(Y,\epsilon) := \{h \in \mathrm{Aut}\,(X)\colon \mu(Y \cap h(Y)) < \epsilon\},$$

where $\epsilon > 0$ and $Y \subseteq X$ is measurable. The *uniform* topology on $\mathrm{Aut}\,(X)$ is a (finer) group topology generated by the bi-invariant metric

$$d_{unif}(g,h) := \mu\{x \in X\colon g(x) \neq h(x)\}. \tag{3.3}$$

(Cf. e.g. [41], pp. 65 and 69–74).

The recent joint result by Thierry Giordano and the present author [32] says that the group $\mathrm{Aut}\,(X)$ with the strong topology has the fixed point on compacta property. (This result admits further generalizations and ramifications, which will be hopefully explored in one or more papers by the same authors, currently in preparation.)

In our view, the entire trend is very significant, indicating that the properties of massive groups in some respect are completely opposite to those of locally compact groups. Moreover, the way the majority of the above results are being established provides an opportunity to link theory of topological groups with an important development in modern analysis and geometry — the *phenomenon of concentration of measure on high-dimensional structures.*

### 3.3. Concentration of Measure

Let us describe the basic idea of concentration phenomenon, which is meaningful in the presence of some sort of proximity between points (usually distance, or else uniformity) and 'size' of sets (measure). We will adopt the setting for analyzing concentration proposed by Gromov and Milman in 1983 [40], which is provided by a metric space $\Omega = (\Omega, \rho)$ equipped with a normalized ($\mu(\Omega) = 1$) positive Borel measure $\mu$. (It should be remarked that this setting is, in all the likelihood, not final, and at least two alternative frameworks for the concentration phenomenon have been proposed recently: an 'ergodic' setting by Gromov [39] and an 'affine' setting, cf. the preprint by Giannopoulos and Milman [31].)

Suppose $\Omega = (\Omega, \rho, \mu)$ is a metric space equipped with measure as above (mm-*space*). Let $A \subseteq \Omega$ be an arbitrary Borel subset containing at least half of all points, that is, $\mu(A) \geq \frac{1}{2}$. Denote by $\mathcal{O}_\epsilon(A)$ the open $\epsilon$-neighbourhood of $A$ in $\Omega$. How massive is $\mathcal{O}_\epsilon(A)$? Or, equivalently, how small — in the sense of measure — is the 'cap' $X \setminus \mathcal{O}_\epsilon(A)$? (Fig. 3.)

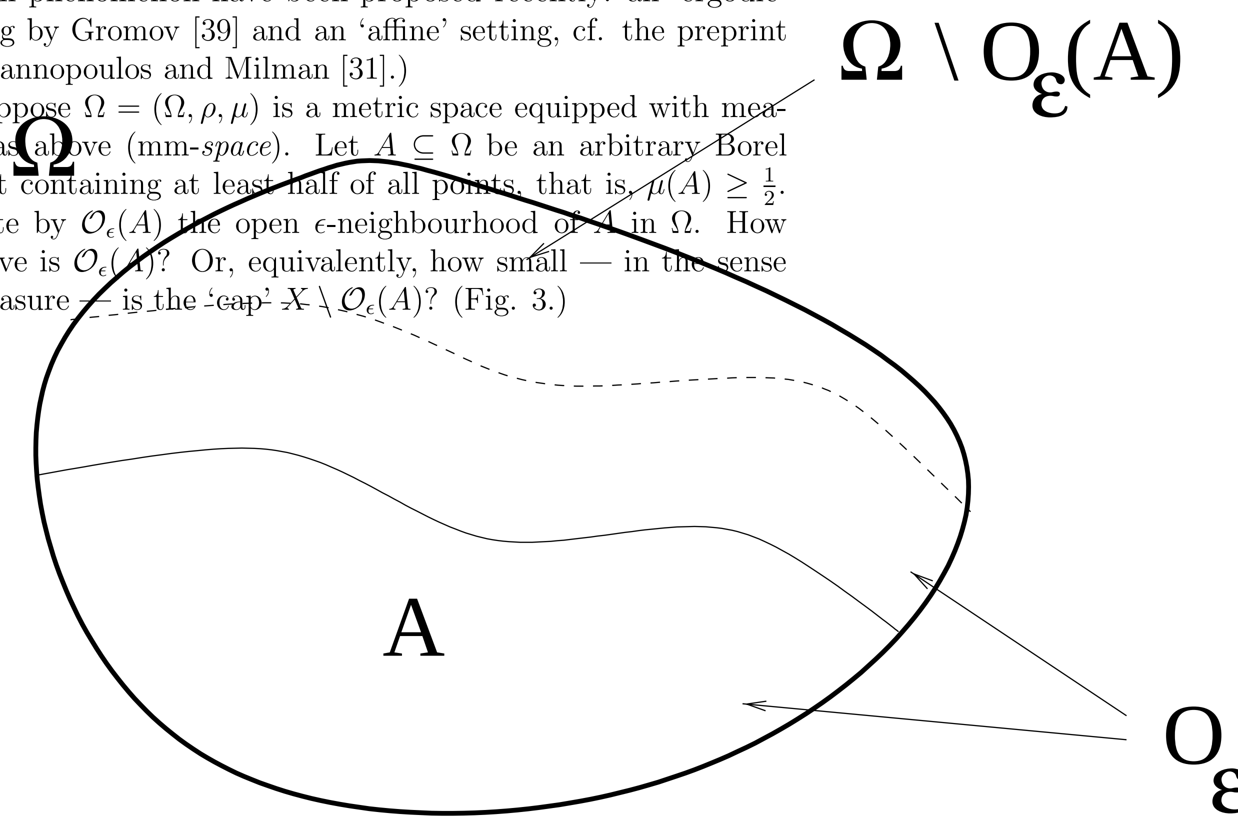


Fig. 3. An illustration to the concentration of measure.

If $\Omega = \mathbb{I}$ is the closed unit interval equipped with the usual distance and the Lebesgue measure, then, by letting $A = [0, \frac{1}{2}]$, one can see that the cap (which is, in this case, the interval $(\frac{1}{2}+\epsilon, 1])$ need not be really small in that it has measure $\frac{1}{2} - \epsilon$. Things do change however when we proceed to higher-dimensional objects. Here is a heuristic way to describe the phenomenon of concentration of measure on structures of high dimension:

*if $\Omega$ is 'high-dimensional' then, typically, the size of the 'cap' $\Omega \setminus \mathcal{O}_\epsilon(A)$ is extremely close to zero already for small values of $\epsilon > 0$.*

In other words, nearly all points of $\Omega$ are $\epsilon$-close to $A$ provided $\mu(A) \geq \frac{1}{2}$.

A convenient way to quantify the concentration phenomenon is to consider the *concentration function* of $\Omega$ which gives the least upper bound on the measures of all 'caps' as above:

$$\alpha_\Omega(\epsilon) = 1 - \inf\left\{\mu\left(\mathcal{O}_\epsilon(A)\right) : A \subseteq \Omega \text{ is Borel and } \mu(A) \geq \frac{1}{2}\right\}. \quad (3.4)$$

For the unit interval $\alpha(\epsilon) = \frac{1}{2} - \epsilon$, which is not very interesting. However, things do look different if we consider, for example, the $n$-spheres $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$, equipped with the geodesic distance and the (unique) normalized ($\mu_n(\mathbb{S}^n) = 1$) rotation-invariant measure, $\mu_n$ (which for $n = 1$ turns into the arc length, for $n = 2$ into the surface area, for $n = 3$ the volume element, and so forth.) The maximal size of the 'cap' is achieved for $A = \mathbb{S}^n_-$, the hemisphere. (This is one of the equivalent forms of the so-called *isoperimetric inequality*.) Now pretty straightforward calculations at the level of a good first-year calculus student enable one to compute the concentration functions $\alpha_{\mathbb{S}^n}$ of the $n$-spheres. Here are their graphs in dimensions $n = 3, 10, 100, 500$. (Fig. 4.)

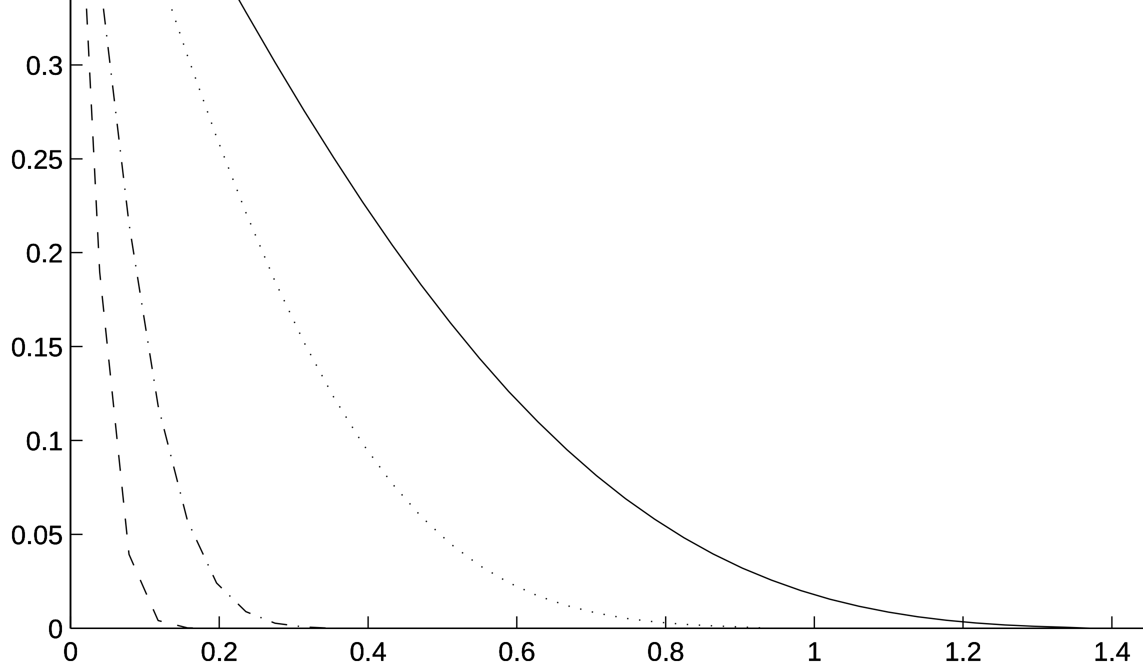


Fig. 4. Concentration functions of $n$-spheres, $n = 3, 10, 100, 500$.

The following result is simple but important. In particular, it explains the origin of the terminology ('concentration of measure.')

**Theorem 3.3.1.** *Let $\Omega = (\Omega, \rho, \mu)$ be a metric space equipped with a normalized Borel measure, and let $f\colon \Omega \to \mathbb{R}$ be a uniformly continuous function. Denote by $\delta = \delta(\epsilon)$ the modulus of uniform continuity of $f$, that is, for every $x, y \in \Omega$ and $\epsilon > 0$ one has $|f(x) - f(y)| < \epsilon$ whenever $\rho(x, y) < \delta(\epsilon)$. Denote by $M$ the median value of $f$ on $\Omega$, that is, a real number with*

$$\mu\{x \in \Omega : f(x) \leq M\} \geq \frac{1}{2}, \ \mu\{x \in \Omega : f(x) \geq M\} \geq \frac{1}{2}.$$

*Then the set of all $x \in \Omega$ such that $|f(x) - M| < \epsilon$ has measure at least*

$$1 - 2\alpha_\Omega(\delta(\epsilon)).$$

If the concentration function $\alpha$ drops off sharply for small values of the argument, then most of the points of the domain $\Omega$ 'concentrate' near one value of $f$. In other words, the function $f$ is, from the probabilistic viewpoint, almost constant.

The asymptotic behaviour of families of spaces with metric and measure, namely the tendency of concentration functions to fall off sharply near zero as the dimension grows (evident in Fig. 4), can be formalised as follows. One says that an infinite family $(\Omega_n, \rho_n, \mu_n)$ of metric spaces equipped with measure is a *normal Lévy family* if

$$\alpha_{\Omega_n}(\epsilon) \leq C_1 e^{-C_2 \epsilon^2 n}.$$

In simpler words, it means that the measures of 'caps' go to zero exponentially fast in dimension for a fixed value of $\epsilon > 0$. For example, $n$-spheres form a normal Lévy family. What is more, and this is very important, 'naturally occurring' infinite families of probabilistic metric spaces are, typically, normal Lévy. Here are just three examples important for what follows.

**Example 3.3.2.** (Glasner [33], and, independently, Furstenberg and B. Weiss.) The family of tori $\mathbb{T}^n$, equipped with the normalized Haar measure and the metric

$$d(x, y) := \frac{1}{n} \sum_{i=1}^{n} |x - y|,$$

where the absolute value is induced through the standard embedding $\mathbb{T} \equiv U(1) \subset \mathbb{C}$, form a normal Lévy family. (This follows from more general results of Talagrand [104].)

**Example 3.3.3.** (Maurey, [61].) The groups of permutations $S_n$ of rank $n$, equipped with the normalized Hamming distance

$$d(\sigma, \tau) := \frac{1}{n} |\{i \colon \sigma(i) \neq \tau(i)|$$

and the normalized counting measure

$$\mu(A) := \frac{|A|}{n!}$$

forms a normal Lévy family, with the concentration functions satisfying the estimate

$$\alpha_{S_n}(\epsilon) \leq \exp(-\epsilon^2 n/64).$$

**Example 3.3.4.** (Gromov and Milman, [40].) The special orthogonal groups $SO(n)$ of rank $n$ consist of all orthogonal $n \times n$ matrices with real entries having determinant $+1$. The family of these groups, equipped with the normalized Haar measure and the uniform metric (that is, the metric induced by the operator norm under the standard embedding $SO(n) \subseteq \mathcal{L}(\mathbb{R}^n)$), form a normal Lévy family.

Listed below are just a few common manifestations of the phenomenon of concentration of measure in mathematical sciences.

• The Law of Large Numbers: the average value of a long sequence of 0s and 1s obtained by tossing a fair coin is typically $\approx \frac{1}{2}$.

• Most of the volume of a high-dimensional Euclidean ball is concentrated near the surface.

• Most of the volume of a high-dimensional unit cube is concentrated near the corners.

• Blowing-Up Lemma in coding theory: if the Hamming cube $\{0,1\}^n$ is partitioned into two subsets of equal size, then almost all binary $n$-strings are close to both subsets.

• Dvoretzky Theorem: If a convex body in a high-dimensional space is cut by a random plane, the section typically looks almost like a circle.

• Two random vectors in a high-dimensional Euclidean space selected independently of each other are typically nearly orthogonal.

Various aspects of the phenomenon of concentration of measure on high-dimensional structures, including all the above examples, are discussed in [40], [67], [68], [69], [104], [31], [39], and [38], Ch. $3\frac{1}{2}_+$.

### 3.4. Lévy Groups

In topological algebra the concentration phenomenon is captured by the concept of a Lévy group. The definition below slightly extends the original one [40] (cf. [33] and [88]) in that the metric is replaced with uniformity.

**Definition 3.4.1.** *We say that a topological group $G$ is a* Lévy group *if there is a family $\mathcal{K}$ of compact subgroups of $G$ with the following properties.*

1. *The family $\mathcal{K}$ is directed by inclusion, that is, for any $F$, $H \in \mathcal{K}$ there is a $K \in \mathcal{K}$ with $F \cup H \subseteq K$.*
2. *The union $\cup\mathcal{K}$ is everywhere dense in $G$.*
3. *Let a family of Borel subsets $A_K \subseteq K$, $K \in \mathcal{K}$ have the property that*

$$\liminf_{K\in\mathcal{K}} \mu_K(A_K) > 0, \tag{3.5}$$

*where $\mu_K$ denotes the normalized Haar measure on $K$. Then for every neighbourhood of zero, $V$, in $G$,*

$$\lim_{K\in\mathcal{K}} \mu_K(K \cap (VA_K)) = 1. \tag{3.6}$$

**Remark 3.4.2.** Zero on the r.h.s. of (3.5) can be replaced, without any loss in generality, by any positive constant $< 1$, for example $\frac{1}{2}$.

**Examples 3.4.3.** 1. The group $L(X, \mathbb{T})$ (Ex. 3.2.4) forms a Lévy group. Simple functions, constant on elements of a sequence of refining partitions of the Lebesgue space $X$, form an increasing sequence of tori having everywhere dense union in the group. Now one applies the observation from Ex. 3.3.2.

2. The group $O(\mathcal{H})_s$ with the strong operator topology (Ex. 3.2.3) is a Lévy group. It follows from Ex. 3.3.4 and the following observation. Let us identify elements of $SO(n)$ with those

orthogonal operators in $\mathcal{H}$ which are represented, with respect to a chosen orthonormal basis in $\mathcal{H}$, by matrices with only finitely many non-zero entries. Then the union of the increasing sequence of the special orthogonal groups of growing finite rank embedded into each other via

$$SO(n) \ni A \mapsto \begin{pmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & A \end{pmatrix} \in SO(n+1) \tag{3.7}$$

is everywhere dense in $O(\mathcal{H})_s$.

A similar argument applies in the case of infinite unitary groups.

3. The group Aut $(X)$ of measure-preserving transformations of a Lebesgue space equipped with the strong topology (Ex. 3.2.6) is Lévy. Indeed, consider a parametrization of $X$ by the closed unit interval with the Lebesgue measure. Call a transformation of the interval $\mathbb{I}$ a *permutation of rank* $n$ if it maps each binary interval of rank $n$ to such an interval by a translation. Then it is well-known in ergodic theory (a corollary of Rokhlin's Lemma) that the collection of all such permutations is everywhere dense in Aut $(X)$ (the so-called Weak Approximation Theorem, cf. e.g. [41], pp. 65–68). The uniform metric on Aut $(X)$ induces the normalised Hamming distance on each group of permutations $S_n$, and by using Ex. 3.3.3, one concludes that Aut $(X)$ is a Lévy group.

In order to establish the fixed point on compacta property for the groups from Examples 3.2.3, 3.2.4, 3.2.6, it is now sufficient to establish the following.

**Theorem 3.4.4.** *Every Lévy group has the fixed point on compacta property.*

This result belongs to Gromov and Milman ([40], Th. 5.3), who stated it in a somewhat more restricted form, later perfected by Glasner [33], Th. 1.2. See also [88], Th. 9.1.

Let us prove Theorem 3.4.4. It is clearly sufficient to establish the existence of a fixed point for the canonical action of $G$ on the greatest ambit $\gamma_{r}(G)$. Furthermore, the compactness considerations easily imply that it is enough to find the common fixed point for an arbitrary finite subset of elements $g_1, g_2, \ldots, g_n$ of $G$. Such a fixed point will certainly exist if the following property is satisfied: for every element $V$ of the unique uniform structure on $\gamma_{r}(G)$, there is a point $x \in \gamma_{r}(G)$ such that all he elements $g_1, g_2, \ldots, g_n$ fail to move $x$ beyond the neighbourhood $V[x]$. One does not lose in generality by assuming that $x \in G$, and instead of the entourage $V$ one can consider, using a common trick in uniform topology, an arbitrary bounded $\mathcal{U}_{r}(G)$-uniformly continuous function $f$ from $G$ to a finite-dimensional Euclidean space. The property we want to establish becomes this: for every such $f$ as above and every $\epsilon > 0$ there is an $x \in G$ such that for all $i$,

$$|f(g_i x) - f(x)| < \epsilon.$$

Assume for the reasons of mere technical simplicity that the Lévy group $G = \cup_{i=1}^{\infty} G_i$ under consideration is separable, so that a net of approximating subgroups can be replaced with an increasing chain, whose union in addition coincides with $G$, and also that $G$ is metrizable, and fix a right-invariant metric $\rho$ generating the topology of $G$.

We denote by $\mu_i$ the normalised Haar measure on the compact group $G_i$. The elements $g_1, g_2, \ldots, g_n$ from the given finite collection are contained in a $G_N$ for $N$ sufficiently large, and thus we can assume by removing the first $N-1$ groups in the sequence that $g_1, g_2, \ldots, g_n \in G_1$. Let $f$ be a $\mathcal{U}_{r}(G)$-uniformly continuous bounded function on $G$ taking values in a finite-dimensional Euclidean space $\mathbb{R}^k$. We will consider the $l_\infty$-norm on the latter space, just to choose any. Denote by $\delta = \delta(\epsilon)$ the modulus of continuity of $f$. Let $f_j$, $j = 1, \ldots, k$ be the components of $f$, and denote for each $i \in \mathbb{N}$ by $M_{i,k}$ the median value of the restriction $f_k|_{G_i}$. Let $\epsilon > 0$ be any. According to Theorem 3.3.1, for all elements $g \in G_i$ from a set of measure at least $1 - 2\alpha_{G_i}(\delta(\epsilon))$ one has

$$|f_{i,k}(g) - M_{i,k}| < \epsilon.$$

Replacing $\epsilon$ with $2\epsilon$, using the compactness of the closed interval, and proceeding to a subsequence if necessary, one can assume that the numbers $M_{i,k} = M_k$ are independent of $i$. Let $M = (M_1, M_2, \ldots, M_k) \in \mathbb{R}^k$. It follows that for each $i \in \mathbb{N}$, for all $g \in G_i$ from a set $S_i$ of measure at least $1 - 2\alpha_{G_i}(\delta(\epsilon))$ one has

$$\|f(g) - M\|_\infty < \epsilon.$$

The translates of the set $S_i$ by elements $g_1^{-1}, \ldots, g_n^{-1}$ have the same measure as $S_i$ and the intersection of all such translates, which we will denote $X_i$, is of measure (computed in the group $G_i$) at least $1 - 2n\alpha_{G_i}(\delta(\epsilon))$. The definition of a Lévy group means that the concentration functions $\alpha_{G_i}$ converge pointwise to zero as $i \to \infty$. In particular, for $i$ sufficiently large, the sets $X_i$ are of positive measure. Any point $x \in X_i$ will then have the desired property.

**Remark 3.4.5.** The proof of the f.p.c. property of the groups from example 3.2.5 (that is, the homeomorphism groups of the closed and the open unit intervals) is established in [89] in a somewhat different manner, using the infinite Ramsey theory, and we do not reproduce it here, cf. e.g. our short survey [88]. Interestingly, it has been repeatedly noted (cf. e.g. [68]) that the Ramsey theorems in combinatorics are very close in spirit to the phenomenon of concentration of measure.

The size of the class of extremely amenable topological groups is immense. The following result (conjectured independently by Gromov and Uspenskij in private discussions with the author) was recently established by the present author.

**Theorem 3.4.6** (Pestov [90])**.** *Let $U$ be an $\omega$-homogeneous generalized Urysohn metric space. Then the group* $\mathrm{Iso}\,(U)$ *has the fixed point on compacta property.*

The proof explores both the phenomenon of concentration of measure and a close link between the extreme amenability of the groups of isometries Iso $(X)$ of sufficiently homogeneous metric spaces $X$ and a Ramsey-type property of $X$.

Theorem 3.4.6 and Uspenskij's Theorem 2.3.2 together imply:

**Corollary 3.4.7.** *Every topological group embeds, as a topological subgroup, into an extremely amenable topological group, that is, a topological group with the fixed point on compacta property.*

Even if one replaces 'extremely amenable' with 'amenable,' the result remains new. Notice that amenability is inherited by topological subgroups of *locally compact* amenable groups; for non-locally compact groups this is no longer true [42], and Corollary 3.4.7 takes this observation to its extreme.

Finally, here is another corollary of Theorem 3.4.6, answering a question from [112].

**Corollary 3.4.8.** *The topological groups* Iso $(\mathbb{U})$ *and* Homeo $(\mathbb{I}^{\omega})$ *are not isomorphic.*

*Proof.* Indeed, the latter group admits a continuous action without fixed points on the compact space $\mathbb{I}^{\omega}$. □

Thus, the two examples of universal Polish groups we are aware of are different indeed.

*Some further questions on concentration in topological groups*
The examples of Lévy groups belonging to Gromov and Milman ($U(\mathcal{H})_s$, Ex. 3.2.3) and to Glasner and Furstenberg–B. Weiss ($L(X, U(1))$, Ex. 3.2.4) share in fact a profound similarity in that both of them are unitary groups of suitable von Neumann algebras equipped with the ultraweak topology [98]: $L_\infty(X)$ in the first case, $L(\mathcal{H})$ in the second.

Moreover, if a von Neumann algebra $W$ is such that the unitary group with the ultraweak topology is Lévy (and therefore, in particular, amenable), then $W$ is hyperfinite (de la Harpe [42] and Paterson [81]) and therefore injective. The following question is natural.

**Problem 3.4.9.** *What are those von Neumann algebras whose unitary groups with the ultraweak topology are Lévy?*

**Problem 3.4.10.** The author understands that (a version of) the following problem was put forward by Furstenberg at least 17 years ago: *Let $G$ be a topological group that is the union of a directed family of compact subgroups. Is it possible to express the property of $G$ being Lévy through the existence of fixed points in some compacta that $G$ acts upon?*

If one interprets the problem as whether or not the Lévy property for topological groups of the above type is equivalent to extreme amenability, then we strongly suspect that the answer is no, though at the moment we do not have any concrete counterexample. However, it is conceivable that the problem admits a wider interpretation, leading to a positive answer.

### 3.5. Extreme Amenability and Left Syndetic Sets

It is worth stressing that though the fixed point on compacta property is formulated in exterior terms (actions on compact spaces), it is an intrinsic property of a topological group $G$ itself. This becomes evident if one looks at the following alternative criterion, which can also be used to establish the fixed point theorems. Recall that a subset $S$ of a group $G$ is *left syndetic,* or (*left*) *relatively dense,* if $FS = G$ for some finite subset $F \subseteq G$.

**Theorem 3.5.1** (Pestov, [88])**.** *A topological group $G$ has the f.p.c. property if and only if for each left syndetic subset $S \subset G$, the set $SS^{-1}$ is everywhere dense in $G$.*

Theorem 3.5.1 was inspired by, and is to be compared with, the following classical result.

**Theorem 3.5.2.** (Følner [28]; Cotlar—Ricabarra [20]; Ellis—Keynes [25]) *An abelian topological group is minimally almost periodic if and only if for each big $S \subseteq G$, the set $S - S + S$ is everywhere dense in $G$.*

To better appreciate the similarity between 3.5.1 and 3.5.2, notice that every abelian topological group with the fixed point on compacta property is minimally almost periodic.

It is in fact unknown if the converse is true! Since the f.p.c. property intuitively feels so much stronger a restriction than minimal almost periodicity, our inability to distinguish between the two properties comes as a surprise.

Both results can be turned the other way round. In particular, the following mirror image of Theorem 3.5.2 yields a criterion for the existence of sufficiently many characters.

**Corollary 3.5.3.** *An abelian topological group $G$ is maximally almost periodic if and only if for every $g \in G$, $g \neq 0$, there exists a big set $S \subseteq G$ such that the closure of $S - S + S$ does not contain $g$.*

In fact, this is a sort of result that gives a fair idea of what would be an acceptable answer to Shtern's question 2.5.14.

It is natural to ask for a non-abelian version of the above, with $S - S + S$ being replaced by $S^{-1}S^2S^{-1}$. While the answer seems to be unknown in the full generality, an important advance is due to Landstadt [55] who established the result for amenable topological groups.

The following particular case of Theorem 3.5.2 is of a special interest in combinatorial number theory.

**Corollary 3.5.4.** *If $S$ is a relatively dense subset of the integers, then $S - S + S$ is a neighbourhood of zero in the Bohr topology on the group $\mathbb{Z}$.*

It remains unknown for a long time [121] if one can replace in the above result $S - S + S$ with $S - S$.

Glasner [33] has observed that a negative answer would follow if one constructs an example of a minimally almost periodic, monothetic topological group without the f.p.c. property. For a simpler explanation of why, see also [88]. No such example is presently known. To construct it, one apparently needs to maintain a very fine balance between minimal almost periodicity and a property that goes in exactly the opposite direction to measure concentration: it is some form of *measure dissipation,* cf. [38].

## 4. Parallels Between Topological and Discrete Groups

### 4.1. Subgroups of Finitely Generated Groups

#### 4.1.1. Higman–Neumann–Neumann Theorem

Here is an example of how actions can be used as a tool in theory of topological groups. Consider the following classical result in group theory (no topology present!).

**Theorem 4.1.1** (Higman-Neumann-Neumann,[45, 79])**.** *Every countable group is isomorphic with a subgroup of a 2-generated group.*

Somewhat unexpectedly, the above result has a direct counterpart for topological groups.

**Theorem 4.1.2** (Morris and Pestov, [74])**.** *Every countable topological group is isomorphic with a subgroup of a group algebraically generated by two elements.*

The following corollary is more or less straightforward and puts the result in a natural topological group wrapping.

**Corollary 4.1.3** ([74])**.** *Every separable topological group is isomorphic with a topological subgroup of a group with two topological generators.*

From here one can deduce without much effort a description of topological subgroups of topologically finitely generated groups. A topological group is called *$\omega$-bounded* if it is covered with countably many translations of every non-empty open subset. This concept is modelled on that of a totally bounded group. A group is $\omega$-bounded if and only if it embeds into the direct product of a family of separable metrizable groups (with the usual product topology).

**Corollary 4.1.4** ([74]). *A topological group embeds into a topologically finitely generated topological group if and only if it is $\omega$-bounded and has weight at most continuum.*

The proof of Theorem 4.1.2 consists of nothing more than injecting a bit of topological dynamics into a proof of the Higman–Neumann–Neumann Theorem 4.1.1 due to Galvin [30]. Here is its outline. Enumerate the group in question with odd positive integers, $G = \{g_1, g_3, \dots, g_{2k+1}, \dots\}$. Let $X$ be a set whose group of permutations contains $G$: $G \hookrightarrow \mathrm{Aut}\,(X)$. Form a new set

$$\tilde{X} = \mathbb{Z} \times \mathbb{Z} \times X \cong \oplus_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}\{(m,n)\} \times X\}.$$

Now define permutations $a$ and $b$ of $\tilde{X}$ by letting (1) $a\cdot(m,n,x) = (m+1,n,x)$, (2) $b\cdot(0,n,x) = (0,n+1,x)$, (3) $b\cdot(m,n,x) = (m,n,xg_m)$ if $m$ is odd, $m > 0$, and $n \geq 0$, and finally (4) making $b$ leave $(m,n,x)$ fixed otherwise.

Embed $G$ into the group of permutations of $\tilde{X}$ by letting it act on $\{0,0\}\times X$ in a way identical to its action on $X$, and on the rest of $\tilde{X}$ in a trivial way (every point is fixed). Straightforward computations show that $G$ is contained in the group $\mathrm{gp}(a,b)$ generated by permutations $a$ and $b$. The Higman–Neumann–Neumann Theorem 4.1.1 is thus proved.

To obtain from the above a proof of Theorem 4.1.2, it suffices to put on $X$ a compact topology so as to make $G$ into a topological subgroup of $\mathrm{Homeo}\,(X)$ (Teleman's theorem!), and to topologize $\tilde{X}$ as the disjoint sum of compacta. Both $a$ and $b$ are now homeomorphisms, and the compact-open topology makes $\mathrm{Homeo}\,\tilde{X}$ into a topological group obviously containing $G$ as a topological subgroup. Q.E.D.

### 4.1.2. Subgroups of Monothetic Groups

What about 1-generated or, as they are most commonly called, *monothetic* topological groups? They are certainly abelian, and so are all their subgroups. Nevertheless, this turns out to be the only additional restriction one has to impose on all potential subgroups of such groups.

**Theorem 4.1.5** (Morris and Pestov, [75]). *Every separable abelian topological group is isomorphic with a topological subgroup of a group with one topological generator.*

One can abelianize Corollary 4.1.4 as well.

**Corollary 4.1.6** ([75]). *A topological group embeds into a monothetic topological group if and only if it is abelian, $\omega$-bounded and has weight at most continuum.*

What is manifest, is that the proof cannot be aped after a discrete case, simply because Theorem 4.1.5 clearly has no discrete counterpart! Instead, the proof is based on an entirely different technique, giving us an opportunity to introduce into consideration free (abelian) topological groups.

### 4.1.3. Free (Abelian) Topological Groups and Free Locally Convex Spaces

Let $X$ be a completely regular $T_1$ topological space. A topological group $F(X)$ is called the *free topological group* on $X$ if it contains a topological copy of $X$ as a distinguished topological subspace in such a way that the following diagram can be made commutative for every continuous mapping $f$ from $X$ to an arbitrary topological group $G$ by means of a unique continuous homomorphism $\bar{f}$:

$$\begin{array}{ccc} X & \hookrightarrow & F(X) \\ & \forall f \searrow & \downarrow \exists \bar{f} \\ & & G \end{array}$$

In a completely similar way, one defines the free *abelian* topological group, $A(X)$, and the *free locally convex space*, $L(X)$. (In the latter case, the morphisms are continuous linear operators between locally convex spaces.)

One of the most immediate observations concerning all three types of objects is their algebraic freedom: $F(X)$ and $A(X)$ are algebraically the free and the free abelian groups on $X$ correspondingly, while $L(X)$ is a vector space spanned by $X$ as an algebraic (Hamel) basis.

The topology on both $A(X)$ and $L(X)$ can be easily described using the following construction going back to Graev [34] and Arens–Eells [4], see also [93, 26, 27]. For a pseudometric $\rho$ on the set $X^\dagger = X \cup \{0\}$ denote by $\bar{\rho}$ the maximal translation invariant pseudometric on $A(X)$ with the property that $\bar{\rho}|X^\dagger = \rho$. The existence of such a pseudometric is obvious, and moreover its value can be computed rather explicitely, or at least in combinatorial terms of manageable complexity. Now it is a matter of an easy exercise, to show that the collection of all pseudometrics of the form $\bar{\rho}$ (called *Graev,* or *maximal, pseudometrics*) determines the topology of the free abelian topological group $A(X)$ as $\rho$ runs through the collection of all compatible pseudometrics on $X \cup \{0\}$. In a similar vein, let $p_\rho$ denote the maximal seminorm on $L(X)$ such that $p_\rho(x - y) = \rho(x, y)$, $x, y \in X^\dagger$. Again, the existence of the seminorm $p_\rho$ is rather straightforward, and the seminorms of this form determine the topology of $L(X)$ if one lets $\rho$ run over all compatible pseudometrics on $X \cup \{0\}$. (Warning: in both cases, it is not enough for $\rho$ to go through *some* collection of generating pseudometrics for the topology of $X$!)

**Remark 4.1.7.** For the (non-abelian) free topological group $F(X)$, no simple description of topology similar to the above is known. The explicit constructions of generating pseudometrics, such as those proposed first by Tkachenko [108] and then others, are pretty hard to work with. However, a similar description is known for the so-called *free SIN group,* $F_{\mathrm{SIN}}(X)$ (also known as the *free balanced group.*) The definition of course mimicks that of the free topological group, where it is assumed that all topological groups under consideration are SIN (= have the left and the right uniform structures coincide). The group $F_{\mathrm{SIN}}(X)$ is also algebraically free over $X$, and its topology is described by the family of all *Graev pseudometrics* on the group $F(X)$. A Graev pseudometric in this case is defined as the maximal bi-invariant pseudometric $\bar{\rho}$ whose restriction to $X^{\dagger} := X \cup \{e\}$ coincides with the given pseudometric $\rho$. Only in exceptional cases do the topologies of $F(X)$ and of $F_{\mathrm{SIN}}(X)$ coincide.

There is no single comprehensive reference to the up-to-date theory of free topological groups. Some pointers can be found in [5, 7, 19, 83].

Listed below (in no particular order) are results about free (abelian) topological groups that the present author likes most. (He hopes to be forgiven for including among them $1\frac{1}{3}$ results of his own.)

- The semi-classical work by Markov [58] and Graev [34].
- Arhangel'skiĭ's results on zero-dimensionality of free topological groups [6].
- Tkachenko's result on the Souslin property of free topological groups on compacta [106].
- Results on completeness and subgroups of free topological groups by Uspenskij [113] and Sipacheva [102, 103].
- Results by Galindo and Hernández on the reflexivity of free abelian topological groups [29].

- Three of the results reproduced below, namely the Tkachenko–Uspenskij Theorem 4.1.8 and Theorems 4.2.1 and 4.2.3.

The following result establishes a remarkable connection between two of the objects so far introduced. It was proved by successive efforts of Tkachenko [107], who announced the result but supplied it with a flawed proof, and Uspenskij [113], who found both the flaw and a correct proof some years later.

**Theorem 4.1.8** (Tkachenko–Uspenskij, 1983/90)**.** *For every pseudometric $\rho$ on the set $X^\dagger = X \cup \{0\}$,*

$$p_\rho | A(X) = \bar{\rho}.$$

*As a corollary, for every topological space $X$ the free abelian topological group $A(X)$ canonically embeds into the free locally convex space $L(X)$ as a closed topological subgroup.*

**Remark 4.1.9.** What is the noncommutative analogue of Tkachenko–Uspenskij Theorem? Or, using the fashionable buzzword, how to *quantize* the above result?

Firstly, we suggest that such a 'quantization' must have to do with the free SIN group $F_{\mathrm{SIN}}(X)$ rather than with the free topological group $F(X)$ (which is really too complicated an object).

Also, notice that the Tkachenko–Uspenskij Theorem is equivalent to the following statement: continuous homomorphisms from the free abelian topological group $A(X)$ to the additive groups of Banach spaces determine the topology of $A(X)$. It is not difficult to verify that the additive topological group of every Banach space embeds, as a topological subgroup, into the unitary group of an abelian $C^*$-algebra, equipped with the induced norm topology. A quantization of the above statement will amount to allowing for all, and not just abelian, $C^*$-algebras.

Finally notice that the unitary group of every $C^*$-algebra embeds, as a topological subgroup, into the unitary group of a Hilbert space equipped with the uniform operator topology. Hence the resulting conjecture.

**Conjecture 4.1.10.** ('Non-commutative Tkachenko–Uspenskij Conjecture') *Continuous homomorphisms from the free balanced topological group $F_{\mathrm{SIN}}(X)$ to the unitary groups $U(\mathcal{H})_u$ of Hilbert spaces with the uniform operator topology determine the topology of $F_{\mathrm{SIN}}(X)$ for every (Tychonoff) topological space $X$.*

If the reader is unconvinced that the above is the 'right' non-commutative version of Theorem 4.1.8, notice that the Tkachenko-Uspenskij theorem can be recast, modulo duality theory for locally convex spaces, in the following equivalent form. Let $I$ denote a fixed convex closed simply connected neighbourhood of zero in the circle rotation group $\mathbb{T} = U(1)$. Call a subset $A \subseteq G$ of an abelian topological group $G$ a *polar set* if for some family $\mathcal{X}$ of continuous characters of $G$ one has

$$A = \bigcap_{\chi \in \mathcal{X}} \chi^{-1}(I).$$

Then Theorem 4.1.8 is equivalent to the statement that the free abelian topological group $A(X)$ admits a neighbourhood base consisting of polar sets. This was proved in [86].

While proceeding from the abelian to non-abelian case, it is natural to replace characters with finite-dimensional unitary representations. Call a subset $A$ of a topological group $G$ a *polar set* if for some family $\Pi = \cup_{n \in \mathbb{N}} \Pi_n$ of continuous finite-dimensional unitary representations of $G$ there are simply connected, convex (in the Riemannian sense) closed neighbourhoods of the identity $I_\pi$ in the groups $U(n)$, where $\pi \colon G \to U(n)$, such that

$$A = \bigcap_{n \in \mathbb{N}} \bigcap_{\pi \in \Pi_n} \pi^{-1}(I_\pi).$$

It is an easy exercise to check that every polar set is invariant. Using the results on the so-called residual finite-dimensionality of free $C^*$-algebras on metric spaces [85], one can prove that the non-commutative Tkachenko–Uspenskij conjecture is equivalent to the existence of a neighbourhood base in the topological group $F_{SIN}(X)$ consisting of polar sets in the above sense. In such a form, the relationship between Conjecture 4.1.10 and Theorem 4.1.8 becomes obvious.

**4.1.4.**

Now we are fully armed to accomplish the proof of Theorem 4.1.5. For simplicity, we will only outline it in the case where the abelian topological group $G$ in question is not only countable, but metrizable as well; the non-metrizable case only requires some extra technical ingenuity but nothing really deep. Denote by $\rho$ a translation-invariant metric generating the topology of $G$. Since $G$ is clearly a topological factor-group of the free group $A(G)$ equipped with the Graev metric $\bar{\rho}$, it is enough to prove the theorem for the metric group $(A(G), \bar{\rho})$ and then divide the monothetic group by the kernel of the factor-homomorphism $A(G) \to G$: indeed, monotheticity is preserved by proceeding to the images under continuous homomorphisms with dense image. According to Tkachenko–Uspenskij Theorem, it suffices to prove the statement for the separable normed space $(L(G), p_\rho)$, which contains $(A(G), \bar{\rho})$ as a topological subgroup, or, equivalently, for the Banach space completion of $(L(G), p_\rho)$, which we will denote for simplicity by $E$. Enumerate by integers a countable everywhere dense subset $\{x_n : n \in \mathbb{N}_+\}$ in $E$. Denote by $H$ the Banach space direct sum of $E$ with the separable Hilbert space $l_2(\mathbb{N}_+ \times \mathbb{N}_+)$, and let $D$ be the subgroup of $H$ generated by all elements of the form

$$(mx_n, e_{m,n}),\ \ m, n \in \mathbb{N}_+.$$

(Here $e_{m,n}$ denote the standard basic vectors in the Hilbert space $l_2(\mathbb{N}_+ \times \mathbb{N}_+)$.) Then one can verify that the Banach space $E$ is isomorphic to a topological subgroup of the topological factor-group $H/D$, and the latter group is clearly generated by the union of all its subgroups isomorphic to the circle rotation group $\mathbb{T}$ (images of all one-dimensional linear spaces passing through elements of $D$!) and therefore $H/D$ is monothetic by the force of the following result which we call Rolewicz Lemma. (Cf. [96], in which note Rolewicz has actually proven the result below — by an accurate, recursive application of the Kronecker Lemma — even if he never stated the result in full generality and instead established it just for a concrete example of a topological group he was constructing.)

**Theorem 4.1.11** (Rolewicz Lemma)**.** *A complete metric abelian topological group $G$ topologically generated by the union of countably infinitely many subgroups topologically isomorphic to the circle group $\mathbb{T} \cong U(1)$ is monothetic (that is, has one topological generator).*

### 4.2. Free Groups: Discrete *vs* Topological

In some respects, the known parallels between discrete and topological groups go surprisingly far, especially for free groups. Recall that two free bases in a free group always have the same cardinality (called the *rank* of the free group). In other words, if $X$ and $Y$ are two sets such that the free groups $F(X)$ and $F(Y)$ are isomorphic, then $|X| = |Y|$. The same holds true for free abelian groups. The following result can be seen as a perfect topological counterpart, where the cardinality of a set is replaced with the dimension of a topological space.

**Theorem 4.2.1** (Pestov, 1982, [82])**.** *Let $X$ and $Y$ be two Tychonoff topological spaces with the property that the free topological groups $F(X)$ and $F(Y)$ are isomorphic.* (*Or: the free abelian topological groups, $A(X)$ and $A(Y)$, are isomorphic.*) *Then $X$ and $Y$ have the same Lebesgue covering dimension:* $\dim X = \dim Y$.

Parallels in this direction go further. It is known that the free group $F_\infty$ on countably infinitely many generators embeds, as a subgroup, into the free group $F_2$ on two generators. For topological groups, the following can be seen as a sensible approximation to the same phenomenon.

**Theorem 4.2.2** (Katz, Morris, Nickolas [50])**.** *If $X$ is a countable CW-complex of finite dimension, then the free topological group $F(X)$ is isomorphic with a topological subgroup of $F(\mathbb{I})$.*

What is the situation in the abelian case? The 'discrete' suggestion is that the rank of a subgroup of a free abelian group $A(X)$ cannot exceed the rank of $A(X)$. For a while it remained unknown whether the free abelian topological group on $A(\mathbb{I}^2)$ embeds, as a topological subgroup, into $A(\mathbb{I})$. The answer turned out to be rather unexpected and requiring much subtler and advanced tools to obtain than for example the noncommutative theorem 4.2.2. The following result demonstrates that the analogy with the discrete case is, after all, not comprehensive.

**Theorem 4.2.3** (Leiderman–Morris–Pestov, [56])**.** *If $X$ is a finite-dimensional metric compactum, then $A(X) \hookrightarrow A(\mathbb{I})$ as a topological subgroup.*

The proof is based on the following deep result, which in its time had answered Hilbert's Problem 13.

**Theorem 4.2.4.** (Kolmogorov Superposition Theorem, [54, 80]). *Every finite-dimensional metric compactum $X$ possesses a* basic system *of continuous functions*

$$f_1, \ldots, f_N \colon X \to \mathbb{I},$$

*meaning that every continuous function $g\colon X \to \mathbb{I}$ can be represented as the sum of compositions*

$$g = \sum_{i=1}^{N} h_i \circ f_i$$

*of the basic functions with suitably chosen continuous functions $h_i \colon \mathbb{I} \to \mathbb{I}$.*

Let us show how to prove Theorem 4.2.3. First of all, it is rather obvious that the space $C_p(\mathbb{I})$ of continuous functions with the pointwise topology admits a continuous linear operator onto the space $C_p(\mathbb{I} \oplus \cdots \oplus \mathbb{I})$ on the disjoint sum of finitely copies of the interval. Using a basic system of functions on a finite-dimensional metrizable compactum $X$, the space $C_p(\mathbb{I} \oplus \cdots \oplus \mathbb{I})$ can be mapped in a linear continuous surjective fashion onto the space $C_p(X)$. An application of a theorem by Arhangel'skiĭ [8] enables one to conclude that the composition operator $C(\mathbb{I}) \to C(X)$ remains continuous with respect to the compact-open (that is, uniform) topologies on both function spaces. According to the Open Mapping Theorem, the continuous linear operator $C(\mathbb{I}) \to C(X)$ is open. Duality theory for locally convex spaces leads one to invert the direction of this operator and to obtain a topological embedding $L(X) \hookrightarrow L(\mathbb{I})$ of the free locally convex spaces. Finally, one invokes the Tkachenko-Uspenskij Theorem together with an observation that the free abelian topological groups on $X$ and on $\mathbb{I}$ sit inside the corresponding free locally convex spaces in the right way to obtain a topological group embedding $A(X) \hookrightarrow A(\mathbb{I})$.

**Remark 4.2.5.** Theorem 4.2.3 suggests that problems about free topological groups can be very difficult and substantial. Imagine that someone has proved the existence of a topological embedding $A(\mathbb{I}^2) \hookrightarrow A(\mathbb{I})$ without prior knowledge of the Kolmogorov Superposition Theorem; such a person would be forced to essentially rediscover the solution to Hilbert's Problem 13 on his/her own. Perhaps something of the kind indeed happens in general topology from time to time, and if anything, this shows the need for us the general topological algebraists to consciously look out for links with other areas of mathematics.

**Remark 4.2.6.** And not just mathematics. There is, it seems, an interesting perspective of linking theory of free topological groups to computing.

Let $X = (X, \rho)$ be a metric space, where the value of the metric $\rho(x, y)$ is interpreted as the *cost* of transporting a unit mass from point $x$ to point $y$. Suppose a unit mass is distributed between and stored at points $x_1, x_2, \cdots, x_n$, with amounts $\lambda_1, \lambda_2, \cdots, \lambda_n$ at each of them, and we want to transport the mass and store it at points $y_1, y_2, \cdots, y_m$, with amount $\mu_1, \mu_2, \cdots, \mu_n$ at each of the corresponding points. The cost of performing such a transportation is known as the *Kantorovich distance* between $\sum_{i=1}^n \lambda_i x_i$ and $\sum_{j=1}^n \mu_j y_j$. The Kantorovich distance plays a singularly important role in a wide range of applied mathematical sciences, from probability theory through information theory to computing and data storage and analysis (for a recent comprehensive treatment, see the two-volume set [92] exclusively devoted to the Kantorovich distance). At the same time, it is not difficult to see that the Kantorovich distance is exactly the metric generated by the maximal norm $p_\rho$ on the free locally convex space. As such, it can be approximated with any given degree of accuracy by the Graev metric on $A(X)$.

The Kantorovich distance can be computed through linear programming in quadratic time in the input size, $n$. An outstanding problem about the Kantorovich distance is the following: does there exist an algorithm for computing the value of the distance in the *linear* time $O(n)$ in the size of the input, $n$? Currently such algorithms are only known for $X = \mathbb{R}$ with the usual distance and also $X = \mathbb{S}^1$, the circle. The accumulated combinatorial techniques for dealing with the Graev metric make free topological group theorists well-poised to tackle this problem for more general metric spaces than the real line, especially in the natural case where $X$ is a finite-dimensional compactum. For example, can the embedding $A(X) \hookrightarrow A(\mathbb{I})$ constructed in the proof of Theorem 4.2.3 be used to achieve an algorithmic speed-up?

In the non-abelian case, the Graev metric on the free group $F(X)$ can be shown to coincide with another distance of great importance in computing: the so-called *string edit distance* between finite words in an alphabet $W$. The value of the string edit distance between two strings (words), $\sigma$ and $\tau$, is defined as the minimal number of insertions, deletions, and replacements necessary to get $\sigma$ from $\tau$. (Example: $d(\mathtt{metric}, \mathtt{track}) = 4$.) Clearly, the string edit distance is recovered from the Graev extension of the discrete metric from $W$ over $F(W)$. This type of distance is of importance in particular in molecular biology [3], and it is not impossible that the fluency of free topological group theorists in manipulating Graev metrics can be used to improve on the existing algorithms for computing the string edit distance and its various modifications.

### 4.3. The Epimorphism Problem

The topic with which I would like to conclude the article provides a nice example where all our main lines of development — actions on compacta, 'massive' groups, free topological groups, and embeddings — converge and work together in harmony.

A morphism $f\colon H \to G$ between two objects in some category is called an *epimorphism* if for all objects $F$ and two arbitrary morphisms $g, h\colon G \to F$, the condition $g \circ f = h \circ f$, cf. the diagram

$$H \xrightarrow{f} G \underset{h}{\overset{g}{\rightrightarrows}} F,$$

implies that $g = h$.

Let us first consider the category of discrete groups and group homomorphisms. Certainly, every homomorphism onto is an epimorphism. It turns out that the converse is also true. The following is apparently a part of the group theory folklore.

**Theorem 4.3.1.** *A group homomorphism $f\colon H \to G$ is an epimorphism if and only if $f$ is onto.*

It is both illuminating and essential for what follows to go through the proof. Let $f$, $H$ and $G$ be as in the statement of the theorem. Assume that $f$ is not onto, and let us show that $f$ is not an epimorphism. To produce a group $F$ and two distinct morphisms $g, h\colon G \to F$ with the property $g \circ f = h \circ f$, we proceed as follows. The image $f(H)$ is a proper subgroup of $G$. Denote by $X = G/f(H)$ the left factor-set, that is, the collection of all left cosets $x \cdot f(H)$, where $x \in G$. Then $|X| > 1$. The natural action of $G$ on $X$ by left translations extends to an action of $G$ on the free group $F(X)$ by group isomorphisms, that is, every element $x \in G$ determines a group isomorphism $y \mapsto x \cdot y$ of $F(X)$, and the resulting mapping from $G$ to the group of automorphisms of $F(X)$ is a homomorphism of groups. Now one can form the *semidirect product*, $G \ltimes F(X)$, corresponding to such an action. This is a group, which is, set-theoretically, the Cartesian product $G \times F(X)$, equipped with the group operation as follows:

$$(x, y)(x', y') := (xx', y(x \cdot y')).$$

The identity is the element $(e_G, e_{F(X)})$, while the inverse of an element $(x, y)$ is simply $(x^{-1}, x^{-1} \cdot (y^{-1}))$. Notice that $G$ forms a subgroup (and at the same time a factor-group) of the semidirect product, under the embedding $x \mapsto (x, e)$. Set $F = G \ltimes F(X)$, and define two homomorphisms from $G$ to $F$ by letting $g$ be the above described embedding $G \hookrightarrow G \ltimes F(X)$, while $h$ sends an element $x \in G$ to its conjugate in $F$ by the element $(e_G, f(H))$, where the subgroup $f(H)$ is viewed as a coset and element of the free basis $X \subset F(X)$. It is easy to see that $g|_H = h|_H$, while in general the two homomorphisms $g$ and $h$ differ (in fact, $g(x) \neq h(x)$ whenever $x \in G \setminus f(H)$). The proof is finished.

What happens in the category of (Hausdorff! – here it is really essential) topological groups and continuous homomorphisms? The most immediate observation is that if $f\colon H \to G$ is a continuous homomorphism with everywhere dense range ($\overline{g(H)} = G$), then $f$ is an epimorphism. It was asked by Karl H. Hofmann in late 1960's whether the converse was true. In other words, must every epimorphism between two topological groups have a dense range? As we have just seen, the answer is 'yes' if $G$ is discrete. It is an obvious exercise to show that the answer is positive if $G$ is abelian. By the end of this section, the reader will be able to prove that the answer is positive if $G$ is locally compact.[1] The general case, however, was only settled (in the negative) a few years ago by means of the following astonishing result. (Cf. [115, 116] for further refinements.)

**Theorem 4.3.2** (Uspenskij, 1993, [114])**.** *Let $x$ be any point of the circle $\mathbb{S}^1$. Then the embedding $\mathrm{St}_x \hookrightarrow \mathrm{Homeo}\,(\mathbb{S}^1)$ is an epimorphism, where $\mathrm{St}_x = \{g|g\cdot x = x\}$ is the isotropy subgroup.*

To understand how such a seemingly improbable thing can happen, it is worth re-examining the proof of Theorem 4.3.1 and finding out to what extent one can have it 'topologized.'

It poses no problem at all, to make the semidirect product of two topological groups into a topological group: the usual product topology will do the trick, provided the action of the first group on the second is continuous, as a map (in our case)

$$G \times F(X) \to F(X). \tag{4.1}$$

(The standard reference for semidirect products of topological groups is [44], 2.6.20.) The factor-space $X = G/\overline{f(H)}$ is of course a topological space, so that it is natural to equip $F(X)$ with the topology of the free topological group on $X$. Here we approach the bottleneck of the argument: the action (4.1) need not be continuous! What is worse, the situation cannot be, in general, remedied by considering group topologies on $F(X)$

[1] Using the following result by de Vries [123]: every continuous action of a locally compact group $G$ on a topological space $X$ is *linearizable,* that is, there exists a continuous action of $G$ on a locally convex space $E$ by isomorphisms and an embedding $X \hookrightarrow E$ as a topological $G$-subspace.

other than the free topology. A convenient framework enabling one to deal with situations of this kind had been developed by Megrelishvili independently from Uspenskij's Theorem 4.3.2 (though published later, [62, 63]), and we will now describe it briefly.

Let $G$ and $F$ be two topological groups. Say that $F$ is a $G$-*group* if $G$ continuously acts on $F$ by automorphisms. More precisely, there is given a continuous action $G \times F \to F$, and every motion $F \ni y \mapsto x \cdot y \in F$ is a group automorphism. (A particular case of this situation arises whenever a topological group $G$ is represented in a normed space $F$.) It is clear how to define morphisms between two $G$-groups: they are continuous homomorphisms commuting with the action of $G$. Now, given a topological group $G$ acting continuously on a topological space $X$, one can define the *free topological* $G$-*group* on $X$ in a standard fashion. Namely, $F_G(X)$ is a topological $G$-group, and there is an (essentially unique) morphism of $G$-spaces $\iota\colon X \to F_G(X)$ with the property that each morphism of $G$-spaces $f$ from $X$ to an arbitrary $G$-group $A$ admits a unique factorization $f = \bar{f} \circ \iota$, where $\bar{f}\colon F_G(X) \to A$ is a morphism between $G$-groups. For example, if $G = \{e\}$ is a trivial group, then $F_G(X)$ turns into the free topological group on $X$. The free topological $G$-group $F_G(X)$ is algebraically generated by the set $\iota(X)$. For a given $G$-space $X$ the free topological $G$-group $F_G(X)$ is the Hausdorff replica of the free group $F(X)$ equipped with the finest group topology such that its restriction to $X$ is coarser than the original topology on $X$ and the action $G \times F(X) \to F(X)$ is continuous. One says that the free topological $G$-group is *trivial* if it is isomorphic to $\mathbb{Z}$ equipped with the discrete topology and the trivial action of $G$, in which case $\iota$ is constant and takes all of $X$ to $1 \in \mathbb{Z}$.

Getting back to our discussion of the proof of Theorem 4.3.1, we can see that the right substitute for the free group $F(X)$ is the free topological $G$-group $F_G(X)$ on the $G$-space $X$. Therefore, everything boils down to the question of nontriviality of $F_G(X)$.

We have practically established the following.

**Theorem 4.3.3** (Pestov, [87])**.** *A continuous homomorphism $f$ between topological groups $H$ and $G$ is an epimorphism if and only if the free topological $G$-group on the $G$-space $X = G/\overline{f(H)}$ is trivial.*

The *Effros microtransitivity theorem* [24] says that if a Polish (= completely metrizable second-countable) topological group $G$ acts transitively on a Polish space $X$, then $X$ is isomorpic, as a $G$-space, with the left factor-space of $G$ by the isotropy subgroup $\mathrm{St}_x$ of an arbitrary element $x \in X$. Now we obtain the following convenient corollary.

**Corollary 4.3.4** ([87])**.** *Let $X$ be a transitive $G$-space such that both the acting group $G$ and the space $X$ are Polish. The following conditions are equivalent:*
*(i) the free topological $G$-group $F_G(X)$ is trivial;*
*(ii) the canonical embedding of the isotropy subgroup $\mathrm{St}_x$ of any point $x \in X$ into $G$ is an epimorphism of Hausdorff topological groups.*

The following result was circulated by Megrelishvili in a preprint form in early 1990's.

**Theorem 4.3.5** (Megrelishvili, [62, 63])**.** *The free topological $\mathrm{Homeo}\,(\mathbb{I})$-group $F_{\mathrm{Homeo}\,(\mathbb{I})}(\mathbb{I})$ is trivial.*

As an obvious corollary, the free topological $\mathrm{Homeo}\,(\mathbb{S}^1)$-group $F_{\mathrm{Homeo}\,(\mathbb{S}^1)}(\mathbb{S}^1)$ is trivial as well, and by invoking Corollary 4.3.4, we obtain a proof of Uspenskij's Theorem 4.3.2.

## References

[1] S. Adams and G. Stuck, *Splitting of non-negatively curved leaves in minimal sets of foliations,* Duke Math. J. **71** (1993), no. 1, 71–92.

[2] A.D. Alexandroff, *On groups with an invariant measure,* C.R. (Doklady) Acad. Sci. URSS (N.S.) **34** (1942), 5–9.

[3] A. Apostolico and R. Giancarlo, *Sequence alignment in molecular biology,* In: Mathematical Support for Molecular Biology, DIMACS Set. Discrete Math. Theoret. Comput. Sci. **47**, Amer. Math. Soc., Providence, RI, 1999, pp. 85–115.

[4] R. Arens and J. Eells, *On embedding uniform and topological spaces,* Pacific J. Math. **6** (1956), 397–403.

[5] A.V. Arhangel'skiĭ, *Relations among invariants of topological groups and their subspaces,* Russ. Math. Surveys **35** (1980), 1–23.

[6] A.V. Arhangel'skiĭ, *Any topological group is a quotient group of a zero-dimensional topological group,* Sov. Math. Dokl. **23** (1981), 615–618.

[7] A.V. Arhangel'skiĭ, *Classes of topological groups,* Russ. Math. Surveys **36** (1981), 151–174.

[8] A.V. Arhangel'skiĭ, *On linear homeomorphisms of function spaces,* Soviet Math. Dokl. **25** (1982), 852–855.

[9] J. Auslander, *Minimal Flows and Their Extensions,* North-Holland Mathematics Studies **153**, North-Holland, Amsterdam–NY–London–Tokyo, 1988.

[10] W. Banaszczyk, *On the existence of exotic Banach–Lie groups,* Math. Ann. **264** (1983), 485–493.

[11] W. Banaszczyk, *Additive Subgroups of Topological Vector Spaces,* Lect. Notes Math. **1466**, Springer-Verlag, Berlin, 1991.

[12] W. Banaszczyk and E. Martín-Peinador, *The Glicksberg theorem on weakly compact sets for nuclear groups,* Ann. New York Acad. Sci. **788** (1996), 34–39.

[13] H. Becker and A.S. Kechris, *The Descriptive Set Theory of Polish Group Actions,* London Mathematical Society Lecture Note Series, **232**, Cambridge University Press, Cambridge, 1996.

[14] J.F. Berglund, H.D. Junghenn, and P. Milnes, *Analysis on Semigroups — Function Spaces, Compactifications, Representations,* Wiley-Interscience Publ., 1989.

[15] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology,* PWN, Warszawa, 1975.

[16] N. Bourbaki, *Lie Groups and Lie Algebras,* Chapters I-III, Springer-Verlag, Berlin a.o., 1989.

[17] W.W. Comfort, *On the existence of group topologies,* Ann. New York Acad. Sci. **659** (1992), 42–50.

[18] W.W. Comfort and K.A. Ross, *Pseudocompactness and uniform continuity in topological groups,* Pacific J. Math. **16** (1966), 483–496.

[19] W.W. Comfort, K.H. Hofmann, and D. Remus, *A survey on topological groups and semigroups,* in: Recent Progress in General Topology (M. Hušek and J. van Mill, eds.), North-Holland, Amsterdam, 1992, 58–144.

[20] M. Cotlar and R. Ricabarra, *On the existence of characters in topological groups,* Amer. J. Math. **76** (1954), 375–388.

[21] W.J. Davis, T. Figiel, W.B. Johnson and A. Pełczyński, *Factoring weakly compact operators,* J. Funct. Anal. **17** (1974), 311–327.

[22] J. Dieudonné, *Sur la complétion des groupes topologiques,* C.R. Acad. Sc. Paris **218** (1944), 774–776.

[23] W.F. Eberlein, *Abstract ergodic theorems and weakly almost periodic functions,* Trans. Amer. Math. Soc. **67** (1949), 217–240.

[24] E.G. Effros, *Transformation groups and $C^*$-algebras,* Ann. Math. **81** (1965), 38–55.

[25] R. Ellis and H.B. Keynes, *Bohr compactifications and a result of Følner,* Israel J. Math. **12** (1972), 314–330.

[26] J. Flood, *Free Topological Vector Spaces,* Ph.D. thesis, Australian National University, Canberra, 1975, 109 pp.

[27] J. Flood, *Free locally convex spaces,* Dissertationes Math. **CCXXI** (1984), PWN, Warczawa.

[28] E. Følner, *Generalization of a theorem of Bogoliuboff to topological abelian groups,* Math. Scand. **2** (1954), 5–18.

[29] J. Galindo and S. Hernández, *Pontryagin-van Kampen reflexivity for free abelian topological groups,* Forum Math. **11** (1999), 399–415.

[30] F. Galvin, *Embedding countable groups in 2-generator groups,* American Math. Monthly **100** (1993), 578–580.

[31] A.A. Giannopoulos and V.D. Milman, *Concentration property on probability spaces,* IHES preprint M/00/07, 26 pp., available from
`http://www.ihes.fr/PREPRINTS/M00/Resu/resu-M00-07.html`

[32] T. Giordano and V. Pestov, a note in preparation.

[33] S. Glasner, *On minimal actions of Polish groups,* Top. Appl. **85** (1998), 119–125.

[34] M.I. Graev, *Theory of topological groups I,* Uspekhi Mat. Nauk (N.S.) **5** (1950), 3–56 (in Russian).

[35] E. Granirer, *Extremely amenable semigroups 2,* Math. Scand. **20** (1967), 93-113.

[36] D.L. Grant, review of the paper [117], Mathematical Reviews, **99i:22003**.

[37] F.P. Greenleaf, *Invariant Means on Topological Groups,* Van Nostrand Mathematical Studies **16**, Van Nostrand – Reinhold Co., NY–Toronto–London–Melbourne, 1969.

[38] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces,* Progress in Mathematics **152**, Birkhäuser, Boston–Basel–Berlin, 1999.

[39] M. Gromov, *Spaces and questions,* in: Geom. Funct. Anal., special volume (2000), to appear.

[40] M. Gromov and V.D. Milman, *A topological application of the isoperimetric inequality,* Amer. J. Math. **105** (1983), 843–854.

[41] P.R. Halmos, *Lectures on Ergodic Theory,* Chelsea Publ. Co., New York, 1960.

[42] P. de la Harpe, *Moyennabilité de quelques groupes topologiques de dimension infinie,* C.R. Acad. Sc. Paris, Sér. **A 277** (1973), 1037–1040.

[43] W. Herer and J.P.R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups,* Math. Ann. **213** (1975), 203–210.

[44] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis. Vol. 1 (2nd ed.),* Springer–Verlag, NY a.o., 1979.

[45] G. Higman, B.H. Neumann and H. Neumann, *Embedding theorems for groups,* J. London Math. Soc. **24** (1949), 247–254.

[46] K.H. Hofmann, *Introduction to the Theory of Compact Groups. Part I,* Tulane University, Department of Mathematics, 1968.

[47] K.H. Hofmann and S.A. Morris, *The Structure of Compact Groups,* de Gruyter Studies in Mathematics **25**, Walter de Gruyter & Co., Berlin, 1998.

[48] G.L. Itzkowitz, *Uniformities and uniform continuity in topological groups,* Lecture Notes in Pure and Applied Math. **134**, Dekker, NY, 1991, pp. 155–178.

[49] M. Katětov, *On universal metric spaces,* General Top. and its Relations to Modern Analysis and Algebra VI: Proc. Sixth Prague Topological Symposium 1986, Heldermann Verlag, Berlin, 1988, 323–330.

[50] E. Katz, S.A. Morris, and P. Nickolas, *Characterization of bases of subgroups of free topological groups,* J. London Math. Soc. **27** (1983), 421–426.

[51] A.S. Kechris, *Descriptive dynamics,* in: Descriptive Set Theory and Dynamical Systems, London Mathematical Society Lecture Note Series, **277**, Cambridge University Press, Cambridge, 2000.

[52] J. Keesling, *Topological groups whose underlying spaces are separable Fréchet manifolds,* Pacific J. Math. **44** (1973), 181–189.

[53] O. Kobayashi, A. Yoshioka, Y. Maeda, and H. Omori, *The theory of infinite-dimensional Lie groups and its applications,* Acta Appl. Math. **3** (1985), 71–106.

[54] A.N. Kolmogorov, *On the representation of continuous functions of many variables,* Amer. Math. Soc. Transl. (2) **28** (1963), 55–61.

[55] M.B. Landstad, *On the Bohr topology in amenable topological groups,* Math. Scand. **28** (1971), 207–214.

[56] A.G. Leiderman, S.A. Morris and V. Pestov, *The free abelian topological group and the free locally convex space on the unit interval,* J. London Math. Soc. **56** (1997), 529–538.

[57] J. Leslie, *Some integrable subalgebras of the Lie algebras of infinite-dimensional Lie groups,* Trans. Amer. Math. Soc. **333** (1992), 423–443.

[58] A.A. Markov, *Three papers on topological groups,* Amer. Math. Soc. Transl. **30** (1950), 120 pp.

[59] G.J. Martin, *The Hilbert–Smith conjecture for quasiconformal actions,* Elect. Res. Announc. of the Amer. Math. Soc. **5** (1999), 66–70.

[60] R.D. Mauldin (ed.), *The Scottish Book,* Birkhäuser, Boston–Basel–Stuttgart, 1981.

[61] B. Maurey, *Constructions de suites symétriques,* C.R. Acad. Sc. Paris, Sér. **A-B 288** (1979), 679–681.

[62] M.G. Megrelishvili, *Free topological groups over (semi)group actions,* Proceedings of the 10th Summer Conference on Topology and Applications (Amsterdam, August 1994), Annals NY Acad. Sci., NY 1996, 164–169.

[63] M.G. Megrelishvili, *Free topological G-groups,* New Zealand J. Math. **25** (1996), 59–72.

[64] M.G. Megrelishvili, *Operator topologies and reflexive representability of groups,* Bar-Ilan University preprint BIUMCS 98/52.

[65] M.G. Megrelishvili, *Every semitopological semigroup compactification of the group* $H_+[0,1]$ *is trivial,* Bar-Ilan University preprint, August 1999, 13 pp.

[66] M. Megrelishvili, P. Nickolas, and V. Pestov, *Uniformities and uniform structures on locally connected groups,* Bull. Austral. Math. Soc. **56** (1997), 279–283.

[67] V.D. Milman, *Diameter of a minimal invariant subset of equivariant Lipschitz actions on compact subsets of* $\mathbb{R}^n$, in: Geometrical Aspects of Functional Analysis (Israel Seminar, 1985–86), Lecture Notes in Math. **1267** (1987), Springer–Verlag, Berlin a.o., pp. 13–20.

[68] V.D. Milman, *The heritage of P. Lévy in geometric functional analysis,* Astérisque **157-158** (1988), 273–301.

[69] V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces,* Lecture Notes in Math. **1200**, Springer-Verlag, 1986.

[70] J. Milnor, *Remarks on infinite-dimensional Lie groups,* in: Relativité, groupes et topologie II. Les Houches, Session XL, Elsevier Sci. Publ., Amsterdam a.o., 1984, pp. 1007–1058.

[71] T. Mitchell, *Topological semigroups and fixed points,* Illinois J. Math. **14** (1970), 630–641.

[72] D. Montgomery and L. Zippin, *Topological Transformation Groups,* Interscience, New York, 1955.

[73] S.A. Morris. Varieties of topological groups...a survey. *Colloq. Math.* **46** (1982), 147–165.

[74] S.A. Morris and V. Pestov, *A topological generalization of the Higman–Neumann–Neumann theorem,* Journal of Group Theory **2** (1998), 181–187.

[75] S.A. Morris and V. Pestov, *Subgroups of monothetic groups,* Journal of Group Theory **3** (2000), 407-417.

[76] K.-H. Neeb, *Holomorphy and convexity in Lie theory,* de Gruyter Expositions in Mathematics, **28**, Walter de Gruyter & Co., Berlin, 2000.

[77] Yu.A. Neretin, *Categories of Symmetries and Infinite-Dimensional Groups,* London Math. Soc. Monographs **16**, The Clarendon Press, Oxford U.P., NY, 1996.

[78] M.H.A. Newman, *A theorem on periodic transformation of spaces,* Oxford Quart. J. Math. **2** (1931), 1–8.

[79] B.H. Neumann and H. Neumann, *Embedding theorems for groups,* J. London Math. Soc. **34** (1959), 465–479.

[80] P.A. Ostrand, *Dimension of metric spaces and Hilbert's problem 13,* Bull. Amer. Math. Soc. **71** (1965), 619–622.

[81] A.T. Paterson, *Amenability,* Math. Surveys and Monographs **29**, Amer. Math. Soc., Providence, RI, 1988.

[82] V.G. Pestov, *The coincidence of the dimensions dim of l-equivalent topological spaces,* Soviet Math. Dokl. **26** (1982), 380-382.

[83] V.G. Pestov, *Universal arrows to forgetful functors from categories of topological algebra,* Bull. Austral. Math. Soc. **48** (1993), 209–249.

[84] V. Pestov, *A remark on embedding topological groups into products,* Bull. Austral. Math. Soc. **49** (1994), 519–521.

[85] V.G. Pestov, *Operator spaces and residually finite-dimensional $C^*$-algebras,* Journal of Functional Analysis **123** (1994), 308–317.

[86] V.G. Pestov, *Free abelian topological groups and the Pontryagin–van Kampen duality,* Bull. Austral. Math. Soc. **52** (1995), 297–311.

[87] V. Pestov, *Epimorphisms of Hausdorff groups by way of topological dynamics,* New Zealand J. Math. **26** (1997), 257–262.

[88] V. Pestov, *Some universal constructions in abstract topological dynamics,* In: Topological Dynamics and its Applications. A Volume in Honor of Robert Ellis, Contemp. Math. **215** (1998), 83–99.

[89] V.G. Pestov, *On free actions, minimal flows, and a problem by Ellis,* Trans. Amer. Math. Soc. **350** (1998), pp. 4149-4165.

[90] V. Pestov, *Ramsey–Milman phenomenon, Urysohn metric spaces, and extremely amenable groups,* SMCS-VUW preprint 00-8, April 2000, 31 pp., available from
`http://arXiv.org/abs/math.FA/0004010`

[91] J. Pym, *A note on $G^{\mathcal{LUC}}$ and Veech's theorem,* Semigroup Forum **59** (1999), 171–174.

[92] S.T. Rachev and L. Rüschendorf, *Mass Transportation Problems. Volume I: Theory. Volume II: Applications,* Springer, NY–Berlin–Heidelberg, 1998.

[93] D.A. Raĭkov, *Free locally convex spaces for uniform spaces,* Mat. Sb. (N.S.) **63** (1964), 582–590 (in Russian).

[94] D. Repovš and E.V. Ščepin, *A proof of the Hilbert–Smith conjecture for actions by Lipschitz maps,* Math. Ann. **2** (1997), 361–364.

[95] W. Roelcke and S. Dierolf, *Uniform Structures in Topological Groups and Their Quotients,* McGraw-Hill, NY, 1981.

[96] S. Rolewicz, *Some remarks on monothetic groups,* Colloq. Math. **13** (1964), 28–29.

[97] W. Ruppert, *Compact Semitopological Semigroups: An Intrinsic Theory,* Lecture Notes Math. **1079**, Springer-Verlag, 1989.

[98] S. Sakai, *$C^*$-Algebras and $W^*$-Algebras,* Springer-Verlag, Berlin–Neidelberg–NY, 1971; Reprinted, Springer, 1998.

[99] S.A. Shkarin, *On universal abelian topological groups,* Mat. Sb. **190** (1999), no. 7, 127–144. (Russian.)

[100] A. Shtern, *Unitary Representation,* in: *Mathematical Encyclopaedia,* Vol. 5, Sov. Encycl., 1984, pp. 508–513 (in Russian).

[101] A. Shtern, *Compact semitopological semigroups and reflexive representability of topological groups,* Russian J. Math. Phys. **2** (1994), 131–132.

[102] O.V. Sipacheva, *Zero-dimensionality and completeness in free topological groups I, II,* Serdica **15** (1989), 119–140; 141–154.

[103] O.V. Sipacheva, *Free topological groups of spaces and their subspaces,* Topology Appl. **101** (2000), 181–212.

[104] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces,* Publ. Math. IHES **81** (1995), 73–205.

[105] S. Teleman, *Sur la représentation linéaire des groupes topologiques,* Ann. Sci. Ecole Norm. Sup. **74** (1957), 319–339.

[106] M.G. Tkachenko, *On the Souslin property in free topological groups over compacta,* Math. Notes **34** (1983), 790–793.

[107] M.G. Tkachenko, *On completeness of free abelian topological groups,* Soviet Math. Dokl. **27** (1983), 341–345.

[108] M.G. Tkachenko, *On topologies of free groups,* Czechoslovak Math. J. **33** (1984), 57–69.

[109] M.G. Tkachenko, *Topological groups for topologists. I, II,* Bull. Soc. Mat. Mexicana **5** (1999), 237–280; **6** (2000), 1–41.

[110] P. Urysohn, *Sur un espace métrique universel,* Bull. Sci. Math. **51** (1927), 43-64, 74-90.

[111] V.V. Uspenskij, *A universal topological group with countable base,* Funct. Anal. Appl. **20** (1986), 160–161.

[112] V.V. Uspenskij, *On the group of isometries of the Urysohn universal metric space,* Comment. Math. Univ. Carol. **31** (1990), 181–182.

[113] V.V. Uspenskiĭ, *Free topological groups of metrizable spaces,* Math. USSR-Izvestiya **37** (1991), 657–680.

[114] V.V. Uspenskij, *The solution of the epimorphism problem for Hausdorff topological groups,* Seminar Sophus Lie **3** 1993, 69–70.

[115] V.V. Uspenskij, *The epimorphism problem for Hausdorff topological groups,* Topology Appl. **57** (1994), 287–294.

[116] V.V. Uspenskij, *Epimorphisms of topological groups and Z-sets in the Hilbert cube,* Proc. 2nd Gauss Symposium (München, 1993), de Gruyter, Berlin, 1995, pp. 733–738.

[117] V.V. Uspenskij, *The Roelcke compactification of unitary groups,* in: Abelian groups, module theory, and topology, Proceedings in honor of Adalberto Orsatti's 60th birthday (D. Dikranjan, L. Salce, eds.), Lecture notes in pure and applied mathematics **201**, Marcel Dekker, New York e.a., 1998, pp. 411-419.

[118] V.V. Uspenskij, *The Roelcke compactification of groups of homeomorphisms,* Topology and its Applications, to appear.

[119] V.V. Uspenskij, *On subgroups of minimal topological groups,* Ohio University 1998 preprint, e-print available at http://arXiv.org/abs/math.GN/0004119.

[120] W.A. Veech, *Topological dynamics,* Bull. Amer. Math. Soc. **83** (1977), 775–830.

[121] W.A. Veech, *The equicontinuous structure relation for minimal abelian transformation groups,* Amer. J. Math. **90** (1968), 723–732.

[122] A.N. Vershik, *Amenability and approximation of infinite groups,* Sel. Math. Sov. **2** (1982), 311–330.

[123] J. de Vries, *On the existence of G-compactifications,* Bull. Acad. Polon. Sci. Math. Astron. Phys. **26** (1978), 275–280.

[124] J. de Vries, *Elements of Topological Dynamics,* Mathematics and Its Applications **257**, Kluwer Academic Publ., Dordrecht–Boston–London, 1993.

[125] S. Wagon, *The Banach–Tarski Paradox,* Cambridge University Press, 1985.

[126] C.-T. Yang, *p-Adic transformation groups,* Michigan Math. J. **7** (1960), 201–218.

School of Mathematical and Computing Sciences, Victoria University of Wellington, P.O. Box 600, Wellington, New Zealand

*E-mail address*: vova@mcs.vuw.ac.nz