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UNIVERSAL ULTRAMETRIC SPACES OF SMALLEST WEIGHT

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Abstract

We modify a construction of A. Lemin and V. Lemin to construct an ultrametric space LW'_τ which is universal (in the sense of isometry) for ultrametric spaces of weight at most τ. Under the singular cardinal hypothesis, a set-theoretic assumption whose negation is related to large cardinals, the weight of LW'_τ is τ for all $\tau > \mathfrak{c}$. This provides a solution to a problem raised by the Lemins.

1. Introduction

An ultrametric space X is called *isometrically universal for ultrametric spaces of weight at most* τ provided that every ultrametric space of weight at most τ can be isometrically embedded into X (for short, we say that X is τ-*universal*). In [3], A. Lemin and V. Lemin construct for every cardinal τ an ultrametric space which they called LW_τ, and proved

Theorem 1. [3, Main Theorem] *The ultrametric space LW_τ is a τ-universal space, and the weight of LW_τ is τ^ω.*

The Lemins point out that if τ is a cardinal such that $\tau^\omega = \tau$, then their space LW_τ has weight τ, which is the smallest possible

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weight for a τ-universal space. This leads to the natural question raised by the Lemins ([3, Problem 1]): If $\mathfrak{c} < \tau < \tau^\omega$, does there exist a τ-universal space having weight smaller than τ^ω? In particular, does there exist one having weight τ? We give two affirmative solutions to this latter problem. We show that there is an unbounded class of cardinals τ satisfying $\mathfrak{c} < \tau < \tau^\omega$ for which there is a τ-universal space of weight τ, and that under the assumption of the singular cardinal hypothesis, for every cardinal satisfying $\mathfrak{c} < \tau < \tau^\omega$ there exists a τ-universal space of weight τ.

We consider only infinite cardinals in this paper, and $\mathfrak{c} = 2^\omega$ denotes the cardinality of the continuum. For a cardinal τ, τ^+ denotes the first cardinal larger than τ, and $cf(\tau)$ denotes the cofinality of τ. We consider the following condition for cardinal numbers $\tau > \mathfrak{c}$:

$$(*) \quad \sum\{\kappa^\omega : \kappa < \tau\} \leq \tau.$$

Clearly (*) is weaker than the condition $\tau^\omega = \tau$. Since $cf(\tau) = \omega$ implies $\tau < \tau^\omega$ by Konig's theorem [1, Theorem 17], we introduce the following condition for discussion of the Lemins' problem:

$$(\dagger) \quad \tau \text{ satisfies } (*) \text{ and } cf(\tau) = \omega$$

Our main result follows from Lemma 2 and Lemma 3:

Theorem 2. *If $\tau > \mathfrak{c}$ and τ satisfies (*), then there exists an ultrametric space LW'_τ which is τ-universal and has weight τ.*

Thus if τ satisfies (†), then LW'_τ provides an affirmative solution to the Lemins' problem. Clearly every strong limit cardinal τ of countable cofinality satisfies (†) (a cardinal τ is a *strong limit cardinal* provided $2^\kappa < \tau$ for every cardinal $\kappa < \tau$). Thus we have an unbounded class of cardinals for which LW'_τ is a τ-universal space of weight τ; so the Lemins' problem is solved for these cardinals. Moreover, we completely solve the Lemins' problem assuming the singular cardinal hypothesis, by proving the following:

Lemma 1. *Under the singular cardinal hypothesis, every $\tau > \mathfrak{c}$ satisfies (*).*

Proof. The result [1, Lemma 8.1] describes the value of κ^λ for any infinite cardinals κ, λ under the assumption of the singular cardinal hypothesis. In case $\kappa > \mathfrak{c}$ and $\lambda = \omega$, the lemma tell us that $\kappa^\omega = \kappa$ or κ^+. Thus, if $\kappa < \tau$ then $\kappa^\omega = \mathfrak{c}, \kappa$ or κ^+, so we have $\kappa^\omega \leq \tau$. □

The hypothesis "$\tau > \mathfrak{c}$" in Theorem 2 can be improved to "$\tau \geq \mathfrak{c}$" since $\mathfrak{c} = \mathfrak{c}^\omega$, but for no cardinal $\tau < \mathfrak{c}$ is there a τ-universal space of weight τ (or of weight less than τ^ω) because the Lemins proved that any ultrametric space that contains an isometric copy of every two-point ultrametric space must necessarily have weight at least $\mathfrak{c}$ [3, Proposition].

The question as to whether every cardinal $\tau > \mathfrak{c}$ satisfies (*) is related to large cardinals. If (*) fails for some cardinal $\tau > \mathfrak{c}$, then the singular cardinal hypothesis is false, and therefore there is an inner model of the universe with a measurable cardinal [2, §29]. In the other direction, using "some very large cardinals," M. Magidor [4] constructed a model satisfying the following two properties: (i) for all $n < \omega$, $2^{\aleph_n} = \aleph_{n+1}$ and (ii) $2^{\aleph_\omega} = \aleph_{\omega+2}$. It follows that (*) fails for the cardinal $\tau = \aleph_{\omega+1} > \mathfrak{c}$ since $\aleph_\omega^\omega = \aleph_{\omega+2}$ in this model (e.g., see [1, Lemma 8.3]). We do not know if the Lemins' problem is related to large cardinals.

In §2, we describe the space LW'_τ, and prove that its weight is τ whenever τ satisfies (*). In §3 we prove that LW'_τ is τ-universal. In §4 we discuss the condition (†), and the remaining unsolved portion of the Lemins' question.

2. A Subspace of the Lemins' τ-universal Space

We first define the Lemins' τ-universal metric space LW_τ. Let $\mathbb{Q}^+$ denote the set of positive rational numbers, and ${}^{\mathbb{Q}^+}\tau$ the set of all function $f : \mathbb{Q}^+ \to \tau$. The Lemins defined

$$LW_\tau = \{f \in {}^{\mathbb{Q}^+}\tau : \exists N(f) \in \mathbb{R} \text{ such that } f(x) = 0 \text{ for all } x > N(f)\},$$

and an ultrametric Δ on LW_τ by the equalities $\Delta(f,f)=0$ and for $f \neq g$, $\Delta(f,g)=\sup\{x : f(x) \neq g(x)\}$. It is easily checked that Δ is an ultrametric on LW_τ.

We define

$$D_\tau = \{f \in LW_\tau : (\exists \alpha < \tau) Range(f) \subset \alpha\},$$

and define the subspace $LW'_\tau \subset LW_\tau$ to be the closure of D_τ in LW_τ:

$$LW'_\tau = cl_{LW_\tau}(D_\tau).$$

Lemma 2. *The weight of LW'_τ equals $\tau \cdot \sum\{\kappa^\omega : \kappa < \tau\}$, and $\tau \cdot \sum\{\kappa^\omega : \kappa < \tau\}$ is either τ or τ^ω. If τ satisfies (*), then the weight of LW'_τ equals τ.*

Proof. The cardinality of D_τ equals $\sum\{|\alpha|^\omega : \alpha < \tau\}$ since

$$D_\tau = \bigcup_{\alpha<\tau} {}^{\mathbb{Q}^+}(\alpha),$$

and clearly $\sum\{|\alpha|^\omega : \alpha < \tau\} = \tau \cdot \sum\{\kappa^\omega : \kappa < \tau\}$. Since D_τ is dense in LW'_τ, the weight of $LW'_\tau \leq |D_\tau|$. We need to show that the weight is not less than $|D_\tau|$, and we do this in two steps. First we show that for every $\kappa < \tau$, there is a discrete subset of LW'_τ of cardinality κ^ω. We proceed as in [3, Main Theorem] and define for each $f \in LW_\kappa$ the function $F_f \in LW_\kappa$ by $F_f(x) = 0$ for $x \leq 1$, and $F_f(x) = f(x-1)$ for $x > 1$. Then if $f \neq g$, we have $\Delta(F_f, F_g) \geq 1$; so $\{F_f : f \in LW_\kappa\}$ is a discrete subset of LW'_τ of cardinality κ^ω. Thus the weight of LW'_τ is at least $\sum\{\kappa^\omega : \kappa < \tau\}$. Now for $\alpha < \tau$ define

$$f_\alpha(x) = \begin{cases} \alpha & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Then $\{f_\alpha : \alpha < \tau\}$ is a discrete subset of LW'_τ of cardinality τ; so the weight of LW'_τ is at least τ.

To see that $\tau \cdot \sum\{\kappa^\omega : \kappa < \tau\}$ is either τ or τ^ω, first note that this cardinal is between τ and τ^ω. Thus if $\tau < \tau \cdot \sum\{\kappa^\omega : \kappa < \tau\}$ then $\tau < \sum\{\kappa^\omega : \kappa < \tau\}$. Hence there exists $\kappa_0 < \tau$ such that $\tau < \kappa_0^\omega$; so $\kappa_0^\omega = \tau^\omega$. Thus $\sum\{\kappa^\omega : \kappa < \tau\} = \tau^\omega$.

The last statement in the Lemma follows from the definition of (*).

Clearly $LW'_\tau = LW_\tau$ if and only if $cf(\tau) > \omega$. □

3. LW'_τ is τ-universal

We now show that LW'_τ is a τ-universal space. Since we work in a subspace of LW_τ, we can refer to the proof in [3] for most of the details.

Lemma 3. *LW'_τ is a τ-universal space.*

Proof. Let (X, d) be an ultrametric space of weight τ. We recall the inductive construction in [3] of the function $i : X \to LW_\tau$ which will be the desired isometry. Essentially, all we do is observe that a minor change in the well-ordering of X allows us to prepare the induction so we can prove that $Range(i) \subset LW'_\tau$. Well-order $X = \{a_\alpha : \alpha < \kappa\}$ (one-to-one) in such a way that $\{a_\alpha : \alpha < \tau\}$ is dense in X (putting this dense set first is the change we need). The function i will be defined by induction on κ and the notation $i(a_\alpha) = f_\alpha$ will be used. Define $i(a_0) = f_0$ to be the constant function in ${}^{\mathbb{Q}^+}\tau$ with constant value 0. Define $i(a_1) = f_1$ by $f_1(x) = 1$ for $0 < x \leq d(a_0, a_1)$ and $f_1(x) = f_0(x)$ for $x > d(a_0, a_1)$. Assume we have defined f_α for $\alpha < \gamma$, where $\gamma < \kappa$, so that for all $\beta < \alpha < \gamma$ we have

(1) $d(a_\alpha, a_\beta) = \Delta(f_\alpha, f_\beta)$
(2) $Range(f_\alpha) \subset \min\{\alpha + 1, \tau\}$

As in [3], we define f_γ in two cases. First put

$$d_\gamma = \inf\{d(a_\alpha, a_\gamma) : \alpha < \gamma\},$$

and note that $d_\gamma = 0$ for all $\tau \leq \gamma < \kappa$.

Case 1. There exists $\beta < \gamma$ such that $d(a_\beta, a_\gamma) = d_\gamma$ (by one-to-one, $d_\gamma \neq 0$; so $\gamma < \tau$ in this case). Then define

$$f_\gamma(x) = \begin{cases} \gamma & \text{if } x \leq d_\gamma \\ f_\beta(x) & \text{if } x > d_\gamma \end{cases}$$

Case 2. If not Case 1, then there exists a sequence of $\alpha_n < \gamma$ such that $d(a_{\alpha_n}, a_\gamma) < d_\gamma + \frac{1}{n}$ (for $n < \omega$).

Case 2(a). If $d_\gamma > 0$ define

$$f_\gamma(x) = \begin{cases} \gamma & \text{if } x \leq d_\gamma \\ f_{\alpha_n}(x) & \text{if } d_\gamma + \frac{1}{n} < x \end{cases}$$

Case 2(b). If $d_\gamma = 0$ define $f_\gamma(x) = f_{\alpha_n}(x)$ for $d_\gamma + \frac{1}{n} < x$. The Lemins prove [3, Theorem 1] that in all cases f_γ is well defined and satisfies (1).

To see that (2) holds for f_γ, we first assume that $\gamma < \tau$, hence $\min\{\gamma + 1, \tau\} = \gamma + 1$. If f_γ is defined by Case 1, we note that since $\beta < \gamma$,

$$Range(f_\gamma) \subset Range(f_\beta) \cup \{\gamma\} \subset (\beta + 1) \cup \{\gamma\} \subset \gamma + 1.$$

If f_γ is defined by Case 2, we have (since $\alpha_n < \gamma < \tau$ for all $n < \omega$) either

$$Range(f_\gamma) \subset \bigcup_{n<\omega} Range(f_{\alpha_n}) \cup \{\gamma\} \subset \bigcup_{n<\omega} (\alpha_n + 1) \cup \{\gamma\} \subset \gamma + 1,$$

or

$$Range(f_\gamma) \subset \bigcup_{n<\omega} Range(f_{\alpha_n}) \subset \bigcup_{n<\omega} (\alpha_n + 1) \subset \gamma + 1.$$

Now we show that (2) holds when $\tau \leq \gamma < \kappa$. In this case we have $\min\{\gamma + 1, \tau\} = \tau$. Since $\{a_\alpha : \alpha < \tau\}$ is dense in X, we have $d_\gamma = 0$ for all $\tau \leq \gamma < \kappa$, and therefore f_γ is defined by Case 2(b). By the induction hypothesis we have

$$Range(f_\gamma) \subset \cup_{n<\omega} Range(f_{\alpha_n}) \subset \cup_{n<\omega} (\min\{\alpha_n + 1, \tau\}) \subset \tau.$$

This completes the induction. To complete the proof of the Lemma, we note that by (2), for $\alpha < \tau$, f_α has bounded range in LW_τ, hence the dense set $\{a_\alpha : \alpha < \tau\}$ is mapped by i into D_τ. Since i is an isometry by (1), hence continuous,

$$i(X) \subset cl_{LW_\tau} D_\tau = LW'_\tau.$$

This completes the proof. □

We remark that the Lemins' proof of their Main Theorem proceeds in two steps. First they define the isometry on a dense subset of X into LW_τ where τ is the density of X, and next they extend the isometry to the whole space X. Our proof does not explicitly use the second step since we construct i to be an isometry on all of X into LW'_τ.

4. Cardinals that Satisfy (†)

As we noted, every strong limit cardinal τ of countable cofinality satisfies (†), hence the class of cardinals satisfying (†) is unbounded. We will describe this class in more detail and enumerate it.

Lemma 4. *(a) A countable sum of cardinals satisfying (†) satisfies (†). (b) For any cardinal κ, the smallest cardinal $\tau > \kappa$ that satisfies (†) is the first singular cardinal greater than κ^ω.*

Proof. The proof of (a) is obvious, and (b) follows from the well-known Hausdorff formula [1, 6.18]

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \aleph_{\alpha+1}$$

and its corollary that if $\kappa = \aleph_\alpha$ satisfies $\kappa^\omega = \kappa$, then similarly $\aleph_{\alpha+1}$ satisfies $(\aleph_{\alpha+1})^\omega = \aleph_{\alpha+1}$. Thus if $\kappa = \aleph_\alpha$, then $\tau = \sup\{\aleph_{\alpha+n} : n < \omega\}$ is the first singular cardinal greater than κ. Moreover, τ satisfies (†). We now show that τ is the smallest cardinal greater than κ satisfying (†): If μ is a cardinal and $\kappa < \mu < \tau$, then either $\mu < \kappa^\omega$, in which case μ does not satisfy (*), or $\mu = \aleph_{\alpha+n}$ for some $n < \omega$, in which case $\mu^\omega = \mu$ so μ has uncountable cofinality by Konig's theorem [1, Theorem 17]. In either case, μ does not satisfy (†).

For a cardinal κ let $s(\kappa)$ denote the first singular cardinal greater than κ^ω. We can enumerate the class of all cardinals satisfying (†) as follows. Let $\kappa_0 = s(\mathfrak{c})$, and for $\alpha > 0$ define

$$\kappa_\alpha = \begin{cases} \sum\{\kappa_\beta : \beta < \alpha\} & \text{if } cf(\alpha) = \omega \\ s(\sum\{\kappa_\beta : \beta < \alpha\}) & \text{otherwise} \end{cases}$$

By Lemma 4, (κ_α) is a strictly increasing enumeration of cardinal numbers satisfying (†). To see that every cardinal τ which satisfies (†) is in this enumeration, let α be the first cardinal such that $\tau \leq \kappa_\alpha$. Thus $\sum\{\kappa_\beta : \beta < \alpha\} \leq \tau$. If $cf(\alpha) = \omega$, then

$$\kappa_\alpha = \sum\{\kappa_\beta : \beta < \alpha\} \leq \tau \leq \kappa_\alpha$$

and thus $\tau = \kappa_\alpha$. If $cf(\alpha) > \omega$, then since $cf(\tau) = \omega$, we have $\sum\{\kappa_\beta : \beta < \alpha\} < \tau$, and since τ satisfies (*), we have $(\sum\{\kappa_\beta : \beta < \alpha\})^\omega \leq \tau$. Finally we have $s(\sum\{\kappa_\beta : \beta < \alpha\}) \leq \tau$ because (as above) any cardinal μ in an interval of cardinals of the form $[\lambda^\omega, s(\lambda^\omega))$ has the property $\mu^\omega = \mu$, which τ does not satisfy. Thus

$$\kappa_\alpha = s(\sum\{\kappa_\beta : \beta < \alpha\}) \leq \tau \leq \kappa_\alpha$$

so again $\tau = \kappa_\alpha$.

By the preceding discussion, we see that the first cardinal for which we cannot answer the Lemins' problem (in ZFC) is the cardinal $\tau = s(\mathfrak{c})^+$. In Magidor's model, $\tau = s(\mathfrak{c})^+ = \aleph_{\omega+1}$, and, as we noted, in his model $\tau^\omega > \tau$. Since $\tau = \aleph_{\omega+1}$ has uncountable cofinality (in fact, is regular) $LW_\tau = LW'_\tau$, so the weight of LW'_τ is $\tau^\omega > \tau$. The following portion of the Lemin's question is therefore still open: Is it true without any assumption outside ZFC, that for every cardinal $\tau > c$ there exists a τ-universal space of weight less than τ^ω, in particular of weight τ?

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