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Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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ON THE STRONG COVERING PROPERTY OF CONTINUA

JANUSZ J. CHARATONIK AND WŁODZIMIERZ J. CHARATONIK

ABSTRACT. The concept of a minimal closed cover of a continuum has been introduced and studied in [11]. We provide a further investigation of this notion. In particular, we show connections with several related notions and answer questions in [11] (concerning hereditarily indecomposable continua and hereditarily locally connected ones). Using the concept we define the strong covering property of continua and study its connections with related concepts.

1. INTRODUCTION

Given a (metric) continuum X, a family $\mathcal{F}$ of nonempty closed subsets of X is said to *cover* X provided that $\cup\mathcal{F} = X$. We denote by $C(X)$ the *hyperspace of* (nonempty) *subcontinua* of X equipped with the Hausdorff metric (see [15, 0.1, p.1]; compare [6, 2, p.9]). By a *Whitney map* for $C(X)$ we mean a mapping $\mu : C(X) \to [0, \infty)$ such that

(1.1) $\mu(\{x\}) = 0$ for each $x \in X$,
(1.2) if $A \subsetneq B$, then $\mu(A) < \mu(B)$.

For each $t \in [0, \mu(X)]$ the preimage $\mu^{-1}(t)$ is called a *Whitney level* for $C(X)$. The reader is referred to [15] and [6] for these and other concepts used in this paper.

A continuum X is said to have the *covering property* (written $X \in CP$) provided that for each Whitney map $\mu : C(X) \to [0, \infty)$

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and for each $t \in [0, \mu(X)]$ no proper subcontinuum of $\mu^{-1}(t)$ covers X (see [8, Section 6, p. 179]; compare also [15, Remark 14.14, p. 417] and [6, Definition 35.3, p. 253, and Theorem 67.1, p. 320]).

The condition of the definition of the covering property which claims that no proper *subcontinuum* of $\mu^{-1}(t)$ covers X has been relaxed in [11] by considering certain special covers of X. Namely, the condition has been replaced by demanding that no proper *compact subset* of $\mu^{-1}(t)$ covers X. The following concept is defined in [11, p. 191].

Definition 2. A subset $\mathcal{A}$ of $C(X)$ is said to be a *minimal closed cover of* X provided that

(2.1) $\mathcal{A}$ is a closed subset of $C(X)$;
(2.2) each element of $\mathcal{A}$ is a nondegenerate proper subcontinuum of X;
(2.3) $\mathcal{A}$ covers X, i.e., $\cup\mathcal{A} = X$;
(2.4) no proper closed subset of $\mathcal{A}$ covers X.

Examples and basic properties of this concept are given in [11]. In particular it is known that if a family of nondegenerate proper subcontinua of X is closed in $C(X)$ and covers X, then it contains a minimal closed cover of X (see [11, Theorem 3, p. 195]).

Definition 3. A continuum X is said to have the *strong covering property* (written $X \in SCP$) provided that for each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in [0, \mu(X)]$ the Whitney level $\mu^{-1}(t)$ for $C(X)$ is a minimal closed cover of X.

Thus, by the definitions,

$$(3.1) \qquad X \in SCP \implies X \in CP \quad \text{for each continuum } X.$$

Remark 4. The opposite implication to (3.1) is not true, because if X is an arc, then $X \in CP$ (see [6, Section 67, (b), p. 319, and Theorem 67.1, p. 320]); on the other hand, each minimal closed cover of X is finite according to [11, Theorem 4, p. 196], so it cannot be equal to any Whitney level $\mu^{-1}(t)$ for some Whitney map μ and for some $t \in [0, \mu(X))$ which obviously is infinite. So $X \notin SCP$.

In the present paper we study minimal closed covers and the strong covering property, especially for indecomposable continua and for hereditarily locally connected ones. We answer some questions asked in [11], (and repeated [6, Section 67, p. 325]).

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2. Hereditarily locally connected continua

Recall that a continuum is *hereditarily locally connected* provided that all of its subcontinua are locally connected. Each *dendrite* (i.e., a locally connected continuum containing no simple closed curve) is hereditarily locally connected, see [16, Corollary 10.5, p. 167]. It is shown in [11, Theorem 7, p. 202] that if for a continuum X all of its minimal closed covers are countable, then X is hereditarily locally connected. The opposite implication is not true (see [11, p. 199]) because the Gehman dendrite G (see [4, the example on p. 42] and Figure 5 in [11, p. 199]) has an uncountable minimal closed cover. The cover is homeomorphic to the Cantor set. In connection with this a question asked in [11, p. 204] if it is true that a continuum is hereditarily locally connected if and only if all its minimal closed covers are totally disconnected. Below we give a negative answer to this question for the implication from hereditary local connectedness of the continuum to total disconnectedness of all its minimal closed covers.

Example 5. *There is a dendrite (thus a hereditarily locally connected continuum) X having the set of all its end points countable, and a minimal closed cover of X containing an arc.*

Proof: Let

$$X_1 = ([0,1]\times\{0\}) \cup \bigcup\Big\{\bigcup\big\{\{\tfrac{2k-1}{2^n}\}\times[0,\tfrac{1}{2^n}] : k\in\{1,2,\dots,2^{n-1}\}\big\} : n\in\mathbb{N}\Big\}.$$

The continuum X_1 is described and pictured in [10, §49, VI, Remark and Fig. 6, p. 247]. Let $X_2 = [1,2]\times\{0\}$ and $X = X_1 \cup X_2$. Thus X is a dendrite having the set $E(X)$ of its end points countable. For $p, q \in X$ let pq be the arc from p to q in X. Let $a = (0,0)$, $b = (1,0)$ and $c = (2,0)$. Define two families of arcs in X by

$$\mathcal{A}_1 = \{cx : x \in E(X)\setminus\{c\}\} \quad\text{and}\quad \mathcal{A}_2 = \{cx : x \in ab\},$$

and note that $\mathcal{A}_2$ is an arc in $C(X)$. Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$.

It is evident that $\mathcal{A}$ covers X, and that each element of $\mathcal{A}$ is an arc. Since $ab \subset \mathrm{cl}(E(X))$, it follows that $\mathcal{A}$ is closed in $C(X)$. Therefore, to prove that it is a minimal closed cover of X we need only to show that condition (2.4) is satisfied. In fact, note that for each $x \in E(X)$ the arc cx is the only element of $\mathcal{A}$ that contains x. Thus if $\mathcal{B}$ is a subcover of $\mathcal{A}$, then $\mathcal{A}_1 \subset \mathcal{B}$. Since $\mathcal{B}$ has to be closed, and since to each point $x \in ab$ there exists a sequence of points $x_n \in E(X)$ with $x = \lim x_n$, we have $\mathcal{A}_2 \subset \mathcal{B}$. Thus $\mathcal{B} = \mathcal{A}$, as needed. □

It is shown in [11, Theorem 7, p. 202] that if a continuum has all its minimal closed covers countable, then it is hereditarily locally connected. We would like to know whether the conclusion holds under a weaker assumption.

Question 6. Let a continuum X have all its minimal closed covers totally disconnected (equivalently: 0-dimensional). Is then X hereditarily locally connected?

The following questions (the first of which is a modification of the question considered by S. Macías in [11, p. 199]) are related to Example 5 and Question 6 above and to Theorem 7 of [11, pp. 202 and 204].

Question 7. Characterize hereditarily locally connected continua (dendrites, in particular) all minimal closed covers of which are countable.

Question 8. Characterize continua (dendrites, in particular) all minimal closed covers of which are totally disconnected.

Recall that continua X having all minimal closed covers of X finite are characterized as graphs (see [11, Theorem 5, p. 198]). Observe that the set $E(X)$ of end points of the dendrite X in Example 5 is not closed. Thus one can ask if there is a dendrite X having the set $E(X)$ closed and such that a minimal closed cover of X contains an arc. Below we give an example of such a dendrite.

Example 9. *The Gehman dendrite has a minimal closed cover which is an arc.*

Proof: To define the needed cover we will use an auxiliary hereditarily locally connected continuum X (see the Figure). The continuum was defined in [3, Example 7, p. 216] (compare also the

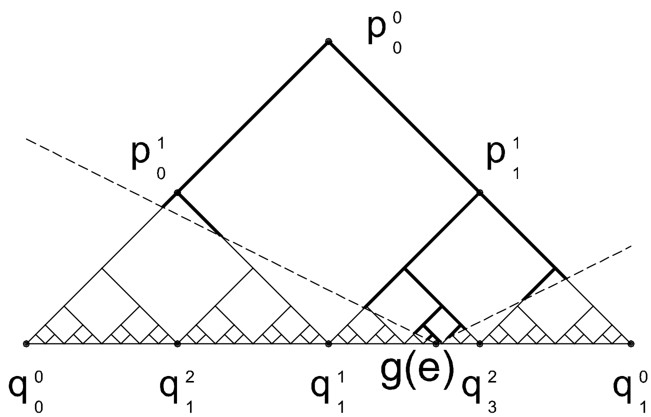

Figure (Example 9)

continuum Z in [14, Example 2.5, p. 782]). Since [3] is not easy to access, we repeat the definition of X for the reader convenience.

In the rectangular coordinates in the plane, for each integer $n \geq 0$ and each $k \in \{0, 1, \ldots, 2^n\}$ put (see the Figure)

$$p_k^n = \left(\tfrac{2k+1}{2^{n+1}}, \tfrac{1}{2^{n+1}}\right) \text{ for } k < 2^n \quad \text{ and } \quad q_k^n = \left(\tfrac{k}{2^n}, 0\right),$$

and let ab stand for the straight line segment joining a and b. Define

$$X = q_0^0 q_1^0 \cup \bigcup \left\{ \bigcup \left\{ p_k^n q_k^n \cup p_k^n q_{k+1}^n : k \in \{0, 1, \ldots, 2^n - 1\} \right\} : n \in \{0\} \cup \mathbb{N} \right\}.$$

Thus X is a (hereditarily locally connected) continuum. Let $\mathcal{C}$ be the standard Cantor middle-third set in the closed unit interval $[0, 1]$, and consider the well known Cantor-Lebesgue step function $\varphi : \mathcal{C} \to [0, 1]$ that identifies the end points of each contiguous interval of the Cantor set (see [9, §16, II, (8) and footnote 1, p. 150] or [17, p. 35]). Note that $[0, 1]$ is isometric with $q_0^0 q_1^0$. Further, we may assume that the set $E(G)$ of the end points of the Gehman dendrite G is isometric with $\mathcal{C}$. Let $\alpha : E(G) \to \mathcal{C}$ and $\beta : [0, 1] \to q_0^0 q_1^0$ be the isometries, and define a mapping $g : G \to X$ by the conditions:

(9.1) $g|E(G) = \beta \circ \varphi \circ \alpha : E(G) \to q_0^0 q_1^0$,

(9.2) $g|\,(G \setminus E(G)) : (G \setminus E(G)) \to X \setminus q_0^0 q_1^0$ is a homeomorphism.

For each $e \in E(G)$ define $x(e) \in [0,1]$ by $g(e) = (x(e), 0)$, and let $L(e)$ be the union of two half lines (the dashed lines in the Figure) emanating from $g(e)$, lying (except $g(e)$) in the upper half plane and determined by the equation

$$y = |\tfrac{1}{2}(x - x(e))|.$$

Let $A(e)$ (the set drawn using thick lines in the Figure) be the set of points of X lying either above or on $L(e)$, i.e.,

$$(x, y) \in A(e) \iff (x, y) \in X \text{ and } y \geq |\tfrac{1}{2}(x - x(e))|.$$

Thus $A(e)$ is a subcontinuum of X. The reader can verify that, for $e \in E(G)$, the sets $g^{-1}(A(e))$ are subcontinua of G and that the family

$$\{g^{-1}(A(e)) : e \in E(G)\}$$

is a minimal closed cover of G. The function $g^{-1}(A(e)) \mapsto x(e)$ establishes a homeomorphism between the family and $[0,1]$. □

So, we have two dendrites X (of Examples 5 and 9) such that for each of them there is a minimal closed cover of X containing an arc. Note that the set of the end points is countable and not closed for the first, while uncountable and closed for the second dendrite. In connection with this the next questions are natural.

Question 10. Does there exist a dendrite X having the set of all its end points countable and closed, such that a minimal closed cover of X contains an arc?

Question 11. Characterize dendrites X with the set $E(X)$ of their end points closed and such that all minimal closed covers of X are totally disconnected.

Recall that properties of dendrites with the closed set of their end points are studied in [2].

3. Indecomposable continua

The following result is known (see [8, Section 6, p. 179] and [15, Theorem 14.14.1, p. 418]).

Theorem 12. *Each hereditarily indecomposable continuum has the covering property.*

A subcontinuum K of a continuum X is said to be *terminal in* X provided that for each continuum L of X the condition $K \cap L \neq \emptyset$

implies $K \subset L$ or $L \subset K$. This concept need not be confused with another one under the same name, e.g. in [15, Definition 1.54, p. 107]. The family of all terminal subcontinua of a given continuum X will be denoted by $\mathrm{Ter}(X)$. The reader is referred to [12, Section 1, p. 535] and [13, Section 1, p. 177] for more information about $\mathrm{Ter}(X)$. In particular, the following equivalence is known (see [7, Remark, p. 85]).

Proposition 13. *A continuum X is hereditarily indecomposable if and only if $\mathrm{Ter}(X) = C(X)$ (i.e., if each subcontinuum of X is terminal in X).*

The following theorem is the basic result of the first part of the paper.

Theorem 14. *If $A \in \mathrm{Ter}(X)$ and $\mathcal{A}$ is a closed cover of X contained in $\mu^{-1}(\mu(A))$, then $A \in \mathcal{A}$.*

Proof: Let $p \in A$. Since $\mathcal{A}$ covers X, there is an element $B \in \mathcal{A}$ with $p \in B$. Thus $p \in A \cap B$. By terminality of A and since $\mu(A) = \mu(B)$ it follows that $A = B \in \mathcal{A}$. This finishes the proof. □

As a consequence of the above theorem we get the main result of this part of the paper.

Corollary 15. *Let a continuum X satisfy the following condition.*

(15.1) *For each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in (0, \mu(X))$ the set $\mathrm{Ter}(X) \cap \mu^{-1}(t)$ is dense in $\mu^{-1}(t)$.*

Then $X \in SCP$.

The next corollary, which has been observed in [11, p. 204], forms a stronger version of Theorem 12 above.

Corollary 16. *Each hereditarily indecomposable continuum has the strong covering property.*

Proof: Really, by Proposition 13, the conclusion follows from Corollary 15. □

Remark 17. Notwithstanding the above corollary is stated in [11, p. 204], the argument given there is incorrect, because it is claimed that any continuum with the covering property has the strong covering property, which is not true by Remark 4.

Example 18. *There is a continuum X containing an arc and which satisfies condition* (15.1)*; hence $X \in SCP$.*

Proof: In a pseudo-arc P replace a point $a \in P$ by an arc A. More precisely, consider a compactification X of the locally compact space $P \setminus \{a\}$ such that an arc A is the remainder of the compactification (see [1, Theorem, p. 35]). Then each subcontinuum of X not intersecting A is a pseudo-arc terminal in X, whence (15.1) is satisfied. Then $X \in SCP$ by Corollary 15. □

Remarks 19. a) A question is asked in [11, p. 204] if the converse to Corollary 16 is true (see also [6, Question 67.8, p. 324]). Example 18 answers this question in the negative.

b) Another question asked in [11, p. 204] is whether or not it is true that if X is a continuum for which all of its minimal closed covers are connected, then X is hereditarily indecomposable (see also [6, Question 67.9, p. 325]). Theorem 20 below gives an affirmative answer to this question: the continuum has to be not only hereditarily indecomposable but even a singleton.

Theorem 20. *Each nondegenerate continuum has a nonconnected minimal closed cover.*

Proof: Let A and B be two closed subsets of a nondegenerate continuum X such that $A \cup B = X$ and that they are minimal with respect to this property, i.e., if $A' \subset A$ and $B' \subset B$ with $A' \cup B' = X$, then $A' = A$ and $B' = B$. Choose points $a \in A \setminus B$ and $b \in B \setminus A$. Let $\mu : C(X) \to [0, \infty)$ be a Whitney map for $C(X)$ and let $t > 0$ be a number such that for each continuum $P \in C(X)$ with $\mu(P) < t$ we have

$$a \in P \implies P \subset A \setminus B \quad \text{and} \quad b \in P \implies P \subset B \setminus A.$$

Define

$$\mathcal{A} = \{P \in C(X) : \mu(P) = t \ \text{ and } \ A \cap P \neq \emptyset\},$$
$$\mathcal{B} = \{P \in C(X) : \mu(P) = \tfrac{t}{2} \ \text{ and } \ B \cap P \neq \emptyset\}.$$

Then $\mathcal{A} \cup \mathcal{B}$ covers X (i.e., $(\cup\mathcal{A}) \cup (\cup\mathcal{B}) = X$), while $\cup\mathcal{A}$ does not contain the point b, and $\cup\mathcal{B}$ does not contain the point a. Moreover, $\mathcal{A} \cap \mathcal{B} = \emptyset$.

According to [11, Theorem 3, p. 195] there exists a minimal closed cover $\mathcal{C}$ of X contained in $\mathcal{A} \cup \mathcal{B}$. Observe that $\mathcal{C} = (\mathcal{A} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{C})$, and that the intersections in the parentheses are nonempty proper subsets of $\mathcal{C}$: the former contains a continuum $P(a) \in \mathcal{A}$ such that $a \in P(a)$, the latter contains a continuum $P(b) \in \mathcal{B}$ such that $b \in P(b)$, and neither $P(a)$ is in $\mathcal{B}$ nor $P(b)$ is in $\mathcal{A}$. This shows that $\mathcal{C}$ is not connected. The proof is complete. □

The rest of this section of the paper is devoted to connections between the strong covering property and various related properties of continua.

Theorem 21. *Consider the following conditions a continuum X may satisfy:*

(21.1) *X is hereditarily indecomposable;*
(21.2) *for each Whitney map $\mu : C(X) \to [0, \infty)$ and for each $t \in (0, \mu(X))$ the set* $\mathrm{Ter}\,(X) \cap \mu^{-1}(t)$ *is dense in $\mu^{-1}(t)$;*
(21.3) $\mathrm{Ter}(X)$ *is a dense subset of $C(X)$;*
(21.4) *$\{X\}$ is an accumulation point of* $\mathrm{Ter}\,(X)$*;*
(21.5) *X is indecomposable;*
(21.6) *$X \in SCP$;*
(21.7) *$X \in CP$;*
(21.8) *X is unicoherent and irreducible.*

Then the following implications hold:

$$\begin{array}{ccccccccc} (21.1) & \Rightarrow & (21.2) & \Rightarrow & (21.3) & \Rightarrow & (21.4) & \Rightarrow & (21.5) \\ & & \Downarrow & & & & & & \\ & & (21.6) & \Rightarrow & (21.7) & \Rightarrow & (21.8) & & \end{array}$$

Proof: The implication (21.1) $\Rightarrow$ (21.2) is a consequence of Proposition 13. The ones (21.2) $\Rightarrow$ (21.3) $\Rightarrow$ (21.4) are obvious.

To show that (21.4) $\Rightarrow$ (21.5) assume (21.4) and suppose that X is decomposable. Let $A, B \in C(X) \setminus \{X\}$ with $X = A \cup B$. If $K \in \mathrm{Ter}(X)$ is such that $H(K, X) < \min\{H(A, X), H(B, X)\}$, where H denotes the Hausdorff metric on $C(X)$, then $K \cap (A \setminus B) \neq \emptyset \neq K \cap (B \setminus A)$. Thus $A \subset K$ and $B \subset K$ by terminality of K, whence $X = A \cup B \subset K$, so $K = X$. Therefore there is no terminal proper subcontinuum of X close to X, which contradicts (21.4).

Implication (21.2) $\Rightarrow$ (21.6) is Corollary 15, and (21.6) $\Rightarrow$ (21.7) according to (3.1). Finally, (21.7) $\Rightarrow$ (21.8) is in [15, the second part of Theorem 14.14.1, p. 418, and Theorem 14.73.1, p. 478]. The proof is then complete. □

Example 22. *There exists a continuum X such that* $\operatorname{Ter}(X)$ *is not a dense subset of $C(X)$, and $\{X\}$ is an accumulation point of* $\operatorname{Ter}(X)$.

Proof: The continuum X is defined as the inverse limit of an increasing sequence of (arc-like) continua with retractions as bonding mappings.

Given an inverse sequence $\{X_n, f_n\}$ of continua X_n with bonding mappings $f_n : X_{n+1} \to X_n$, where the set $\mathbb{N}$ of positive integers is taken as the directed set of indices, we denote by $X_\infty = \varprojlim \{X_n, f_n\}$ its inverse limit (see [16, Chapter II, p. 17-35]).

Let $X_1 = [0, 1]$. If X_n is defined for some $n \in \mathbb{N}$, let X_{n+1} be a compactification of the half-line $[0, \infty)$ having X_n as the remainder (see [1, Theorem, p. 35]), and let $f_n : X_{n+1} \to X_n$ be the natural retraction. Thus each X_n is an arc-like continuum, so the inverse limit $X_\infty = \varprojlim \{X_n, f_n\}$ also is arc-like and all these continua can be seen as embedded in the plane $\mathbb{R}^2$ (see e.g. [12, Propositions 7 and 10, p. 537]). Therefore

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X_\infty \subset \mathbb{R}^2.$$

Furthermore, the embedding can be chosen in such a way that assumptions (1) and (2) of [16, Theorem 2.10, p. 23] are satisfied. Thereby X_∞ is homeomorphic to $\operatorname{cl}(\bigcup\{X_n : n \in \mathbb{N}\})$, whence

$$X_\infty = \operatorname{Lim} X_n. \tag{22.1}$$

Besides, it is evident from the above construction that each X_n is terminal in X_{n+1}; moreover, it is the only nondegenerate proper terminal subcontinuum of X_{n+1}. Put, for shortness, $X = X_\infty$. Consequently, if $F_1(X)$ stands for the hyperspace of singletons, then

$$\operatorname{Ter}(X) = F_1(X) \cup \{X_n : n \in \mathbb{N}\} \cup \{X\}.$$

Thus $\operatorname{Ter}(X)$ is a boundary subset of $C(X)$, while $\{X\}$ is an accumulation point of $\operatorname{Ter}(X)$ by (22.1). □

Remarks 23. We will discuss the problem of what implications of ones considered in Theorem 21 can be reversed.

1) Since (15.1) is another name of (21.2), Example 18 shows that (21.2) does not imply (21.1).

2) Since the only terminal subcontinua in the dyadic solenoid X are the singletons and the whole space, (21.5) does not imply (21.4).

3) Example 22 shows that (21.4) does not imply (21.3).

4) It is stated in Remark 4 that (21.7) does not imply (21.6). The same example (viz. an arc) shows that no one of the conditions (21.1)-(21.6) is implied by (21.7).

5) An example of an irreducible and unicoherent continuum without having CP is shown in [5, Example 5.7, p. 501]. Thus (21.8) does not imply (21.7).

6) The authors do not have any examples showing that the other implications in Theorem 21 cannot be reversed.

The following question seems to be particularly interesting.

Questions 24. Are the conditions (21.2), (21.3) and (21.6) equivalent? If not, what implications besides those established in Theorem 21 are true?

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J. J. Charatonik

Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: jjc@hera.math.uni.wroc.pl

Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D. F., México
E-mail address: jjc@gauss.matem.unam.mx

W. J. Charatonik

Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: wjcharat@hera.math.uni.wroc.pl

Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409-0020
E-mail address: wjcharat@umr.edu