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MOORE SPACE COMPLETION REMAINDERS OF Q

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ABSTRACT. This paper answers a question of Gruenhage, Fitzpatrick and Ott raised in the Proceedings of the American Mathematical Society (volume 117, Number 1, January 1993) by giving necessary and sufficient conditions for a Moore space to be a completion remainder of the rational numbers.

1. INTRODUCTION

Completeness is one of the most important properties of a space for topology and analysis. Completeness implies the Baire property, which is one of the most well-used topological characteristics of a space. Every metric space can be densely embedded in a complete metric space. However, it is not clear which metric spaces are spaces into which some metric spaces can be densely embedded. For a given metric space X it is less clear still which metric spaces are the complements of X in these dense embeddings. For instance, it was unknown for a time whether there was a three point completion of the reals. A completion remainder for a metric space X is a space Y so that for some complete space Z, X is densely embedded in Z, and Y is homeomorphic to the complement of X in Z. If Z is a metric space, then by complete we mean that Z is complete with respect to some metric. If Z is a Moore space we mean that Z is complete with respect to some development, and this will be defined later. It should be noted that there are multiple versions of

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completeness that can be used for Moore spaces which all imply the Baire property, and would all be equivalent to completeness in a metric space. We will use the most standard definition. Whipple [8] gave an alternate characterization of this definition of completeness and completability depending on sequences.

Fitzpatrick, Gruenhage and Ott [4] have shown that the metric space completion remainders of the real numbers are the non-empty Polish spaces. The completion remainders of the rational numbers in the class of metric spaces were shown to be the nowhere locally compact, separable, complete metric spaces.

It was an open question, however, which Moore spaces were completion remainders of Q. Not all Moore spaces can be densely embedded in complete Moore spaces, and in fact not all Moore spaces can be densely embedded in a Moore space with the Baire property, including many Moore spaces with dense metric subspaces, and even Moore spaces with developments satisfying axiom C at a dense set of points. Rudin [1] described the first Moore space which could not be densely embedded in a complete Moore space, Reed [7] showed that a Moore space can be densely embedded in a developable Hausdorff space with the Baire property if and only if it has a development satisfying axiom C at a dense set of points. Fitzpatrick [3] showed that a Moore space has a dense metrizable subset if and only if it has a σ-discrete π-base. Fearnley [2] gave an example of a Moore space with a σ-discrete π-base which cannot be densely embedded in a Moore space with the Baire property.

2. Preliminaries

Let S be a topological space. Let $G = (G_1, G_2, ...)$ be a sequence of open covers for S. For any set T in S, we let $st(T, G_n)$ denote the union of all elements of G_n which intersect T (or if T is a single point then the set of elements of G_n containing T as an element). Then G is a development for S if for every point $p \in S$ and every open set U in S so that $p \in U$, there is a positive integer n so that $st(p, G_n) \subset U$. We say S is developable if there is a development for S, and we say S is a Moore space if S is both developable and regular. Throughout this paper it is assumed that every element of every member of every development is also non-empty. We will define a Moore space S to be complete with respect to a development G if for any descending sequence (A_i) of closed sets in S so that for each integer i, $A_i \subset g_i$ for some $g_i \in G_i$, the intersection $\bigcap_{i \in N} A_i \neq \varnothing$. Note that any development which refines a complete development is also complete, and that if there is a complete development for a space then there is a nested complete development for the space, or in other words a development $G = (G_1, G_2, ...)$ so that for each positive integer i, $G_{i+1} \subset G_i$. A space S has the Baire property if every countable collection of dense open subsets of S has a dense intersection in S. A collection B of non-empty open sets is a π-base for S if for each open set U in S there is some open $V \in B$ such that $V \subset U$. A development G for a space S satisfies axiom C at a point p if for each open set U with $p \in U$, there is some positive integer n and $g_n \in G_n$ such that $p \in g_n$ and $st(g_n, G_n) \subset U$. By the Moore metrization theorem [6], any Hausdorff space which has a development satisfying axiom C at every point is metrizable.

Medvedev [5] showed that for any cardinal κ, all σ-discrete metric spaces having the property that every non-empty open set has cardinality κ are homeomorphic. This space is called $Q(\kappa)$. In particular, every countable metric space which has no isolated points is homeomorphic to Q.

Throughout this paper, we will refer to a sequence of sets (T_i), as being closure nested if for each positive integer i, $\overline{T_{i+1}} \subset T_i$. We refer to a sequence (T_i) of sets as being eventually closure nested if for some positive integer n, if $i > n$ then $\overline{T_{i+1}} \subset T_i$. In a space S with development G then we will refer to T_i as being star nested if for each n, $st(T_{n+1}, G_{n+1}) \subset T_n$, and define eventually star nested similarly. We also say that the members of a descending sequence of sets T_i are eventually contained in the set U if for some positive integer j, $T_j \subset U$. We define a collection C of closure nested sequences of open sets to be *eventually thin* if for each point q, and for each open O so that $q \in O$, there is an open set V so that $q \in V \subset O$ and for each sequence $(U_n) \in C$, there is some n so that either U_n is contained in V or disjoint from V. In this definition it should be emphasized that for a given $q \in O$ the open set V described is the same open set for every sequence $(U_n) \in C$, though the value of n so that either U_n is contained in V or disjoint from V may be different for different sequences. In a metric space the following is equivalent to nowhere local compactness and having a metric which is nowhere locally complete. In a Moore space this condition implies nowhere local compactness and is similar to the existence of a nowhere locally complete development. We will refer to this condition as being densely incomplete. We define S to be *densely incomplete* if there is a function F assigning to every open set U a closure nested sequence of non-empty open sets $F(U) = (U_n)$ so that $U_1 \subset U$ and $\bigcap_{n \in N} U_n = \varnothing$, having the additional property that the range of F is eventually thin.

3. Characterization of the completion remainders of Q

Theorem 3.1. *A Moore space S is a completion remainder of Q if and only if S is complete, separable, and densely incomplete.*

This theorem will be proven in stages. We will begin with necessity and finish with sufficiency.

Theorem 3.2. *If a Moore space S is a completion remainder of Q then S is complete, separable and densely incomplete.*

Proof: By the definition of completion remainder, we may pick a complete Moore space Z having the property that Q can be densely embedded in Z, and S is homeomorphic to the complement of Q in Z. From now on we will refer to S and Q as the complementary subsets of Z which they are homeomorphic to. Let G be a nested development for Z with respect to which Z is complete.

We begin this proof by noting that each point of Z has a basis of open sets whose boundaries do not intersect Q. This is true because Q is zero dimensional, or more directly because $\{p\} \cup Q$ is a countable first countable regular space, and therefore metrizable. Thus, since there are uncountably many different epsilon radius neighborhoods about p and only countably many points of Q, there is an epsilon radius neighborhood $N_\epsilon(p)$ about p whose boundary does not intersect Q. If V is an open set in Z whose intersection with $p \cup Q$ is $N_\epsilon(p)$ then since Q is dense in Z, the boundary of V does not intersect Q either.

Moore has shown [6] that every G_δ-set in a complete Moore space is complete. Since S is a dense G_δ subset of Z, we know that S is complete.

We know that S is separable because S is a dense subset of a first countable separable space. We know S is dense in Z because a non-empty open set in Z containing no points of S would be an open subset of Q. Since no non-empty open subset of Q is complete and every open subset of a complete Moore space is complete, it follows that S must be dense in Z. It remains only to show that S is densely incomplete. We may choose, for each $q \in Q$, a closure nested sequence of open sets $(O_q(n))$ containing q, so that for any open set V in Z such that $q \in V$, there is some n such that $O_q(n) \subset V$.

For each open set U in S, we pick an open set U^* in Z such that $U^* \cap S = U$. Well order the points of Q. Then, for each open set U in S, define $F(U)$ as follows. Let $q(U)$ be the first element of Q which is contained in U^*. Then let n_U be the first positive integer such that $O_{q(U)}(n_U) \subset U^*$. Then, we define $F(U) = (O_{q(U)}(n+n_U) \cap S)$ where n varies over the natural numbers.

It remains to show that F satisfies all the conditions for S to be densely incomplete. First, note that since $\bigcap_{n \in N} O_{q(U)}(n+n_U) = q$ in Z, it follows that $\bigcap_{n \in N}(O_{q(U)}(n+n_U) \cap S) = \varnothing$. Also, each member of each element of the range of F is non-empty since S is dense in Z.

We must still show that the range of F is eventually thin. Let $p \in U$, where U is open in S. Then, as was discussed in the proof of completeness, we may choose an open set V in Z so that $p \in V \subset \overline{V} \subset U^*$ in Z and the boundary of V does not intersect Q. Let $W = V \cap S$. Then $W \subset U$ and for any $q \in Q$ we know that either $q \in V$ or $q \in (Z \setminus \overline{V})$. If $q \in V$ then for some positive integer i, $O_q(i) \subset V$, and so $(O_q(i) \cap S) \subset W$. If $q \in (Z \setminus \overline{V})$ then for some positive integer i, $O_q(i) \subset (Z \setminus \overline{V})$, and hence $(O_q(i) \cap S) \cap W = \varnothing$. Hence, the members of $(O_q(n) \cap S)$ are either eventually contained in W or eventually contained in $S \setminus \overline{W}$. It follows that the range of F is eventually thin, and so S is densely incomplete. □

We will now prove sufficiency. This sufficiency condition is closely analogous to a sufficient condition for a Moore space to be a completion remainder of $Q(\kappa)$ which we will describe later.

Theorem 3.3. *If S is a Moore space which is complete, separable and densely incomplete, then S is a completion remainder of Q.*

The proof will be in three steps. The first step is the construction of Z, the completion of S, the second step is in showing that Z is a Moore space, and the third step is to construct a development with respect to which Z is complete.

Proof: Step 1: Construction of Z.

We will construct a complete Moore space Z having both S and Q as dense subsets by defining certain closure nested sequences in S to be additional points. This proof will require the construction of several different developments and modifications of closure nested sequences.

We begin with an inductive construction of a sequence of closure nested sequences of open sets in S which have empty intersections. Let G be a complete nested development for S. Since S is separable and first countable, S has a countable π-base. Hence, we may choose open sets $B_1, B_2, ...$ which are a π -base for S. We let F be the function prescribed by the definition of densely incomplete. We define $U_1(1)$ to be an element of G_1 whose closure is contained in B_1. Then for all positive integers $n > 1$ we define the $n+1$st member of $F(U_1(1))$ to be $U_1(n)$. Note that the sequence $(U_1(n))$ is closure nested and has empty intersection.

We proceed by induction. Suppose that we have already defined $(U_m(n))$ if $m \leq j$, where n ranges over the natural numbers indexing the members of the mth sequence, and that for each $i \leq j$ the sequence $(U_i(n))$ is closure nested, has empty intersection, and $U_i(1) \subset B_i$. Also, suppose that for each $i, k \leq j$, and each $m, n \in N$, one of $U_i(m) \subset U_k(n)$, or $U_i(m) \cap U_k(n) = \varnothing$, or $U_k(n) \subset U_i(m)$ is true. Finally, suppose that whenever $i \leq k \leq j$, the set $U_k(i) \in G_i$.

We will now define $(U_{j+1}(n))$, where n ranges over the natural numbers. Let k_1 be the first integer so that $\overline{U_1(k_1)} \not\supseteq B_{j+1}$. We define $O_1 = B_{j+1} \setminus \overline{U_1(k_1)}$. In general, for $i > 1$ we let k_i be the first integer such that $\overline{U_i(k_i)} \not\supseteq O_{i-1}$ and let $O_i = O_{i-1} \setminus \overline{U_i(k_i)}$. We choose $U_{j+1}(1) \in G_1$ so that $\overline{U_{j+1}(1)} \subset (O_j \cap (\bigcap\{\ U_i(k) | i \leq j$ and $k < k_i\}\ ))$. For all positive integers n such that $1 < n \leq j+1$ we choose $U_{j+1}(n) \in G_n$ so that $\overline{U_n} \subset U_{n-1}$. Then, for all $n > j+1$ we let $U_{j+1}(n)$ be the nth member of the sequence $F(U_{j+1}(j+1))$.

In this way we have defined the sequences $(U_i(n))$ for all $i \leq j$. Each such sequence is closure nested and has empty intersection, and for all positive integers i it follows that $U_i(1) \subset B_i$. Note that if $n < j+1$ then by construction, if $U_n(m)$ intersects $U_{j+1}(1)$ then $U_n(m)$ contains $U_{j+1}(1)$ and thus $U_{j+1}(k)$ for all $k > 1$. Hence, it follows that for each $i, k \leq j+1$, and each $m, n \in N$, one of

$U_i(m) \subset U_k(n)$, or $U_i(m) \cap U_k(n) = \varnothing$, or $U_k(n) \subset U_i(m)$ is true. Also, whenever $i \le k \le j+1$, the set $U_k(i) \in G_i$.

There are four properties about the closure nested sequences $(U_m(n))$ which will be important to our construction. These properties will be listed for reference as follows.

(1) For any $i \ne j$ there is some positive integer n so that $\overline{U_i(n)} \cap \overline{U_j(n)} = \varnothing$.

(2) For any $U_m(n)$ and any positive integer i, there is some positive integer j such that either $U_i(j) \subset U_m(n)$ or $U_i(j) \cap U_m(n) = \varnothing$. In other words, the members of $(U_i(k))$ are either eventually contained in $U_m(n)$ or the members of $(U_i(k))$ are eventually contained in $S \setminus U_m(n)$.

(3) Every open set U in S contains the first member of one of these closure nested sequences.

(4) If $i \le j$ then $U_j(i) \in G_i$. So, the upper right triangle of the array mentioned before consists of development open sets of a corresponding index.

We can now define the space Z. We let Z consist of S and for each positive integer m, we let the sequence $(U_m(n))$ be a point of Z. We refer to each such sequence as a sequence point of Z. The topology on Z is constructed by letting, for every open set U in S, the set $U^* = U \cup \{\ (U_m(n)) \in Z \mid m \in N$ and the members of $(U_m(n))$ are eventually contained in $U\}$ be open in Z. Note that this defines a topology for Z, and that S is a subspace of Z.

Step 2: The space Z is a Moore space in which the complement of the subspace S is homeomorphic to Q.

We make our first modification to the development G for S, and we also modify the sequence points that we just constructed. The purpose for this modification is to make the closure nested sequences become eventually star nested sequences with respect to a new development. This will be used to construct a development for Z.

We will inductively define new sequences $V_n(m)$ and a new development $K = (K_1, K_2, ...)$. We will define both simultaneously since the constructions are dependent on one another. We begin by letting $V_1(1) = U_1(2)$. We will use integers $r_m(n)$ to index subsequences of our original closure nested sequences. We define $r_1(1) = 1$. For each $m > 1$ choose an integer $r_m(1)$ so that $U_m(r_m(1))$ is either contained in $V_1(1)$ or disjoint from $V_1(1)$. Then for all $m > 1$ we define $V_m(1) = U_m(r_m(1) + 1)$. Then, we define $K_1 = \{ g_1 \cap U_1(1) \mid g_1 \in G_1\} \cup \{ g_1 \cap (S \setminus \overline{V_1(1)}) \mid g_1 \in G_1\} \cup \{ V_m(1) \mid m \in N\}$. We have defined the first member of the development and the first member of each $V_m(n)$ sequence.

Suppose we have defined K_i and $V_m(i)$ for all $i \leq j$. Suppose further that we have defined $r_m(i)$ for all $i \leq j$, so that for each positive integer m, $r_m(1) < r_m(2) < ... < r_m(j)$ and $V_m(i) = U_m(r_m(i)+1)$. Furthermore, suppose that for all $i < j$, if $j > t \geq i$ then $st(V_i(t+1), K_{t+1}) \subset V_t$. Also, suppose that for all $i, t \leq j$ and all positive integers m, $V_m(i)$ is either contained in $V_t(i)$ or is disjoint from $\overline{V_t(i)}$. Finally, suppose that for all $i \leq j$, the sets $\overline{V_1(i)}, \overline{V_2(i)}, ..., \overline{V_i(i)}$ are pairwise disjoint.

To illustrate these conditions graphically, list the $V_m(n)$ in an array as before where the nth row in the array is listing the $V_m(n)$ open sets with n fixed, and the mth column in the array is listing the $V_m(n)$ open sets with m fixed. Then all elements below the diagonal of the array would have their stars in the corresponding development stage contained in the open set immediately above them in the array list. Also, each open set to the right of the diagonal on each row is either contained in one of the open sets to the left of or on the diagonal, or it is disjoint from all open sets on that row on or to the left of the diagonal. Finally, all members of the array on or to the left of the diagonal on each row would have disjoint closures.

We now give the inductive step. For each integer $i, 0 < i \leq j+1$, choose $r_i(j+1) > r_i(j)$ so that $\overline{U_1(r_1(j+1))}, ..., \overline{U_{j+1}(r_{j+1}(j+1))}$ are pairwise disjoint. Then, for each positive integer $m > j+1$, if for some $i \leq j+1$, the members of $(U_m(n))$ are eventually contained in $V_i(j+1)$ then choose $r_m(j+1) > r_m(j)$ so that $U_m(r_m(j+1)) \subset V_i(j+1)$. Otherwise choose $r_m(j+1) > r_m(j)$ so that $U_m(r_m(j+1))$ is disjoint from $V_i(j+1)$, for all $i \leq j$. Then for each positive integer m, we define $V_m(j+1) = U_m(r_m(j+1)+1)$.

Finally, we define $K_{j+1} = (\bigcup_{i=1}^{j+1}\{ g_{j+1} \cap U_i(r_i(j+1)) \mid g_{j+1} \in G_{j+1}\}) \cup (\{ g_{j+1} \setminus (\bigcup_{i=1}^{j+1} \overline{V_i(j+1)}) \mid g_{j+1} \in G_{j+1}\}) \cup \{ V_m(j+1) \mid m \in N\}$. We note that each element of the induction holds for the $j+1$st row and development stage. If any $k_{j+1} \in K_{j+1}$ intersects some $V_i(j+1)$ then either $k_{j+1} = V_m(j+1)$ for some m, in which case $k_{j+1} \subset V_i(j+1)$, or k_{j+1} is the intersection of a member of

G_{j+1} with $U_i(r_i(j+1))$. Either way, $k_{j+1} \subset U_i(r_i(j+1)) \subset V_i(j)$, so $st(V_i(j+1), K_{j+1}) \subset V_i(j)$ as desired. The other properties of the induction are immediate from the construction.

We must show that K is a development for S. Let $p \in U$, an open set in S. Then, since G is a nested development, for some n, if $m \geq n$ then $st(p, G_m) \subset U$. For some $m \geq n$, p is not an element of $\bigcup_{i=1}^{n} V_i(m)$. For all $i \geq n$, $V_i(m)$ is a subset of an element of G_n, and so if $p \in V_i(m)$ then $V_i(n) \subset U$. All elements of K_m are either contained elements of G_m or are of the form $V_i(m)$ for some i. Hence, all elements of K_m which contain p are contained in U. Therefore, $st(p, K_m) \subset U$ and K is a development.

Each sequence $(V_m(n))$ is a subsequence of $(U_m(n))$. Hence properties 1-3 for the $(U_m(n))$ sequences also hold for the $V_m(n)$ sequences. Also, the following are true.

(5) For each positive integer m, $V_m(n)$ is eventually star nested with respect to the development K. Specifically, for every positive integer i, if $t \geq i$ then $st(V_i(t+1), K_{t+1}) \subset V_i(t)$.

(6) If $i \leq j$ then for some $g_i \in G_i$, $V_j(i) \subset g_i$. So, the upper right triangle of the array mentioned before is contained in development open sets of a corresponding index. We further note the that then for every positive integer n, if $j \geq i$ then $V_j(n)$ is a subset of an element of G_i.

For each positive integer m, we also refer to the sequence $(V_m(n))$ as a sequence point since it is a subsequence of $(U_m(n))$. Note that if we define $U^* = U \cup \{ (V_m(n)) \in Z \mid m \in N$ and the members of $(V_m(n))$ are eventually contained in $U\}$ then the sequence points $(V_m(n))$ contained in the new open set U^* are the subsequences of exactly the sequence points $(U_m(n))$ contained in U^* in the earlier definition of the topology on Z. Hence, the two spaces are homeomorphic and the definitions may be used interchangeably. Note also that every open set in Z is equal to U^* for some open set U in S.

We define $K_n^* = \{ k_n^* \mid k_n \in K_n\}$. Then we claim that $K^* = (K_1^*, K_2^*, ...)$ is a development for Z. Let $p \in U^*$, an open set in Z. Then either $p \in S$ or $p = (V_m(n))$ for some positive integer m. If $p \in S$ then there is some positive integer n so that $st(p, K_n) \subset U$. Hence, by definition $st(p, K_n^*) \subset U^*$. If $p = (V_m(n))$ then there is some positive integer $j \geq m$ such that $V_m(j) \subset U$. Then by property (5) we know that $st(V_m(j+1), K_{j+1}) \subset V_m(j) \subset U$, so since any open in Z containing p must intersect $V_m(j+1)$, we conclude that $st(p, K^*_{j+1}) \subset U^*$. Hence, K^* is a development for Z.

Next, we wish to show that Z is regular. As before, we let $p \in U^*$, an open set in Z. Then either $p \in S$ or $p = (V_m(n))$ for some positive integer m. If $p \in S$ then since the set of all the sequence points of Z is eventually thin, we may choose an open set V in S so that $p \in V \subset \overline{V} \subset U$ and for each positive integer i the members of the sequence $(V_i(n))$ are either eventually contained in V or disjoint from V. Hence, there are no sequence points on the boundary of V^*. Thus, $\overline{V^*} \subset U^*$. If $p = (V_m(n))$ then for some positive integer $j > m$, we know that $V_m(j) \subset U$. So, then $st(V_m(j+1), K_{j+1}) \subset U$, and thus $st(V_m(j+1)^*, K^*_{j+1}) \subset U^*$, and hence $p \in V_m(j+1)^* \subset \overline{V_m(j+1)^*} \subset U^*$. Hence, Z is regular, and so Z is a Moore space.

Finally, we wish to show that Q can be densely embedded in Z. We claim that the subspace $Z \setminus S$ of sequence points of Z is homeomorphic to Q. By a theorem of Medvedev [5], this is true if and only if $Z \setminus S$ is metrizable, countable, and has no isolated points. We know $Z \setminus S$ is countable by definition, and hence also metrizable because it is a countable subspace of a Moore space. Let $(V_m(n)) \in (Z \setminus S)$. Let $U^* \cap (Z \setminus S)$ be an open set in $Z \setminus S$ so that $(V_m(n)) \in (U^* \cap (Z \setminus S))$. Then for some positive integer j, it follows that $V_m(j) \subset U$, and by construction $V_m(j) \neq \varnothing$. Hence, since $V_1(1), V_2(1), ...$ is a π -base for S by construction, there is some positive integer i so that $V_i(1)$ is contained in $V_m(j)$. Thus, $(V_i(n)) \in (U^* \cap (Z \setminus S))$, and so $Z \setminus S$ has no isolated points, and is homeomorphic to Q. From now on, we will refer to $Z \setminus S$ as Q.

Step 3. The space Z is complete.

As in the proof of necessity, we replace the development K^* with a development the elements of whose members have boundaries which do not intersect Q. For each positive integer n, let W_n be the set of all open subsets w_n of Z having the property that the boundary of w_n does not intersect Q, and also $w_n \subset k^*_n$ for some $k^*_n \in K^*_n$. We claim the Z is complete with respect to the development $W = (W_1, W_2, ...)$. Note that for the same reasons given in the proof of necessity, W is a development for Z. Let (T_i) be a decreasing sequence of closed sets in Z such that for every positive integer i there is some $w_i \in W_i$ so that $T_i \subset w_i$. Note that by construction, w_i is either contained in $V_m(i)^*$ for some positive integer m, or w_i is contained in some $k_i^* \in K_i^*$, where $k_i^* \subset g_i^*$ for some $g_i \in G_i$, since all elements of K_i not contained in $\{ V_m(i) \mid m \in N\}$ are subsets of elements of G_i.

Either, for infinitely many integers i it is the case that $w_i \subset g_i^*$ for some $g_i \in G_i$, or not. If so, then the sequence $(T_i \cap S)$ is a decreasing sequence of closed sets in S such that for each i there is some $g_i \in G_i$ so that $(T_i \cap S) \subset g_i$. This is true since G_i is a nested development. Hence, since S is complete with respect to G,we know that $\bigcap_{i\in N}(T_i \cap S) \neq \varnothing$, so $\bigcap_{i\in N} T_i \neq \varnothing$.

If it is not the case that for infinitely many integers i, $w_i \subset g_i^*$ for some $g_i \in G_i$, then for infinitely many integers i we know that $w_i \subset V_m(i)^*$ for some positive integer m. In that case either there is some positive integer j so that $w_i \subset V_j(i)^*$ for infinitely many integers i or not. If so then $(V_j(n)) \in \bigcap_{i\in N} T_i$. If not then there are infinitely many integers m so that $w_i \subset V_m(i)^*$. Then we can pick a positive integer i_1 so that $w_{i_1} \subset g_{i_1}^*$ for some $g_1 \in G_1$. This is true because every $V_m(i_1)$ is a subset of some element of G_1 by property (6). Let m_1 be the positive integer so that $w_{i_1} \subset V_{m_1}(i_1)$. Since there are infinitely many integers m so that $w_i \subset V_m(i)^*$, there is some $i_2 > i_1$ and some $m_2 \geq 2$ such that $w_{i_2} \subset V_{m_2}(i_2)^*$. Then, since $m_2 \geq 2$, we know that for some $g_2 \in G_2$, $V_{m_2}(i_2) \subset g_2$ by property (6).

Inductively, suppose we have chosen i_n and m_n for all $m \leq j$, so that $i_1 < i_2 < ... < i_{j-1} < i_j$, and $w_n \subset V_{m_n}(i_n)^*$, and $V_{m_n}(i_n) \subset g_n$ for some $g_n \in G_n$. Then we can pick $i_{j+1} > i_j$ and some $m_{j+1} \geq j+1$ so that $w_{i_{j+1}} \subset V_{m_{j+1}}(i_{j+1})^*$. Then, since $m_{j+1} \geq j+1$, by (6) we know that $V_{mj+1}(i_{j+1}) \subset g_{j+1}$ for some $g_{j+1} \in G_{j+1}$. Hence, for each positive integer n, we know that $T_{i_n} \cap S \subset g_n$ for some $g_n \in G_n$. Hence, since G is complete, $\bigcap_{n\in N}(T_{i_n} \cap S) \neq \varnothing$. Thus, since (T_i) is a descending sequence of sets, $\bigcap_{n\in N} T_n = \bigcap_{n\in N} T_{i_n} \supset \bigcap_{n\in N}(T_{i_n} \cap S)$, and so $\bigcap_{n\in N} T_n \neq \varnothing$. Hence, W is a complete development for Z as desired. This completes the proof. □

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