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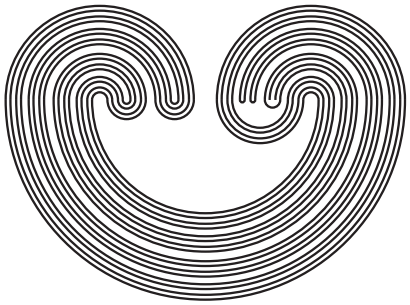

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COMBINATORIAL AND TOPOLOGICAL ASPECTS OF MEASURE-PRESERVING FUNCTIONS

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Abstract. We study measure-preserving functions between Lebesgue measurable subsets of the real line. We use particular bijections of the interval $[0, 1)$, called shifts, to approximate from below the set of measure-preserving maps on $[0, 1)$. This construction is similar to the method used in ergodic theory to obtain special transformations by cutting and stacking. In our approach we provide the set of shifts with an algebraic symmetric structure, which allows us to investigate the topic from both a combinatorial and a topological point of view. It is interesting the interplay between these two aspects of the problem.

1. Introduction

Measure-preserving functions have been widely studied in the literature, both from a theoretical and an applied point of view (see, e.g., [5, 7, 8, 11, 12, 14] and references therein). The traditional approach focuses primarily on the measure algebra homomorphisms that they induce.

In this paper we analyze the subject from a quite different perspective, aiming to show how a more constructive and explicit study of measure-preserving functions can shed light on the whole area. For concreteness, we concentrate our attention on maps between measurable subsets of the real line $\mathbb{R}$ having the same finite positive

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Lebesgue measure. Our investigation is centered on the topological and metric structure (as opposed to the purely measure theoretic structure) of the underlying measure spaces. Specifically, our goal is to try to "understand from below" the general features of measure-preserving maps. This procedure of approximation is performed using some rather elementary functions, called "shifts". Then, every measure-preserving function can be expressed as the limit (in the topology of convergence in measure) of a finite composition of shifts.

Shifts are similar to the so-called "interval exchange transformations", well known in the literature in ergodic theory (see, for instance, [2, 5]). Nonetheless, our approach departs from the traditional one in three respects. First, the environment in which our analysis is carried out is the underlying measure space, and not the associated measure algebra. Second, the topologies we put on the spaces of functions are different from those already studied in the literature (see, e.g., [5, 7, 12]). Finally, we endow the set of shifts with a natural algebraic symmetric structure, which turns out to be quite useful in order to obtain a new representation of measure-preserving maps.

Our first goal is to obtain, using a similar construction, similar results. Indeed, within the set of all shifts, it is possible to identify a countable subfamily $\mathcal{R}$, formed by the so called "rational" shifts. Then, the group $\mathcal{S}$ (under composition) generated by rational shifts witnesses the separability of the space of measure-preserving functions endowed with the topology of convergence in measure. Therefore, $\mathcal{S}$ plays in this context the same role that the set of interval exchange transformations plays in the space of automorphisms of the associated measure algebra endowed with the so-called "neighborhood topology" (cf. [7]).

On the other hand, the use of the group $\mathcal{S}$ as a tool to describe from below measure-preserving functions presents some rather unique advantages. In fact, there exists a natural correspondence between rational shifts and finite permutations of $\mathbb{N}$. This correspondence between analytical and combinatorial objects is useful in two ways.

First of all, it allows us to express every measure-preserving function as a countable sequence of finite permutations of natural numbers. As consequence, we could get some insight into the space

of measure-preserving maps by using combinatorial properties of permutations of natural numbers.

At the same time, analytical properties of shifts might provide us with geometrical methods to determine the order of specific elements of symmetric groups. Furthermore, the above correspondence can also be used to obtain some combinatorial metrics on (equivalence classes of) finite permutations of $\mathbb{N}$ by simply translating the usual distances in the corresponding spaces of functions.

The paper is organized as follows. In Section 2 we introduce the basic objects of our investigation, namely, the measure-preserving functions. The shifts on the interval $[0, 1)$ are defined in Section 3. Section 4 is devoted to the study of the group $\mathcal{S}$ generated by rational shifts. Here we show that $\mathcal{S}$ is the directed colimit of a directed system of the symmetric groups. In Section 5 we analyze the shifts from a group-theoretical point of view, determining the order of any shift. Section 6 is devoted to an investigation of suitable metric structures that the set of measure-preserving functions on the interval $[0, 1)$ can be endowed with. In particular, we prove that this set, endowed with the distance induced by convergence in measure, is a complete separable metric space. In Section 7 we deal with the subspace of measure-preserving bijections, and we show that its algebraic and topological structures are compatible. Specifically, we prove that this space is a Polish group. Conclusions group final remarks, open problems and future directions of research.

2. Preliminaries

We begin by recalling the definition of measure-preserving maps (with respect to the Lebesgue measure μ).

Definition 2.1. Let $X, Y \subset \mathbb{R}$ be two measurable sets of finite positive measure. A *measure-preserving function* (or, more simply, a *homomorphism*) of X into Y is a map $\gamma : X \to Y$ such that $\gamma^{-1}(M) \subset X$ is measurable and $\mu(\gamma^{-1}(M)) = \mu(M)$, whenever $M \subset Y$ is measurable. The map γ is said to be an *isomorphism* if, in addition, γ is a bijection. A homomorphism of X into itself is called an *endomorphism*; an isomorphism of X onto itself is called an *automorphism*.

Notice that the definition of homomorphism can be weakened by requiring only that $\gamma^{-1}(I) \subset X$ is measurable and $\mu(\gamma^{-1}(I)) =$

$\mu(I \cap Y)$, for every open interval $I \subset \mathbb{R}$. Furthermore, it is well known that isomorphism between sets of positive measure can be characterized in a quite useful manner (see, e.g., [13, 14]):

Theorem 2.2. *Let $X, Y \subset \mathbb{R}$ be two measurable sets of finite positive measure, and let $\gamma : X \to Y$ be an injective map. Then the following conditions are equivalent:*

(i) *for every open interval $I \subset \mathbb{R}$, the set $\gamma^{-1}(I)$ is measurable, and $\mu(\gamma^{-1}(I)) = \mu(I \cap Y)$;*
(ii) *γ is surjective, and for every open interval $I \subset \mathbb{R}$, the set $\gamma(I \cap X)$ is measurable, with $\mu(\gamma(I \cap X)) = \mu(I \cap X)$;*
(iii) *γ is an isomorphism from X onto Y;*
(iv) *γ is surjective, and γ^{-1} is an isomorphism from Y onto X.*

Theorem 2.2 says, in particular, that injective homomorphisms are isomorphisms, and that isomorphisms carry measurable sets into measurable sets.

As usual, we identify functions which are equal almost everywhere. Then, every equivalence class of isomorphisms is uniquely determined by a bijection which carries nullsets into nullsets. Moreover, two isomorphisms which differ on a set of positive measure give rise to different equivalence classes. It is understood that Theorem 2.2 still holds, *mutatis mutandis*, if we replace the sentence "γ is injective" by the sentence "$[\gamma]$ contains an injective element", where $[\cdot]$ stands for equivalence class in the relation of equality almost everywhere.

In the following, we denote the space of equivalence classes of measure-preserving functions from X to Y by $\mathrm{Hom}(X, Y)$; in addition, $\mathrm{Iso}(X, Y)$ is the set of $[\gamma] \in \mathrm{Hom}(X, Y)$ such that γ is an isomorphism of X onto Y. Further, we set $\mathrm{End}(X) = \mathrm{Hom}(X, X)$, and $\mathrm{Aut}(X) = \mathrm{Iso}(X, X)$.

The space $\mathrm{Aut}(X)$ is a group with respect to the ordinary composition of functions (denoted by $\circ$). The unit in $(\mathrm{Aut}(X), \circ)$ is $[e]$, where e is the identity map $e(x) = x$. If γ is an automorphism of X, the inverse of $[\gamma] \in \mathrm{Aut}(X)$ is $[\gamma^{-1}]$. The space $(\mathrm{End}(X), \circ)$ is a monoid with unit $[e]$. Note also that the inclusion $\mathrm{Iso}(X, Y) \subset \mathrm{Hom}(X, Y)$ is in general proper, as the following example shows.

Example 2.3. Let $X = Y = [0, 1]$, and consider the function $\beta(x) = 1 - |1 - 2x|$. Since for every interval $I \subset [0, 1]$ we have

$\mu(I) = \mu(\beta^{-1}(I))$, we get that $[\beta] \in \mathrm{End}([0,1])$. Assume now that $[\beta] \in \mathrm{Aut}([0,1])$. Then there exists nullsets N_1, $N_2 \subset [0,1]$ such that $\beta : [0,1] \setminus N_1 \to [0,1] \setminus N_2$ is a bijection. Set $M = [0,1/2] \setminus N_1$. Since M is measurable, it must be $\mu(M) = \mu(\beta(M))$. But $\beta_{|M}(x) = 2x$, which entails $2\mu(M) = \mu(\beta(M))$, a contradiction.

Our aim is to investigate the general features of homomorphisms and isomorphisms on measurable subsets of $\mathbb{R}$ having finite positive measure. The following well known result shows that there is no loss of generality if we restrict our attention to endomorphisms and automorphisms of the unit interval.

Theorem 2.4. *Let X, $Y \subset \mathbb{R}$ be two measurable sets having the same finite positive measure. Then there exists an isomorphism γ between X and Y.*

In the sequel we will therefore focus our attention on the following objects:

$$\mathcal{E} = \mathrm{End}([0,1)) \qquad \text{and} \qquad \mathcal{A} = \mathrm{Aut}([0,1)) \,.$$

We will also follow the usual convention of denoting the equivalence class $[\gamma] \in \mathcal{E}$ by γ, choosing a bijective representative γ whenever $[\gamma] \in \mathcal{A}$.

3. The shifts

In this section we single out some particular elements of $\mathcal{A}$, called "shifts". They will be the building blocks in the construction of an algebraic object which "approximates from below" the spaces $\mathcal{A}$ and $\mathcal{E}$.

A very similar construction is well known in ergodic theory, under the name of cutting and stacking method (see [5]). This method, originally due to von Neumann and Kakutani, is very useful in order to construct transformations of various types (ergodic, mixing, measure-preserving, etc.). Furthermore, this representation has been used to prove approximation results of the type we are interested in. Namely, the automorphisms of the associated measure algebra can be endowed with suitable topologies (see [3, 7]), and some interesting separability results can be proved in these topological spaces. More recently, it has been shown (see [2]) that an abstract cutting and stacking construction can be realized as

an "interval exchange transformation". Therefore, we may think of interval exchange transformations as the building blocks used to approximate from below every automorphism of the corresponding measure algebra.

As it will soon become apparent, our definition of shift is very similar to that of interval exchange transformation. Nevertheless, in spite of the evident similarities, our approach is justified by several reasons. In fact, we will establish a correspondence between shifts and particular permutations of the natural numbers. This will in turn provide us with a geometric way to compute their group-theoretic order. Moreover, shifts will be used to generate an algebraic object, the group $\mathcal{S}$, which will allow us to obtain some approximation results having the same flavor as those already mentioned. In this investigation, we will endow the space $\mathcal{E}$ with topologies which, in our setting, seem to be more appropriate than the one used in the literature for the associated measure algebra. The advantage of our construction is that the group $\mathcal{S}$ is a purely combinatorial object, being the directed colimit of a system of the symmetric groups S_n, and therefore the approximation results that we get for spaces of functions have a combinatorial representation as well. Note that, by the lifting theorem on the representability of automorphisms of measure algebras (see, for instance, [9], p.118), all the results obtained on the underlying measure space still hold in the associated measure algebra.

Definition 3.1. A *shift* is a map of the form $\gamma_{a,b}^h : [0,1) \to [0,1)$ defined by

$$\gamma_{a,b}^h(x) = \begin{cases} x + h & \text{if} \quad x \in [a,b) \\ x - \max\{b-a, h\} & \text{if} \quad x \in [a+h, b+h) \setminus [a,b) \\ x & \text{otherwise}\,, \end{cases}$$

where the parameters $a, b, h \in [0,1]$ satisfy $0 \le a + h < b + h \le 1$. We call $[a,b)$ and $[a+h, b+h)$ the *basic interval* and the *shifted interval* of $\gamma_{a,b}^h$, respectively. A shift is said to be *total* if the two intervals do not overlap (i.e., $h \ge b - a$), and *partial* otherwise. A shift $\gamma_{a,b}^h$ is *rational* if the parameters a, b, h are all rational numbers. The set of all [total, partial] rational shifts will be denoted by $\mathcal{R}$ $[\mathcal{R}_{tot}, \mathcal{R}_{par}]$.

Remark 3.2. It is an immediate consequence of the definition that every shift is an automorphism of $[0,1)$. Notice also that every nonidentity partial rational shift can be written as a composition of total rational shifts. More generally, any nonidentity rational shift has many representations as a composition of (more than one) other rational shifts, e.g.,

$$\gamma^{\frac{1}{2}}_{\frac{1}{8},\frac{3}{8}} \;=\; \gamma^{\frac{1}{2}}_{\frac{1}{8},\frac{1}{4}} \circ \gamma^{\frac{1}{2}}_{\frac{1}{4},\frac{3}{8}} \;=\; \gamma^{\frac{1}{4}}_{\frac{1}{8},\frac{3}{8}} \circ \gamma^{\frac{1}{4}}_{\frac{3}{8},\frac{5}{8}} \circ \gamma^{\frac{1}{4}}_{\frac{1}{8},\frac{3}{8}} \,.$$

Its representation in the form $\gamma^h_{a,b}$ is, however, unique (instead, for the identity, we have $e = \gamma^0_{a,b}$ for each $0 \leq a < b \leq 1$). Finally observe that the graph of a shift is composed by a finite number of segments (not necessarily of the same length), closed to the left and open to the right, all having slope equal to 1. Therefore, also the graph of any finite composition of shifts will be of the same type.

In the sequel, we are mostly interested in rational shifts. In fact, we focus our attention on the group (under composition) generated by $\mathcal{R}$; we will denote this group by $\mathcal{S}$. The importance of this algebraic object is based both on the simple nature of its elements as functions on $[0,1)$ (which allows us to give a purely combinatorial representation of it: see Sections 4 and 5) and on its close topological relationship to $\mathcal{E}$ and $\mathcal{A}$ (see Sections 6 and 7).

Remark 3.3. The following chain of inclusions holds:

$$\mathcal{R} \subset \mathcal{S} \sqsubset \mathcal{A} \subset \mathcal{E}\,,$$

where $\sqsubset$ stands for *subgroup*. Notice that $\mathcal{S}$ is not normal in $\mathcal{A}$. Indeed, if we let $\beta = \gamma^h_{a,b}$, with $(b-a) \in \mathbb{Q}$ and $h \in \mathbb{R}\setminus\mathbb{Q}$, and define

$$\alpha(x) = \begin{cases} \dfrac{\beta(2x)}{2} & \text{if} \quad x \in [0,1/2) \\ x & \text{if} \quad x \in [1/2,1)\,, \end{cases}$$

then $\alpha \in \mathcal{A}$, $\gamma^{\frac{1}{2}}_{0,\frac{1}{2}} \in \mathcal{R} \subset \mathcal{S}$, but $(\alpha^{-1} \circ \gamma^{\frac{1}{2}}_{0,\frac{1}{2}} \circ \alpha) \notin \mathcal{S}$. Finally, observe that $\mathcal{E}$ is not a group under compositions of functions, since there are endomorphisms whose equivalence classes do not contain bijections (cf. Example 2.3).

4. The group $\mathcal{S}$ generated by rational shifts

We now start analyzing the group-theoretic structure of $\mathcal{S}$ providing a representation theorem for $\mathcal{S}$ (in this section) and determining the order of its generators (in the next section).

As the graph of any rational shift suggests, we may regard any element of $\mathcal{R}$ as a finite permutation of natural numbers by properly subdividing $[0,1)$ in intervals of equal length, and collapsing each interval to a point. In other words, any rational shift can be identified with a particular element of S_n, the *symmetric group on n letters*, for a suitably chosen $n \in \mathbb{N}$. This easy but crucial observation is the starting point for an attempt to represent the group $\mathcal{S}$ by means of the symmetric groups S_n, $n \in \mathbb{N}$. In the following we show that indeed $(\mathcal{S}, \circ)$ is a directed colimit of the groups S_n.

We briefly recall from [10] the categorial definition of directed colimit for a directed family of morphisms. Let $(I, \preceq)$ be a *directed set of indices*, namely, a poset with the property that for each $i, j \in I$ there is $k \in I$ such that both $i \preceq k$ and $j \preceq k$. Let $\mathbf{A}$ be a category and $(A_i)_{i\in I}$ a family of $\mathbf{A}$-objects. For each pair $(i,j) \in I^2$ such that $i \preceq j$, assume we are given an $\mathbf{A}$-morphism $f_i^j : A_i \to A_j$ satisfying the following two properties: $f_i^i = id_{A_i}$ (the identity morphism on A_i) for every $i \in I$, and $f_i^k = f_j^k \circ f_i^j$ whenever $i \preceq j \preceq k$. Then the I-indexed family of pairs $(A_i, f_i^j)_I$ is said to be a *directed family of morphisms* over $(I, \preceq)$. Finally, a *directed colimit* of the family $(A_i, f_i^j)_I$ is a pair $(A, (f_i)_{i\in I})$ such that the $\mathbf{A}$-morphisms $f_i : A_i \to A$ satisfy the following two properties:

(i) $f_i = f_j \circ f_i^j$ whenever $i \preceq j$;
(ii) for every other pair $(B, (g_i)_{i\in I})$, where $g_i : A_i \to B$ are $\mathbf{A}$-morphisms such that $g_i = g_j \circ f_i^j$ if $i \preceq j$, there exists a unique $\mathbf{A}$-morphism $f : A \to B$ satisfying $f \circ f_i = g_i$ for every $i \in I$.

In other words, a directed colimit of $(A_i, f_i^j)_I$ is an initial object in the category $\mathbf{C}$ obtained from $\mathbf{A}$ taking as $\mathbf{C}$-objects the pairs $(A, (f_i)_{i\in I})$ as above, and defining $\mathbf{C}$-morphisms from $(A, (f_i)_{i\in I})$ to $(B, (g_i)_{i\in I})$ as those $\mathbf{A}$-morphisms from A to B that make the relative diagrams commute.

We will now apply the above notions in the category of groups. Partially order $\mathbb{N}$ by divisibility, denoting $m \preceq n$ if m divides n.

Then, $(\mathbb{N}, \preceq)$ is a directed set. For each pair $(m,n) \in \mathbb{N}^2$ of comparable naturals, say $m \preceq n$ with $n = tm$, define a map $\iota_m^n : S_m \longrightarrow S_n$ as follows:
if $\sigma = \begin{pmatrix} 1 & \dots & m \\ k_1 & \dots & k_m \end{pmatrix} \in S_m$, let

$$\iota_m^n(\sigma) = \begin{pmatrix} 1 & \dots & t & t+1 & \dots & t\,m \\ t(k_1-1)+1 & \dots & t(k_1-1)+t & t(k_2-1)+1 & \dots & t(k_m-1)+t \end{pmatrix}.$$

One can easily verify that $\iota_m^n(\sigma)$ is the permutation in S_n given by

$$(i-1)\,t + j \;\longmapsto\; (\sigma(i)-1)\,t + j$$

for $i = 1, \dots, m$ and $j = 1, \dots, t$. Hence, ι_m^n is a well defined injection of S_m into S_n.

Lemma 4.1. *$(S_n, \iota_m^n)_{\mathbb{N}}$ is a directed system over the directed set $(\mathbb{N}, \preceq)$.*

Proof: Straightforward verification of the properties of a directed system. □

Next, we associate to each finite permutation of $\mathbb{N}$ a finite composition of rational shifts. Given $n \in \mathbb{N}$, consider the decomposition of $[0,1)$ in n subintervals of equal length $[(i-1)/n, i/n)$, $i = 1, \dots, n$. For each permutation $\sigma \in S_n$, define a map γ_σ in $[0,1)$ by

$$\gamma_\sigma(x) = x + \frac{\sigma(i) - i}{n}$$

if $x \in [(i-1)/n, i/n)$. Note that γ_σ is a finite composition of rational shifts. Therefore, the maps $\phi_n : S_n \to \mathcal{S}$ given by

$$\phi_n(\sigma) = \gamma_\sigma$$

are well defined for each $n \in \mathbb{N}$. The next lemma collects some properties of the family of maps $(\phi_n)_{n \in \mathbb{N}}$.

Lemma 4.2. *The following hold:*

(i) *ϕ_n is a group monomorphism for each $n \in \mathbb{N}$;*
(ii) *$\phi_m = \phi_n \circ \iota_m^n$ whenever $m \preceq n$;*
(iii) *for each $\gamma \in \mathcal{R}_{tot}$ there exists an involution σ in some S_n such that $\gamma = \phi_n(\sigma)$;*
(iv) *for each $\gamma \in \mathcal{S}$ there exist $n \in \mathbb{N}$ and $\sigma \in S_n$ such that $\gamma = \phi_n(\sigma)$.*

Proof: To prove (i), fix $n \in \mathbb{N}$ and σ_1, $\sigma_2 \in S_n$, and let $x \in [0,1)$. Then there exists $i \in \{1,\dots,n\}$ such that $x \in [(i-1)/n, i/n)$. Observe that, if $k \in \{1,\dots,n\}$, the point $x + (k-i)/n$ lies in the interval $[(k-1)/n, k/n)$. Then, denoting $j = \sigma_2(i)$, we get

$$\gamma_{\sigma_1 \circ \sigma_2}(x) \,=\, x + \frac{j-i}{n} + \frac{\sigma_1(j)-j}{n} \,=\, \gamma_{\sigma_1}\Big(x + \frac{j-i}{n}\Big) \,=\, \gamma_{\sigma_1} \circ \gamma_{\sigma_2}(x)\,.$$

Therefore, the equality $\gamma_{\sigma_1 \circ \sigma_2} = \gamma_{\sigma_1} \circ \gamma_{\sigma_2}$ holds for each σ_1, $\sigma_2 \in S_n$, that is, ϕ_n is a group homomorphism. It is apparent that ϕ_n is injective.

For (ii), let $n = tm$ for some $t \in \mathbb{N}$. To show that the equality $\gamma_\sigma = \gamma_{\iota_m^n(\sigma)}$ holds for each $\sigma \in S_m$, let $x \in [(i-1)/m, i/m) \subset [0,1)$, for some $i \in \{1,\dots,m\}$. Since there exists $l \in \{1,\dots,t\}$ such that

$$x \in \Big[\frac{(i-1)t + l - 1}{n}, \frac{(i-1)t + l}{n}\Big)\,,$$

the definition of $\iota_m^n(\sigma)$ gives

$$\gamma_{\iota_m^n(\sigma)}(x) = x + \frac{((\sigma(i)-1)t + l) - ((i-1)t + l)}{n} = x + \frac{\sigma(i) - i}{m} = \gamma_\sigma(x)\,,$$

as wanted.

To prove (iii), observe that any $\gamma_{a,b}^h \in \mathcal{R}_{tot}$ can be written as $\gamma_{i/n, j/n}^{k/n}$, where n is the least common multiple of the denominators of a, b, h, and $0 < j - i \le k < n$. Then, $\gamma_{a,b}^h = \phi_n(\sigma)$, where σ is the permutation given by the following product of disjoint transpositions in S_n:

$$(i+1, i+1+k)\ (i+2, i+2+k)\ \dots\ (j, j+k)\,.$$

Finally, (iv) is an immediate consequence of (iii). □

Then we have:

Theorem 4.3. *$(\mathcal{S}, (\phi_n)_\mathbb{N})$ is a directed colimit of the directed system $(S_n, \iota_m^n)_\mathbb{N}$ in the category of groups.*

Proof: We already know from Lemma 4.2 (i) that all the maps ϕ_n are group homomorphisms. Moreover, Lemma 4.2 (ii) shows that the commutativity condition

$$\phi_m = \phi_n \circ \iota_m^n$$

holds for all $m \preceq n$. Therefore, we are left to prove that $(\mathcal{S}, (\phi_n)_\mathbb{N})$ satisfies the following universal mapping property: for every group

G, and for every family of group homomorphisms $(\psi_n)_{\mathbb{N}}$, $\psi_n : S_n \to G$, such that $\psi_m = \psi_n \circ \iota_m^n$ whenever $m \preceq n$, there exists a unique homomorphism $\psi : \mathcal{S} \to G$ satisfying the relations $\psi \circ \phi_n = \psi_n$ for every $n \in \mathbb{N}$. We define ψ to fulfill the above commutativity conditions. For each $n \in \mathbb{N}$ and each $\gamma \in \mathcal{S}$, set

$$\psi(\gamma) = \psi_n(\sigma)\,,$$

where $\sigma \in S_n$ is such that $\phi_n(\sigma) = \gamma$. In order to prove the result, it suffices to show that ψ is a group homomorphism, uniqueness being obvious. We start proving that ψ is a well-defined map from $\mathcal{S}$ into G. Indeed, let $\sigma_n \in S_n$ and $\sigma_m \in S_m$ be such that $\phi_n(\sigma_n) = \phi_m(\sigma_m)$. Setting $p = \mathrm{lcm}(n, m)$, and exploiting the commutativity condition, we have

$$\phi_p(\iota_n^p(\sigma_n)) = \phi_n(\sigma_n) = \phi_m(\sigma_m) = \phi_p(\iota_m^p(\sigma_m))\,.$$

Since ϕ_p is left-cancellable by Lemma 4.2 (i), we get that

$$\iota_n^p(\sigma_n) = \iota_m^p(\sigma_m)\,,$$

which in turn entails

$$\psi_n(\sigma_n) = \psi_p(\iota_n^p(\sigma_n)) = \psi_p(\iota_m^p(\sigma_m)) = \psi_m(\sigma_m)\,,$$

as desired. Finally, we show that ψ is a homomorphism. Let $\gamma_1, \gamma_2 \in \mathcal{S}$. Then, by Lemma 4.2 (iv), there exist $n_i \in \mathbb{N}$ and $\tau_i \in S_{n_i}$ $(i = 1, 2)$ such that $\gamma_i = \phi_{n_i}(\tau_i)$. Setting $n = \mathrm{lcm}(n_1, n_2)$, let σ_i be the elements of S_n given by $\sigma_i = \iota_{n_i}^n(\tau_i)$. Then we have

$$\phi_n(\sigma_i) = \phi_n \circ \iota_{n_i}^n(\tau_i) = \phi_{n_i}(\tau_i) = \gamma_i\,.$$

Recalling now the definition of ψ, and the fact that both ϕ_n and ψ_n are homomorphisms, we conclude that

$$\psi(\gamma_1 \circ \gamma_2) = \psi_n(\sigma_1 \circ \sigma_2) = \psi_n(\sigma_1) \circ \psi_n(\sigma_2) = \psi(\gamma_1) \circ \psi(\gamma_2)\,.$$

This completes the proof. □

Recall from [1] that there is a canonical way to associate a colimit to any directed family of morphisms in the category of groups (and, more generally, in any algebraic *construct* having a certain form). Then, if we denote by $(S_{[\infty]}, (\alpha_n)_{\mathbb{N}})$ the *canonical* directed colimit of the system $(S_n, \iota_m^n)_{\mathbb{N}}$, Theorem 4.3 yields the following result:

Corollary 4.4. *$\mathcal{S}$ and $S_{[\infty]}$ are isomorphic groups.*

Proof: Colimits of the same (directed) system are essentially unique (see [1], Propositions 11.29 and 11.7). In particular, since both $(\mathcal{S}, (\phi_n)_{\mathbb{N}})$ and $(S_{[\infty]}, (\alpha_n)_{\mathbb{N}})$ are colimits of the same directed system in the category of groups, there exists a group isomorphism $\alpha : S_{[\infty]} \to \mathcal{S}$ such that $\phi_n = \alpha \circ \alpha_n$ for each $n \in \mathbb{N}$. $\square$

In light of the above result, it is interesting to analyze in some detail which type of topological structure the group $S_{[\infty]}$ can be endowed with.

Remark 4.5. By construction, the underlying set of $S_{[\infty]}$ is built as a quotient of the disjoint union of the sets S_n, since its elements are equivalence classes of finite permutations of $\mathbb{N}$ under the relation $\sim$ defined as follows: for each $\sigma_m \in S_m$ and $\sigma_n \in S_n$

$$\sigma_m \sim \sigma_n \iff \exists\, p \succeq m, n \text{ such that } \iota_m^p(\sigma_m) = \iota_n^p(\sigma_n)\,.$$

It is now easy to check that the isomorphism α in the proof of Corollary 4.4 is actually given by $[\sigma] \mapsto \gamma_\sigma$.

Remark 4.6. Each class in $S_{[\infty]}$ intersects each S_n in at most one element. Therefore, we can identify $[\sigma] \in S_{[\infty]}$ with its $\preceq$-minimal representative. More precisely, for each finite permutation σ, denote

$$N_\sigma = \{n \in \mathbb{N} : [\sigma] \cap S_n \neq \emptyset\}$$

and define for each $n \in N_\sigma$

$$\sigma^{(n)} = [\sigma] \cap S_n\,.$$

Then, if we let $n_\sigma = \min N_\sigma$, the map

$$[\sigma] \longmapsto (\sigma^{(n_\sigma)}, n_\sigma)$$

is a well defined injection of $S_{[\infty]}$ into $\bigcup_{n\in\mathbb{N}}(S_n, n)$.

The above identification turns out to be a useful device to compare different elements of $S_{[\infty]}$. Given $[\sigma], [\tau] \in S_{[\infty]}$, denote $n_{\sigma,\tau} = \operatorname{lcm}\{n_\sigma, n_\tau\}$. Then, we can compare $[\sigma]$ and $[\tau]$ by looking at their representatives in the $\preceq$-minimally indexed common symmetric group, that is, $\sigma^{(n_{\sigma,\tau})}, \tau^{(n_{\sigma,\tau})} \in S_{n_{\sigma,\tau}}$.

Example 4.7. Let $\sigma \in S_8$ and $\tau \in S_9$ be the following involutions (here we use their cycle representation):

$$\sigma = (1\,5)\,(2\,6)\,(3\,7)\,(4\,8) \qquad \text{and} \qquad \tau = (1\,7)\,(2\,8)\,(3\,9)\,(4)\,(5)\,(6)\,.$$

Then, we have:

- $n_\sigma = 2\,,\quad \sigma^{(2)} = (1\,2)\,,\quad n_\tau = 3\,,\quad \tau^{(3)} = (1\,3)\,(2)\,;$
- $n_{\sigma,\tau} = 6\,,\quad \sigma^{(6)} = (1\,4)\,(2\,5)\,(3\,6)\,,\quad \tau^{(6)} = (1\,5)\,(2\,6)\,(3)\,(4)\,;$
- $\gamma_\sigma = \gamma_{0,\frac{1}{2}}^{\frac{1}{2}}\,,\quad \gamma_\tau = \gamma_{0,\frac{1}{3}}^{\frac{2}{3}}\,,\quad [\sigma] \neq [\tau]\,.$

We now use the identification of Remark 4.6 to define some interesting metrics in $S_{[\infty]}$. They are combinatorial versions of some well known metrics for spaces of functions, and are obtained by means of the isomorphism between $S_{[\infty]}$ and $\mathcal{S}$.

Proposition 4.8. *The maps d_1, d_2 and d_3 defined by*

$$\begin{aligned} d_1([\sigma],[\tau]) &= \frac{1}{n_{\sigma,\tau}} \max\{|\sigma^{(n_{\sigma,\tau})}(i) - \tau^{(n_{\sigma,\tau})}(i)| \,:\, i = 1,\dots,n_{\sigma,\tau}\}\,,\\ d_2([\sigma],[\tau]) &= \frac{1}{(n_{\sigma,\tau})^2} \sum_{i=1}^{n_{\sigma,\tau}} |\sigma^{(n_{\sigma,\tau})}(i) - \tau^{(n_{\sigma,\tau})}(i)|\,,\\ d_3([\sigma],[\tau]) &= \end{aligned}$$

$$\frac{1}{(n_{\sigma,\tau})^2} \sum_{i=1}^{n_{\sigma,\tau}} \Big(|\sigma^{(n_{\sigma,\tau})}(i) - \tau^{(n_{\sigma,\tau})}(i)| + |(\sigma^{-1})^{(n_{\sigma,\tau})}(i) - (\tau^{-1})^{(n_{\sigma,\tau})}(i)| \Big)\,,$$

are metrics in $S_{[\infty]}$.

Proof: The correspondence $[\sigma] \mapsto \gamma_\sigma$ gives at once

$$\begin{aligned} d_1([\sigma],[\tau]) &= \sup\{|\gamma_\sigma(t) - \gamma_\tau(t)| : t \in [0,1)\}\,,\\ d_2([\sigma],[\tau]) &= \int_0^1 |\gamma_\sigma(t) - \gamma_\tau(t)|\, d\mu(t)\,,\\ d_3([\sigma],[\tau]) &= \int_0^1 \Big(|\gamma_\sigma(t) - \gamma_\tau(t)| + |\gamma_\sigma^{-1}(t) - \gamma_\tau^{-1}(t)| \Big)\, d\mu(t)\,, \end{aligned}$$

the right-hand sides being well known metrics on $\mathcal{A}$. □

Therefore all the analytic results that we obtain in Section 6 and Section 7 can be translated into a combinatorial form. In particular, the following fact is worth being mentioned (see Theorem 6.1):

Theorem 4.9. *Every endomorphism of $[0,1)$ is the limit of finite permutations of $\mathbb{N}$.*

Remark 4.10. The "internal" relationships of the natural numbers are expressed by the group S_∞ of permutations of $\mathbb{N}$. From this point of view, each $n \in \mathbb{N}$ gives a finite picture of these relations

via its symmetric group S_n. Ordering the groups S_n into the $\preceq$-directed system $(S_n, \iota_m^n)_{\mathbb{N}}$ sheds new light onto their limit behavior. Each S_n is a finite approximation of S_∞, but not all the information provided by S_n is relevant to this aim. The effective contribution of $n \in \mathbb{N}$ to S_∞ is in fact contained into the subset $S_n^{'}$ of S_n given by

$$S_n^{'} = \{\sigma \in S_n : n_\sigma = n\}\,.$$

Indeed, only the $\preceq$-minimal elements of S_n add new information, since any $\sigma \in S_n \setminus S_n^{'}$ is an element of $S_k^{'}$, for some $k \precneqq n$.

5. Some numerical properties of shifts

Here we study the order of shifts as elements of the group $\mathcal{A}$. As a consequence, we get some insight into the group $\mathcal{S}$ by determining the order of its generators. The next step would be the study of the order of particular compositions of rational shifts. This problem presents some serious difficulties, and it is currently under investigation.

Determining the order of total shifts is an easy task. We know already from Lemma 4.2 (iii) that the order a rational total shift is 2. We will show that this fact immediately extends to all total shifts. For what concerns partial shifts, some of them have finite order, but there exist also torsion-free partial shifts. Obviously, all rational partial shifts do have finite order. Nonetheless, the class of partial shifts with finite order is actually much larger, as we will see.

In the sequel, we classify all shifts and determine their order accordingly. We begin with two easy observations.

Remark 5.1. Any shift $\gamma_{a,b}^h \neq e$ moves both the points in the basic interval $[a, b)$ and those in the shifted interval $[a + h, b + h)$, but it leaves all the other points of $[0, 1)$, if any, fixed. It is an immediate consequence of the definition of shifts that any natural power of $\gamma_{a,b}^h$ has at least the same fixed points as $\gamma_{a,b}^h$. Thus, for each $n \in \mathbb{N}$, we have (with obvious notation)

$$\mathrm{fix}((\gamma_{a,b}^h)^n) \supset [0,1) \setminus ([a,b) \cup [a+h, b+h))\,.$$

Remark 5.2. It is apparent that the order of any shift $\gamma^h_{a,b}$ (denoted by $\mathrm{ord}(\gamma^h_{a,b})$) is invariant under translations of the basic interval $[a, b)$, namely,

$$\mathrm{ord}(\gamma^h_{a,b}) = \mathrm{ord}(\gamma^h_{a+c,b+c})$$

for any $c \in (-1, 1)$ such that $0 \leq a + c \leq b + c \leq 1$. In particular, $\mathrm{ord}(\gamma^h_{a,b}) = \mathrm{ord}(\gamma^h_{0,b-a})$. Therefore, the order of a shift $\gamma^h_{a,b}$ will depend only on two parameters: its *length* $l = b - a$ and its *height* h.

This last remark motivates the following definition.

Definition 5.3. For any shift $\gamma^h_{a,b}$, define its *ratio* by $\mathrm{rat}(\gamma^h_{a,b}) = h/(b - a)$.

Observe that $\gamma^h_{a,b}$ is total if and only if $\mathrm{rat}(\gamma^h_{a,b}) \in [1, \infty)$, and $\gamma^h_{a,b}$ is partial if and only if $\mathrm{rat}(\gamma^h_{a,b}) \in [0, 1)$. In particular, $\gamma^h_{a,b} = e$ if and only if $\mathrm{rat}(\gamma^h_{a,b}) = 0$.

The following fact is an easy consequence of the definition of partial shift.

Lemma 5.4. *Let $\gamma^h_{a,b} \neq e$ be a partial shift. Then, the following two conditions are equivalent for any positive integer n:*

(i) *there are $r, s \in \mathbb{N}$ such that $r + s \leq n$ and $r(b - a) = sh$;*
(ii) $\mathrm{ord}(\gamma^h_{a,b}) \leq n$.

The next result establishes for any shift a precise relationship between the order and the ratio.

Theorem 5.5. *Given any shift $\gamma^h_{a,b} \neq e$, the following hold:*

(i) *if $\mathrm{rat}(\gamma^h_{a,b}) \in [1, \infty)$, then $\mathrm{ord}(\gamma^h_{a,b}) = 2$;*
(ii) *if $\mathrm{rat}(\gamma^h_{a,b}) \in (0, 1) \cap \mathbb{Q}$, then $\mathrm{ord}(\gamma^h_{a,b}) = p + q$, with $p/q = \mathrm{rat}(\gamma^h_{a,b})$ and $\gcd(p, q) = 1$;*
(iii) *if $\mathrm{rat}(\gamma^h_{a,b}) \in (0, 1) \cap (\mathbb{R} \setminus \mathbb{Q})$, then $\mathrm{ord}(\gamma^h_{a,b})$ is infinite.*

Proof: (i) is an immediate consequence of Remark 5.1 and the definition of $\gamma^h_{a,b}$.

For (ii), let $\gamma^h_{a,b}$ be such that

$$\mathrm{rat}(\gamma^h_{a,b}) = \frac{h}{b - a} = \frac{p}{q} \in (0, 1) \cap \mathbb{Q},$$

where $\gcd(p,q) = 1$. Then, $\gamma^h_{a,b}$ is a nonidentity partial shift satisfying Lemma 5.4 (i) with $r = p$, $s = q$ and $n = p + q$. Thus, $\mathrm{ord}(\gamma^h_{a,b}) \leq p + q$. To finish the proof of (ii), we show that it cannot be $\mathrm{ord}(\gamma^h_{a,b}) < p + q$. Otherwise, Lemma 5.4 (ii) holds with $n = p+q-1$, hence there should be $r, s \in \mathbb{N}$ such that $r+s \leq p+q-1$ and $r(b-a) = sh$. This is obviously impossible.

Finally, let

$$\mathrm{rat}(\gamma^h_{a,b}) = \frac{h}{b-a} \in (0,1) \cap (\mathbb{R} \setminus \mathbb{Q}) .$$

In this case Lemma 5.4(i) fails for any $n \in \mathbb{N}$. Thus, the order of $\gamma^h_{a,b}$ is unbounded. □

Remark 5.6. Theorem 5.5 provides some information about the group $\mathcal{S}$, since it allows us to determine the order of any element of $\mathcal{R}$. It would be interesting to derive a formula that gives the order of the composition of particular types of rational shifts. Indeed, in force of the correspondence $\sigma^{(n)} \mapsto \gamma_\sigma$, if γ_σ can be written as composition of these particular shifts, then the formula would automatically give the order of the permutation $\sigma \in S_n$. Not surprisingly, determining the order of a composition of rational shifts is not an easy task. Even in the simple case of a composition of two total rational shifts, the problem is far from being elementary. We have, however, some partial results related to compositions of the form $\gamma^{q/n}_{0,\,p/n} \circ \gamma^{s/n}_{0,\,r/n}$, with $p \leq q$ and $r \leq s$. In fact, it is possible to classify these compositions according to the reciprocal position of the basic and shifted intervals of the two shifts, and derive a formula giving the order of their composition in each of these cases. This topic will be developed in a forthcoming paper.

6. Topological properties of the sets $\mathcal{A}$ and $\mathcal{E}$

In [7] Halmos studies from a topological point of view the automorphisms of the measure algebra of measurable sets on the unit interval, in the hope to shed new light onto some important topics in ergodic theory. To this aim, he endows the group G of automorphisms of the measure algebra with three topologies: the "neighborhood topology", the "metric topology"and the "uniform topology". Besides several ergodic results, he proves some theorems which are purely topological in nature. In particular, he shows that if G is

endowed with the neighborhood topology (but not with the other two, which, however, coincide on G and are stronger than the first), the set of "permutations of intervals" is a countable dense set in G.

Here we obtain a similar result on the underlying measure space, but using a topology which is in our opinion more expressive of the real proximity of (equivalence classes of) functions. Specifically, we show that, if we endow $\mathcal{E}$ with the topology of convergence in measure, the group $\mathcal{S}$ generated by rational shifts witnesses the separability of the space of endomorphisms. We remark again that, by the lifting theorem, all the results we obtain are valid in the associated measure algebra.

Denote by $\mathcal{M}$ the set of Lebesgue measurable $\mathbb{R}$-valued functions on $[0,1)$. It is well known that $\mathcal{M}$ endowed with the distance

$$d(f_1, f_2) = \int_0^1 \frac{|f_1(t) - f_2(t)|}{1 + |f_1(t) - f_2(t)|}\, d\mu(t)$$

is a complete metric space, and the topology induced on $\mathcal{M}$ by this metric is the topology of *convergence in measure*. Here and below we denote this topology by τ_d.

It is clear that $(\mathcal{E}, d)$ is a closed subspace of $(\mathcal{M}, d)$, and hence $(\mathcal{E}, d)$ is a complete metric space. Note also that $(\mathcal{E}, \circ)$ is a topological monoid with respect to the topology τ_d. The next result shows that $(\mathcal{E}, \tau_d, \circ)$ is indeed a Polish monoid.

Theorem 6.1. *The space $\mathcal{S}$ is τ_d-dense in $\mathcal{E}$; that is, every endomorphism of $[0,1)$ is the limit (in the topology of convergence in measure) of a sequence of automorphisms, each of which is a compositions of finitely many rational shifts.*

Proof: Let $\gamma \in \mathcal{E}$. We construct a sequence $\{\gamma_n\}$ in $\mathcal{E}$ which converges in measure to γ. For any $n \in \mathbb{N}$, and any $k \in \{1, \ldots, 2^n\}$, denote the disjoint sets

$$A_k^n = \gamma^{-1}\Big([\frac{k-1}{2^n}, \frac{k}{2^n})\Big)\,.$$

Clearly, $\mu(A_k^n) = 1/2^n$. Due to the outer regularity of the Lebesgue measure, there exist open sets $O_k^n \supset A_k^n$ such that

$$\mu(O_k^n \setminus A_k^n) \leq \frac{1}{n 2^{2n+1}}\,.$$

Since an open set is a countable union of open intervals, it is easy to find sets $B_k^n \subset O_k^n$ which are finite union of intervals with rational endpoints, such that

$$\frac{1}{n2^{2n+1}} < \mu(O_k^n \setminus B_k^n) \leq \frac{1}{n2^{2n}}\,.$$

In particular, $\mu(B_k^n) < 1/2^n$. Observing that $A_k^n \cap B_k^n = A_k^n \setminus (O_k^n \setminus B_k^n)$, we have that

$$\mu(B_k^n \cap A_k^n) \geq \frac{1}{2^n}\Big(1 - \frac{1}{n2^n}\Big)\,.$$

Concerning the case $k \neq j$, we get at once the estimate

$$\mu(B_j^n \cap A_k^n) \leq \mu(O_k^n \cap O_j^n) \leq \frac{1}{n2^{2n}}\,.$$

At this point, we introduce the disjoint sets

$$W_k^n = B_k^n \setminus \bigcup_{j \neq k} B_j^n\,.$$

Again, $\mu(W_k^n) < 1/2^n$. Notice also that $\mu(W_k^n) \in \mathbb{Q}$. Moreover, exploiting the two above inequalities,

$$\mu(W_k^n \cap A_k^n) \geq \mu(B_k^n \cap A_k^n) - \sum_{j \neq k} \mu(B_j^n \cap A_k^n) \geq \frac{1}{2^n}\Big(1 - \frac{1}{n}\Big)\,.$$

Notice that each set W_k^n is a finite disjoint union of intervals with rational endpoints. Then we can re-define (without affecting the measure) W_k^n in such a way that each interval is half-open on the right. Finally, we set

$$W_0^n = [0,1) \setminus \bigcup_{k=1}^{2^n} W_k^n\,.$$

It is apparent that W_0^n is of the form $\bigcup_{i=1}^m [a_i, b_i)$ (with a_i, $b_i \in \mathbb{Q}$ and $a_{i+1} > b_i$). Up to splitting conveniently some of the intervals $[a_i, b_i)$, we can write W_0^n as

$$W_0^n = \bigcup_{k=1}^{2^n} W_{0k}^n\,,$$

where each W_{0k}^n is still a finite disjoint union of half-open intervals on the right with rational endpoints, and such that

$$\mu(W_{0k}^n) = \frac{1}{2^n} - \mu(W_k^n)\,.$$

Indeed, to build W_{01}^n proceed as follows: let r be the smallest index such that

$$\sum_{i=1}^{r}(b_i - a_i) \geq 1/2^n - \mu(W_1^n)\,.$$

If equality occurs, then let $W_{01}^n = \bigcup_{i=1}^{r}[a_i - b_i)$. Otherwise there must exist a rational number $c \in [a_r, b_r)$ such that

$$\sum_{i=1}^{r-1}(b_i - a_i) + [a_r, c) = 1/2^n - \mu(W_1^n)$$

(if $r = 1$ the sum disappears). Thus we define $W_{01}^n = \bigcup_{i=1}^{r-1}[a_i - b_i)\bigcup[a_r, c)$ (or $[a_1, c)$ if $r = 1$). To build W_{0k}^n once W_{0j}^n are given for $j = 1, \ldots, k-1$, just apply the same argument to the set $W_0^n \setminus \bigcup_{j=1}^{k-1} W_{0j}^n$. Writing now, for $k = 1, \ldots, 2^n$,

$$W_k^n = \bigcup_{i=1}^{n_k}[\alpha_i^k, \beta_i^k) \qquad \text{and} \qquad W_{0k}^n = \bigcup_{i=1}^{m_k}[a_i^k, b_i^k)$$

(with n_k, $m_k \in \mathbb{N}$ and α_i^k, β_i^k, a_i^k, $b_i^k \in \mathbb{Q}$), and setting for convenience α_0^k, β_0^k, a_0^k, $b_0^k = 0$, we define the automorphism γ_n in the following manner:

$$\gamma_n(x) = \begin{cases} x - \alpha_i^k + \dfrac{k-1}{2^n} + \displaystyle\sum_{j=0}^{i-1}(\beta_j^k - \alpha_j^k) & \text{if } x \in [\alpha_i^k, \beta_i^k) \\ x - a_r^k + \dfrac{k-1}{2^n} + \displaystyle\sum_{j=0}^{n_k}(\beta_j^k - \alpha_j^k) + \sum_{j=0}^{r-1}(b_j^k - a_j^k) & \text{if } x \in [a_r^k, b_r^k)\,, \end{cases}$$

with

$$i \in \{1, \ldots, n_k\} \qquad \text{and} \qquad r \in \{1, \ldots, m_k\}\,.$$

By construction,

$$\gamma_n(W_k^n) \subset [\frac{k-1}{2^n}, \frac{k}{2^n})\,.$$

Denoting

$$U_n = \left\{x \in [0,1) : |\gamma(x) - \gamma_n(x)| \leq \frac{1}{2^n}\right\},$$

we get that $U_n \supset W_k^n \cap A_k^n$ for every $k = 1, \dots, 2^n$, and therefore

$$U_n \supset \bigcup_{k=1}^{2^n} (W_k^n \cap A_k^n)\,.$$

Being the union disjoint, we conclude that

$$\mu(U_n) \geq \sum_{k=1}^{2^n} \mu(W_k^n \cap A_k^n) \geq 1 - \frac{1}{n}\,,$$

which implies the thesis. □

Remark 6.2. Since from every sequence converging in measure it is possible to extract a subsequence converging pointwise almost everywhere to the same limit, we conclude that every endomorphism of $[0,1)$ is the pointwise almost everywhere limit of finite compositions of rational shifts.

Remark 6.3. For $p \in [1,\infty)$, denote by $\|\cdot\|_p$ the norm in $L^p([0,1),\mu)$. It is apparent that, for every γ, $\beta \in \mathcal{E}$,

$$\frac{1}{2}\|\gamma - \beta\|_1 \leq d(\gamma,\beta) \leq \|\gamma - \beta\|_1\,,$$

and

$$\|\gamma - \beta\|_1 \leq \|\gamma - \beta\|_p \leq 2^{(p-1)/p}\|\gamma - \beta\|_1^{1/p}\,.$$

As consequence, for every $p \in [1,\infty)$, the topology τ_d and the norm topology $\tau_{\|\cdot\|_p}$ of L^p coincide on $\mathcal{E}$. Moreover, it is rather easy to check that, if $\gamma \in \mathcal{S}$, then

$$\int_0^1 (\gamma(t))^p\, d\mu(t) = \frac{1}{1+p}\,.$$

Then, in force of Theorem 6.1 and of the equality of the two topologies, we get that, for every $p \in [1,\infty)$, $\mathcal{E}$ is a closed subset of L^p, which is contained in $S(0\,,\, 1/(1+p)^{1/p})$, the sphere of radius $1/(1+p)^{1/p}$ centered at zero.

Remark 6.4. Consider now $\overline{\mathcal{E}}^{w,p}$, that is, the closure of $\mathcal{E}$ in the weak topology $\tau_{w,p}$ of L^p. Then, for all $1 \leq p < \infty$, we have:

(i) $\overline{\mathcal{E}}^{w,p}$ is contained in the closed annulus of radii $1/2$ and $1/(1+p)^{1/p}$ centered at zero;
(ii) $\mathcal{E} = \overline{\mathcal{E}}^{w,p} \cap S(0\,,\, 1/(1+p)^{1/p})$ for $p > 1$;
(iii) $\tau_{w,p} \neq \tau_{\|\cdot\|_p}$ on $\mathcal{E}$.

For (i), let $\gamma \in \overline{\mathcal{E}}^{w,p}$. Then, $\|\gamma\|_p \geq 1/2$. Indeed, let $\{\gamma_\iota\}_{\iota\in I}$ be a net in $\mathcal{E}$ converging to γ in $\tau_{w,p}$. Then, since the constant function 1 belongs to the norm dual of $L^p([0,1),\mu)$, Remark 6.3 yields

$$\|\gamma\|_1 = \int_0^1 \gamma(t)\, d\mu(t) = \lim_{\iota\in I} \int_0^1 \gamma_\iota(t)\, d\mu(t) = \lim_{\iota\in I} \|\gamma_\iota\|_1 = \frac{1}{2}\,,$$

whence $\|\gamma\|_p \geq 1/2$ by Hölder inequality. On the other hand, we also have $\|\gamma\|_p \leq 1/(1+p)^{1/p}$, owing to Remark 6.3 and to the fact that norm closure and weak closure coincide for any convex set.

To prove (ii), let $\gamma \in \overline{\mathcal{E}}^{w,p} \cap S(0\,,\, 1/(1+p)^{1/p})$. Then $\gamma_\iota \to \gamma$ in $\tau_{w,p}$ for some net $\{\gamma_\iota\}_{\iota\in I}$ in $\mathcal{E}$. Note that, for each $\iota \in I$, $\|\gamma_\iota\|_p = 1/(1+p)^{1/p} = \|\gamma\|_p$, since $\gamma \in S(0\,,\, 1/(1+p)^{1/p})$. By the uniform convexity of $L^p([0,1),\mu)$ for $p > 1$, we obtain $\gamma_\iota \to \gamma$ in $\|\cdot\|_p$, whence, being $\mathcal{E}$ closed in $\|\cdot\|_p$, $\gamma \in \mathcal{E}$. This proves one inclusion in (ii). The other inclusion is a consequence of Remark 6.3.

Finally, assume that $\tau_{w,p} = \tau_{\|\cdot\|_p}$ on $\mathcal{E}$ for some $1 \leq p < \infty$. Then, for every $d-$closed set $\mathcal{C} \subset \mathcal{E}$ and every $r > p$, we have the inclusions

$$\mathcal{C} \;\subset\; \overline{\mathcal{C}}^{w,r} \subset\; \overline{\mathcal{C}}^{w,p} =\; \overline{\mathcal{C}}^{p} =\; \mathcal{C}\,.$$

Thus, $\tau_d = \tau_{w,r}$ on $\mathcal{E}$. Notice that L^r is reflexive (since $r > 1$), hence the weak and the weak* topologies coincide on L^r. Then, Banach-Alaoglu theorem implies that the closed unit ball of L^r is weakly compact, hence, by our assumption, τ_d-compact. On the other hand, it is easy to check that the sequence $\gamma_n = \beta_n^{-1}$, with β_n as in Example 6.5 below, has no convergent subsequences (see also the proof of Proposition 7.4). A contradiction.

We have seen that the elements of the closure of $\mathcal{S}$ are not necessarily automorphisms. So it is a natural question to ask what happens if we take the closure of $\mathcal{S}$ in L^∞ (denoted by $\overline{\mathcal{S}}^\infty$), that is, in the topology of uniform convergence almost everywhere. A first answer is that $\overline{\mathcal{S}}^\infty \not\subset \mathcal{A}$.

Example 6.5. Consider the endomorphism $\beta(x) = 1 - |1 - 2x|$ of Example 2.3, and construct the sequence $\beta_n \in \mathcal{S}$ as follows: for $x \in [k/2^{n+1}, (k+1)/2^{n+1})$, with $k = 0, \ldots, 2^{n+1} - 1$, let

$$\beta_n(x) = \begin{cases} x + \dfrac{k}{2^{n+1}} & \text{if} \quad k < 2^n \\ x + 2 - \dfrac{(3k+1)}{2^{n+1}} & \text{if} \quad k \geq 2^n\,. \end{cases}$$

It is straightforward to check that β_n converges uniformly to β.

Furthermore, $\overline{\mathcal{S}}^{\infty} \not\supset \mathcal{A}$, as the following example shows.

Example 6.6. Let $T \subset (0,1)$ be a measurable set with the following property: for every $0 \leq a < b \leq 1$,

$$0 < \mu([a,b] \cap T) < b - a\,.$$

Define $f : [-1,1) \to [-1,1)$ to be

$$f(x) = \begin{cases} x & \text{if } x \in T \\ x-1 & \text{if } x \in [0,1) \setminus T \\ -x-1 & \text{if } x \in -T \\ -x & \text{if } x \in (-1,0) \setminus -T \\ 0 & \text{if } x = -1\,. \end{cases}$$

Finally, set for each $x \in [0,1)$

$$\gamma(x) = \frac{1}{2}\Big[f(2x-1)+1\Big]\,.$$

It is easily verified that $\gamma \in \mathcal{A}$. On the other hand, using the properties of T, it is immediate to see that, given any interval $I \subset [0,1)$ with rational endpoints, and any function $g : I \to [0,1)$ of the form $g(x) = x + c$, with $c \in [0,1) \cap \mathbb{Q}$, there exists an element $\sigma \in \mathcal{S}$ such that $\sigma_{|I} = g$. It follows that

$$\|\gamma - \sigma\|_\infty \geq \operatorname*{ess\,sup}_{x \in I} |\gamma(x) - g(x)| \geq \frac{1}{4}\,,$$

whence

$$\|\gamma - \mathcal{S}\|_\infty = \inf_{\gamma_0 \in \mathcal{S}} \|\gamma - \gamma_0\|_\infty \geq \frac{1}{4}\,.$$

Thus $\gamma \notin \overline{\mathcal{S}}^{\infty}$.

7. $\mathcal{A}$ as a topological group

As already pointed out, it is possible to endow the group G of automorphisms of the associated measure algebra with some topologies which make it into a Polish group. Here we prove a similar result for the group $\mathcal{A}$ of automorphisms of $[0,1)$, endowed with the topology of convergence in measure. We begin with two lemmas, the first of which is stated in a general form for later convenience.

Lemma 7.1. *Let $X, Y, Z \subset \mathbb{R}$ be three measurable sets having the same positive measure. Let $\{\gamma_n\}_{n\in\mathbb{N}}, \gamma \in \mathrm{Hom}(X,Y)$, and $\{\beta_n\}_{n\in\mathbb{N}}, \beta \in \mathrm{Hom}(Y,Z)$. Then:*

(i) $\beta_n \to \beta$ *implies* $\beta_n \circ \gamma \to \beta \circ \gamma$;
(ii) $\gamma_n \to \gamma$ *implies* $\beta \circ \gamma_n \to \beta \circ \gamma$.

The above convergences are in measure.

Proof: To prove (i), select $\varepsilon > 0$, and denote for each $n \in \mathbb{N}$

$$A_n = \{y \in Y : |\beta_n(y) - \beta(y)| > \varepsilon\} .$$

By hypothesis, $\mu(A_n) \to 0$ as $n \to \infty$. Since, for each $x \in X$, $x \in \gamma^{-1}(A_n)$ if and only if $|\beta_n \circ \gamma(x) - \beta \circ \gamma(x)| > \varepsilon$, we get

$$\gamma^{-1}(A_n) = \{x \in X : |\beta_n \circ \gamma(x) - \beta \circ \gamma(x)| > \varepsilon\} .$$

The result now follows from the fact that $\gamma \in \mathrm{Hom}(X,Y)$.

Next we prove (ii). Notice that, since $\mu(X) < \infty$, $\gamma_n \to \gamma$ in $L^1(X,\mu)$. We will show that for every $f \in L^1(Y,\mu)$, $f \circ \gamma_n \to f \circ \gamma$ in $L^1(X,\mu)$, which implies the thesis. For each $n \in \mathbb{N}$ consider the linear maps $T_n : L^1(Y,\mu) \to L^1(X,\mu)$, defined by $T_n f = f \circ \gamma_n$. Moreover, let $T : L^1(Y,\mu) \to L^1(X,\mu)$ be such that $Tf = f \circ \gamma$. Therefore, it suffices to show that $T_n f \to Tf$ in $L^1(X,\mu)$ for every $f \in L^1(Y,\mu)$. Now, since the functions γ_n are measure preserving, it results (cf. [13], Proposition 4.3)

$$\|T_n f\|_1 = \int_X |f(\gamma_n(x))|\, dx = \int_Y |f(y)|\, dy = \|f\|_1 ,$$

hence the sequence $\{\|T_n\|\}_{n\in\mathbb{N}}$ is bounded. Therefore, by Banach-Steinhaus theorem, it suffices to prove convergence for a dense set of maps. In our case, the family of characteristic functions of $(-\infty, a) \cap Y$, for $a \in \mathbb{R}$, will do. Since $\mu(X) < \infty$, it is enough to show convergence in measure. Fix $0 < \varepsilon < 1$, and consider the sets

$$A_n = \{x \in X : |f \circ \gamma_n(x) - f \circ \gamma(x)| > \varepsilon\} ,$$

where $f = \chi_{(-\infty,a)\cap Y}$, for some $a \in \mathbb{R}$. Furthermore, set

$$B_n = \{x \in X : |\gamma_n(x) - \gamma(x)| > \varepsilon\}$$

(observe that $\mu(B_n) \to 0$ as $n \to \infty$ by hypothesis) and

$$C_n = \{x \in X :\ \gamma_n(x) < a \le \gamma(x) \quad \text{or} \quad \gamma(x) < a \le \gamma_n(x)\} .$$

It is apparent that $A_n \subset C_n$. Moreover, if $x \in C_n \setminus B_n$, then $|\gamma(x) - a| \leq \varepsilon$, i.e., $\gamma(x) \in [a-\varepsilon, a+\varepsilon] \cap Y$. Therefore, since $\gamma \in \mathrm{Hom}(X,Y)$, we conclude that $\mu(C_n \setminus B_n) \leq 2\varepsilon$. Finally, from the inequality

$$\mu(A_n) \leq \mu(C_n \setminus B_n) + \mu(B_n)\,,$$

we obtain

$$\limsup_{n\to\infty} \mu(A_n) \leq \limsup_{n\to\infty} \mu(C_n \setminus B_n) + \lim_{n\to\infty} \mu(B_n) \leq 2\varepsilon\,.$$

Now let $\varepsilon \to 0$. □

Lemma 7.2. *Let $\{\gamma_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}} \in \mathcal{A}$, and let e be the identity of $\mathcal{A}$. Then the following hold:*

(i) $\gamma_n \to e$ *and* $\beta_n \to e$ *imply* $\beta_n \circ \gamma_n \to e$ *;*
(ii) $\gamma_n \to e$ *implies* $\gamma_n^{-1} \to e$ *.*

The above convergences are in d.

Proof: We prove (i), and leave to the reader the easier proof of (ii). Select $\varepsilon > 0$, and denote for each $n \in \mathbb{N}$

$$A_n = \{x \in [0,1) : |\gamma_n(x) - x| > \varepsilon\}$$

and

$$B_n = \{y \in [0,1) : |\beta_n(y) - y| > \varepsilon\}\,.$$

It is immediate to check the validity of the equality

$$\gamma_n^{-1}(B_n) = \{x \in [0,1) : |\beta_n \circ \gamma_n(x) - \gamma_n(x)| > \varepsilon\}\,.$$

Set for each $n \in \mathbb{N}$

$$C_n = \{x \in [0,1) : |\beta_n \circ \gamma_n(x) - x| > 2\varepsilon\}\,.$$

Then the inclusion

$$C_n \subset A_n \cup \gamma_n^{-1}(B_n)$$

holds; thus

$$\mu(C_n) \leq \mu(A_n) + \mu(\gamma_n^{-1}(B_n)) = \mu(A_n) + \mu(B_n)\,.$$

By hypothesis $\mu(A_n) + \mu(B_n)$ converges to zero as n goes to infinity, whence (i) follows from the last inequality. □

We are now able to prove the main result of the section.

Theorem 7.3. $(\mathcal{A}, \tau_d, \circ)$ *is a topological group.*

Proof: Assume we are given two sequences γ_n and β_n in $\mathcal{A}$, with $\gamma_n \to \gamma \in \mathcal{A}$ and $\beta_n \to \beta \in \mathcal{A}$. We have to show that $\beta_n \circ \gamma_n^{-1} \to \beta \circ \gamma^{-1}$ as $n \to \infty$. For each $n \in \mathbb{N}$, we have

$$\beta_n \circ \gamma_n^{-1} = \beta \circ (\beta^{-1} \circ \beta_n) \circ (\gamma_n^{-1} \circ \gamma) \circ \gamma^{-1} .$$

From Lemma 7.1 (ii) we obtain $\beta^{-1} \circ \beta_n \to e$ and $\gamma^{-1} \circ \gamma_n \to e$, whence, by Lemma 7.2 (ii), $\gamma_n^{-1} \circ \gamma \to e$. Therefore, Lemma 7.2 (i) entails the convergence

$$\beta^{-1} \circ \beta_n \circ \gamma_n^{-1} \circ \gamma \longrightarrow e .$$

Finally, apply twice Lemma 7.1 to get

$$\beta \circ (\beta^{-1} \circ \beta_n \circ \gamma_n^{-1} \circ \gamma) \circ \gamma^{-1} \longrightarrow \beta \circ \gamma^{-1} ,$$

that is,

$$\beta_n \circ \gamma_n^{-1} \longrightarrow \beta \circ \gamma^{-1} ,$$

as desired. □

At this point it would be interesting to investigate whether $\mathcal{E}$ can be given the structure of topological group with respect to some binary operation which extends the group structure of $\mathcal{A}$. However, the answer is negative.

Proposition 7.4. *There exists no binary operation which makes $(\mathcal{E}, \tau_d)$ into a topological group for which $(\mathcal{A}, \tau_d, \circ)$ is a subgroup.*

Proof: Assume that such an operation $* : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ exists, that is, $(\mathcal{A}, \circ) \sqsubset (\mathcal{E}, *)$. Select the element $\beta \in \mathcal{E}$ defined by $\beta(x) = 1-|1-2x|$, and consider the approximating sequence $\beta_n \in \mathcal{S} \subset \mathcal{A}$ as in Example 6.5. Since $\mathcal{E}$ is a topological group, β_n^{-1} must converge to an element $\gamma \in \mathcal{E}$, which is the $*$-inverse of β. On the other hand, as we remarked before, the convergence of β_n to β is actually uniform. As a consequence (recall that $\beta_n^{-1} \in \mathcal{S}$ as well), the graph of γ is contained in the set

$$\left\{(x,y) \in [0,1)\times(0,1) : y = \frac{x}{2}\right\} \cup \left\{(x,y) \in [0,1)\times(0,1) : y = -\frac{x}{2}+1\right\} .$$

We conclude that γ is an injective element of $\mathcal{E}$, and therefore, by Theorem 2.2, $\gamma \in \mathcal{A}$. Hence the $*$-inverse and the $\circ$-inverse of γ coincide, that is, $\beta = \gamma^{-1} \in \mathcal{A}$. This leads to a contradiction, since $\beta \notin \mathcal{A}$. □

Remark 7.5. It is actually possible to show even more, namely, that $\mathcal{A}$ admits no extension $\mathcal{A}'$ which is a topological group in the topology induced by convergence in measure. Indeed, it is immediate to check that the sequence β_n^{-1} of Proposition 7.4 is not a Cauchy sequence.

According to Theorem 7.3, $(\mathcal{A}, d, \circ)$ is a metric group; it is not complete as a metric space (since its d-completion is $(\mathcal{E}, d, \circ)$ by Theorem 6.1). It is possible, however, to construct a complete metric d_* on $\mathcal{A}$ which is topologically equivalent to d. Namely, such a metric is given by

$$d_*(\gamma_1, \gamma_2) = d(\gamma_1, \gamma_2) + d(\gamma_1^{-1}, \gamma_2^{-1})\,.$$

This metric has the property that d-Cauchy sequences which do not converge to an element of $\mathcal{A}$ are not d_*-Cauchy. In particular, $\mathcal{S}$ is d_*-dense in $\mathcal{A}$. Hence we have:

Corollary 7.6. *$(\mathcal{A}, \tau_d, \circ)$ is a Polish group.*

Note that $(\mathcal{A}, \tau_d, \circ)$ is isomorphic to the Polish group $\mathrm{Aut}_{\bar\mu}(\mathfrak{A})$, where $(\mathfrak{A}, \bar\mu)$ is the *measure algebra* associated to $([0,1), \mu)$ (cf. [6, 7, 9]).

In the light of the results proved in the last two sections, the statement "$(\mathcal{S}, \circ)$ approximates $\mathcal{E}$ and $\mathcal{A}$ from below" can now be given a precise meaning: $(\mathcal{S}, \tau_d, \circ)$ is a topological group which has the topological monoid $(\mathcal{E}, \tau_d, \circ)$ as its completion with respect to the metric d, and the Polish group $(\mathcal{A}, \tau_d, \circ)$ as its completion with respect to the (topologically equivalent) metric d_*.

The analysis we have carried out so far concerns endomorphisms and automorphisms of $[0,1)$. As already pointed out, this is hardly a limitation. First, notice that there is no loss of generality if we consider sets of the form $[0,a)$, for every $a > 0$. Moreover, given any two measurable sets X and Y of the same measure $a > 0$, there exist by Theorem 2.4 $\gamma \in \mathrm{Iso}(X, [0,a))$ and $\beta \in \mathrm{Iso}([0,a), Y)$ such that

$$\mathrm{Hom}(X,Y) = \beta \circ a\mathcal{E} \circ \gamma \qquad \text{and} \qquad \mathrm{Iso}(X,Y) = \beta \circ a\mathcal{A} \circ \gamma\,,$$

where $a\mathcal{E} = \{a\gamma : \gamma \in \mathcal{E}\}$ and $a\mathcal{A} = \{a\gamma : \gamma \in \mathcal{A}\}$. Furthermore, if we define

$$\mathcal{R}^{X,Y} = \beta \circ a\mathcal{R} \circ \gamma \qquad \text{and} \qquad \mathcal{S}^{X,Y} = \beta \circ a\mathcal{S} \circ \gamma\,,$$

then, in view of Lemma 7.1, all the results of the paper hold if we replace $\mathcal{R}$, $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{E}$ with $\mathcal{R}^{X,Y}$, $\mathcal{S}^{X,Y}$, $\mathrm{Iso}(X,Y)$ and $\mathrm{Hom}(X,Y)$, respectively.

8. Conclusions

The purpose of this paper is to analyze some notions of approximation from below in the realm of measure-preserving functions. This topic has already been carefully studied in the literature, and the existing investigations have focused on concepts and tools which are very similar to those used in this paper. Nevertheless, our approach differs from the classical one in several aspects, since different are motivations and goals.

As stated in [7], the original motivation for the introduction of some notions of approximation into the theory of measure-preserving functions is ergodic, namely, to develop new tools in order to investigate some of the unsolved problems in ergodic theory. As consequence, the goal of these investigations has been to endow the space of automorphisms of the associated measure algebra with suitable topologies, in order to give a more precise meaning to the metamathematical statement "in general a measure-preserving transformation is ergodic". It has been shown (see [7, 8, 12]) that this statement is true in a Baire-categorial sense (i.e., the set of ergodic transformations is comeager) for some rather natural topologies. In the course of these topological investigations the concept of "permutation of intervals" has played an important role, since the set of such transformations is dense in the space of automorphisms endowed with the so-called neighborhood topology (see [7], Theorem 3).

Later on, a generalization of the notion of permutation of intervals, known as cutting and stacking method, has been fully developed. Again, the motivation for this approach is ergodic. In fact, the cutting and stacking method has been mainly (but not only, see, e.g., [2]) used to construct specific transformations on the unit interval: ergodic, ergodic and measure-preserving, mixing, weakly mixing but not mixing, etc. (see [5], Chapter 6).

On the other hand, the motivation for our approach is completely different. We aim at finding, if possible, some similarities between

the sets $\mathbb{R}$ and $\mathbb{N}$ with respect to their "internal" structure. Specifically, we wish to determine an algebraic object which plays for $\mathbb{R}$ the same role that the group $\mathcal{S}_\infty$ of permutations plays for $\mathbb{N}$. Our claim is that this object can be identified with $\mathcal{A}$, the group of automorphisms of $[0,1)$. With this goal in mind, we have determined an algebraic object - the group $\mathcal{S}$ - and a notion of approximation - the topology of convergence in measure - in order to describe from below the group $\mathcal{A}$. One should read the results obtained in Sections 4, 6 and 7 in this perspective: Theorem 4.3 provides us with a purely combinatorial representation of the algebraic object which is used for the description, whereas Theorem 6.1 and Theorem 7.3 show that the approximation procedure is effective for a meaningful topology.

Unfortunately, the main objective of our investigation is far from being achieved. In fact, the analogy between $\mathbb{R}$ and $\mathbb{N}$ would be strengthened if there were a (hopefully natural) group embedding of $\mathcal{S}_\infty$ into $\mathcal{A}$. But the existence of such an embedding seems to be difficult to establish. A possible way to approach indirectly this problem could be to view both $\mathcal{S}_\infty$ and $\mathcal{A}$ as embedded in the larger group of authomorphisms of $\mathrm{Pow}(\mathbb{N})/\mathrm{Fin}$, assuming suitable axioms (e.g., CH).

On the other hand, our approach has produced some interesting combinatorial byproducts, which we aim to investigate further. First of all, we will try to make more effective the representation of automorphisms as limit of finite permutations, using the already existing results on permutation groups as a guideline. In fact, the conjugacy classes in finite symmetric groups are well characterized by finite collections of natural numbers, which express the cycle structure of each class. In a similar way, we hope to succeed in giving a discrete (countable) spectrum for a special class of automorphisms. Note that the same approach has given useful results from a quite different perspective, using the fact that an automorphism, under appropriate conditions, can be interpreted as an operator from L^2 into itself generated by its eigenvalues (see, for instance, [15]).

Furthermore, we will also try to take advantage of the correspondence between elements of $\mathcal{S}$ and finite permutations of the natural numbers, with the objective of obtaining geometric methods to compute the order of particular elements of symmetric groups. Specifically, we seek to determine a formula that gives the order of

a (particular) element γ in $(\mathcal{S}, \circ)$ in terms of the parameters of the shifts of which γ is the composition. Such a formula would then automatically give the order of the corresponding permutation in S_n. This would be interesting in the light of the fact that many properties of the maximum order of an element in S_n (regarded as a function of n) are unknown (see, e.g., [4], Chapter 5). Notice that Theorem 5.5 is a result of this type, since every partial shift is a composition of total shifts of a certain type. Moreover, as observed at the end of Section 5, we have already some partial results related to other particular cases (composition of two total shifts having a certain form), and it seems realistic to extend these results to somewhat more complicated elements of $\mathcal{S}$. We hope that (i) the directed-colimit connection between the group $\mathcal{S}$ and the finite permutation groups $\mathcal{S}_n$ and (ii) the close topological relationship of $\mathcal{S}$ to $\mathcal{A}$ and $\mathcal{E}$ may provide additional tools for considering this problem.

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