Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY
PROCEEDINGS
Volume 25, Spring 2000
Pages 167-180

LINEARLY OPAQUE HOMEOMORPHISMS OF $\mathbb{R}^n$

DENISE M. HALVERSON AND DAVID G. WRIGHT

Abstract. We show that there is a homeomorphism of $\mathbb{R}^n$ onto itself such that the image of each line in $\mathbb{R}^n$ meets every line in $\mathbb{R}^n$. In addition we generalize this result to certain classes of non-linear curves and certain types of non-linear projections.

1. Introduction

In this paper we demonstrate the construction of a homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ isotopic to the identity such that the image of each line in $\mathbb{R}^n$ meets every line in $\mathbb{R}^n$ for $n \geq 2$. The fact that such homeomorphisms exist was first demonstrated by A.V. Kuzḿinykh in [2]. However, the construction is lenghty and relies heavily upon analytical methods. It is our goal to construct such a homeomorphism using more geometrical methods. In addition we provide a stronger generalization of this result than that given in [2]. In particular we show that for certain non-linear classes of curves $\mathcal{T}$ and $\mathcal{S}$ in $\mathbb{R}^n$ there are self-homeomorphisms of $\mathbb{R}^n$ isotopic to the identity such that the image of any curve in $\mathcal{T}$ meets every curve in $\mathcal{S}$.

2000 *Mathematics Subject Classification.* Primary 57N3; Secondary 57N45, 54C10.

Key words and phrases. homeomorphism, cantor sets.

2. Preliminaries

We first begin by introducing some terminology that will be used to restate the problem in a manner that lends itself to generalizations. For the remainder of this paper we will assume that n is fixed and that $\mathcal{L}$ denotes the set of all straight lines in $\mathbb{R}^n$.

Definition 2.1. Let $\mathcal{S}$ be a set of subsets in a space Y. A subset $Z \subset Y$ is said to be *$\mathcal{S}$-opaque in Y* if every element of $\mathcal{S}$ meets Z. A subset Z of $\mathbb{R}^n$ is said to be *linearly opaque* in $\mathbb{R}^n$ if Z is $\mathcal{L}$-opaque in $\mathbb{R}^n$.

Definition 2.2. Let $\mathcal{Z}$ be a set of subsets of a space X and $\mathcal{S}$ be a set of subsets of a space Y. Then a map $\psi : X \to Y$ is *$\mathcal{S}$-opaque with respect to $\mathcal{Z}$* if for every $Z \in \mathcal{Z}$, $\psi(Z)$ is $\mathcal{S}$-opaque in Y. A map $\psi : X \to \mathbb{R}^n$ is said to be *linearly opaque with respect to $\mathcal{Z}$* if ψ is $\mathcal{L}$-opaque with respect to $\mathcal{Z}$.

Let $S_r = \{x \in \mathbb{R}^n | \; \|x\| = r\}$. For $T \subset \mathbb{R}^+ = [0, \infty)$, let $S_T = \bigcup_{r \in T} S_r$.

Definition 2.3. The angular distance between two points $A, B \in \mathbb{R}^n - O$ is

$$\theta(A, B) = m\angle AOB$$

where O denotes the origin of $\mathbb{R}^n$. The r-level angular distance between two sets X and Y meeting S_r is

$$\theta_r(X, Y) = \inf\{\theta(A, B) \mid A \in S_r \cap X,\; B \in S_r \cap Y\}$$

Definition 2.4. The angular diameter of a nonempty set $X \subset \mathbb{R}^n$ is

$$\omega(X) = \sup\{\theta(A, B) \mid A, B \in X - O\}$$

If $X = \{O\}$, then $\omega(X) \equiv 0$.

Let

$$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}^n \;\mid f \text{ is proper}\}$$

and for $[a, b] \subset \mathbb{R}^+$ let

$$\mathcal{G}([a, b]) = \{f : [a, b] \to S_{[a,b]} \mid f \text{ is continuous and } f(t) \in S_t\}$$

Let $\mathcal{G}^*([a, b])$ denote the subset of $\mathcal{G}([a, b])$ whose images are straight line segments. Given a function $f \in \mathcal{F}$, define $f^+ : [0, \infty) \to \mathbb{R}^n$ such that $f^+(t) = f(t)$ and $f^- : [0, \infty) \to \mathbb{R}^n$ such that $f^-(t) = f(-t)$.

3. Main Construction

In this section we generalize a method of J.J. Dijkstra presented in [1] in order to prove our general theorem which will be the basis of all subsequent results. We begin by describing certain basic rotations of S_r in $\mathbb{R}^n$. For points $A, B \in S_r$ such that $A \neq -B$ there is a well defined rotation of $\mathbb{R}^n$ taking A to B which fixes the subspace of $\mathbb{R}^n$ orthogonal to the plane determined by A, O and B. Let $\phi[A, B]$ be the restriction of this rotation to S_r.

Theorem 3.1. *Suppose Λ is a compact subset of $\mathcal{G}([a, b]) \times \mathcal{G}([a, b])$ with the property that for any $(f, g) \in \Lambda$ and $t \in [a, b]$,*

$$\theta(f(t), g(t)) < \pi$$

Then there is a homeomorphism $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following:

(1) *Ψ is level preserving in the sense that $\Psi(S_t) = S_t$.*
(2) $\Psi|_{\mathbb{R}^n - S_{[a,b]}} = id$
(3) *Ψ is isotopic to the identity map.*
(4) *For any $(f, g) \in \Lambda$, $\Psi f([a, b]) \cap g([a, b]) \neq \emptyset$.*

Proof: Let C be a cantor set in the interior of $[a, b]$. Let $\psi : C \to \Lambda$ be a surjection. Let $C^* = (\mathbb{R}^+ - (a, b)) \cup C$. Define $\Psi : S_{C^*} \to S_{C^*}$ such that

$$\Psi|_{S_t} = \begin{cases} \phi[f(t), g(t)] & \text{if } t \in C \text{ and } \psi(t) = (f, g) \\ id & \text{if } t \in \mathbb{R}^+ - (a, b) \end{cases} \tag{1}$$

Thus any extension of Ψ satisfies conditions (2) and (4). To show that Ψ can be extended to all of $\mathbb{R}^n$ we first note that it follows from the compactness of Λ that there is a $\delta > 0$ such that for all $(f, g) \in \Lambda$ and $t \in [a, b]$

$$\theta(f(t), g(t)) \leq \pi - 2\delta$$

Choose $\eta > 0$ so that whenever $s, t \in C$, $|s - t| \leq \eta$, $\psi(t) = (f, g)$ and $\psi(s) = (f', g')$, then

$$\theta(f(t), f'(s)) < \delta \quad \text{and} \quad \theta(g(t), g'(s)) < \delta$$

Note that for all but a finite number of components $[c, d]$ of $\overline{\mathbb{R}^n - C^*}$, $|c - d| \leq \eta$.

Suppose that $[c, d]$ is a component of $\overline{\mathbb{R}^n - C^*}$ missing $\{a, b\}$ such that $|c - d| \leq \eta$, $\psi(c) = (f, g)$ and $\psi(d) = (f', g')$. Let $\alpha : [c, d] \to S_1$

be a minimal geodesic path from $\frac{f(c)}{c}$ to $\frac{f'(d)}{d}$ and $\beta : [c,d] \to S_1$ be a minimal geodesic path from $\frac{g(c)}{c}$ to $\frac{g'(d)}{d}$. Define $\tilde{\alpha} : [c,d] \to S_{[c,d]}$ such that $\tilde{\alpha}(t) = t\alpha(t)$ and $\tilde{\beta} : [c,d] \to S_{[c,d]}$ such that $\tilde{\beta}(t) = t\beta(t)$. Note that

$$\begin{aligned}
\theta(\tilde{\alpha}(t), \tilde{\beta}(t)) &\le \theta(\tilde{\alpha}(t), f(c)) + \theta(f(c), g(c)) + \theta(g(c), \tilde{\beta}(t)) \\
&= \theta(\alpha(t), \frac{f(c)}{c}) + \theta(f(c), g(c)) + \theta(\frac{g(c)}{c}, \beta(t)) \\
&\le \theta(\frac{f'(d)}{d}, \frac{f(c)}{c}) + \theta(f(c), g(c)) + \theta(\frac{g(c)}{c}, \frac{g'(d)}{d}) \\
&= \theta(f'(d), f(c)) + \theta(f(c), g(c)) + \theta(g(c), g'(d)) \\
&< \delta + (\pi - 2\delta) + \delta = \pi
\end{aligned}$$

Therefore we may extend Ψ to $S_{[c,d]}$ such that

$$\Psi|_{S_t} = \phi[\tilde{\alpha}(t), \tilde{\beta}(t)] \tag{2}$$

Now suppose that $[c,d]$ is a component of $\overline{\mathbb{R}^n - C^*}$ missing $\{a,b\}$ such that $|c-d| > \eta$, $\psi(c) = (f,g)$, $\psi(d) = (f',g')$ and m is the midpoint of $[c,d]$. Let $\mu : [c,m] \to S_1$ be a minimal geodesic path from $\frac{g(c)}{c}$ to $\frac{f(c)}{c}$ and $\nu[m,d] \to S_1$ be a minimal geodesic path from $\frac{f'(d)}{d}$ to $\frac{g'(d)}{d}$. Define $\tilde{\mu} : [c,m] \to S_{[c,m]}$ such that $\tilde{\mu}(t) = t\mu(t)$ and $\tilde{\nu} : [m,d] \to S_{[m,d]}$ such that $\tilde{\nu}(t) = t\nu(t)$. Extend Ψ to $S_{[c,d]}$ such that

$$\Psi|_{S_t} = \begin{cases} \phi[\frac{t}{c} f(c), \tilde{\mu}(t)] & \text{if } t \in [c,m] \\ \phi[\frac{t}{d} f'(d), \tilde{\nu}(t)] & \text{if } t \in [m,d] \end{cases} \tag{3}$$

Note that $\Psi|_{S_m}$ is the identity map.

Now let $[a,d]$ and $[c,b]$ be the two remaining components of $\overline{\mathbb{R}^n - C^*}$. Suppose $\psi(d) = (f,g)$ and $\psi(c) = (f',g')$. Let $\mu : [a,d] \to S_1$ be a minimal geodesic path from $\frac{f(d)}{d}$ to $\frac{g(d)}{d}$ and $\nu[c,b] \to S_1$ be a minimal geodesic path from $\frac{g'(c)}{c}$ to $\frac{f'(c)}{c}$. Define $\tilde{\mu} : [a,d] \to S_{[a,d]}$ such that $\tilde{\mu}(t) = t\mu(t)$ and $\tilde{\nu} : [c,b] \to S_{[c,b]}$ such that $\tilde{\nu}(t) = t\nu(t)$. Extend Ψ to $S_{[a,d] \cup [c,b]}$ such that

$$\Psi|_{S_t} = \begin{cases} \phi[\frac{t}{d} f(d), \tilde{\mu}(t)] & \text{if } t \in [a,d] \\ \phi[\frac{t}{c} f'(c), \tilde{\nu}(t)] & \text{if } t \in [c,b] \end{cases} \tag{4}$$

It should be clear that Ψ is isotopic to the identity map by a level preserving isotopy. In particular, given $A, B \in S_t$ such that $A \neq -B$, define $\Phi[A,B] : S_t \times [0,1]$ so that for $\tau \in [0,1]$ then $\Phi[A,B]|_{S_t \times \tau} = \phi[A,C]$ where C is the unique point between A and B on S_t such that $m\angle AOC + m\angle COB = m\angle AOB$ and $m\angle AOC = \tau(m\angle AOB)$. For $t \in [a,b]$ define A_t and B_t to be the points such that $\Psi|_{S_t} \equiv \phi[A_t, B_t]$ in equations (1), (2), (3) and (4). Then an isotopy between Ψ and the identity map on $\mathbb{R}^n$ is given by $\tilde{\Psi} : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ such that

$$\tilde{\Psi}|_{S_t \times [0,1]} = \begin{cases} \Phi[A_t, B_t] & \text{if } t \in [a,b] \\ id & \text{otherwise} \end{cases} \tag{5}$$

□

Theorem 3.2. *Suppose the following are given:*

(1) $\{R_1, R_2, R_3, \ldots\}$ *such that* $0 < R_1 < R_2 < R_3 < \ldots$.
(2) $\{\delta_1, \delta_2, \delta_3, \ldots\}$ *such that* $0 < \delta_k < R_{k+1} - R_k$.
(3) Λ_k *is a compact subset of* $\mathcal{G}([R_k, R_k+\delta_k]) \times \mathcal{G}([R_k, R_k+\delta_k])$ *with the property that for any* $(f,g) \in \Lambda_k$ *and* $t \in [R_k, R_k + \delta_k]$,
$$\theta(f(t), g(t)) < \pi$$
(4) Ω *is a set of ordered pairs of subsets of* $\mathbb{R}^n$ *with the property that for each* $(Z, Z') \in \Omega$, *there is a* k *and* $(f,g) \in \Lambda_k$ *such that* $im(f) \subset Z$ *and* $im(g) \subset Z'$.

Then there is a homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ *satisfing the following:*

(1) Ψ *is level preserving in the sense that* $\Psi(S_t) = S_t$
(2) $\Psi|_{\mathbb{R}^n - \bigcup S_{[R_k, R_k+\delta_k]}} = id$
(3) Ψ *is isotopic to the identity map*
(4) $\Psi(Z) \cap Z' \neq \emptyset$ *for any* $(Z, Z') \in \Omega$.

Proof: Let Ψ_k be the homeomorphism satisfying the conclusions of Theorem 3.1 for Λ_k. Then $\lim_{i\to\infty} \Psi_i \circ \ldots \circ \Psi_3 \circ \Psi_2 \circ \Psi_1$ is the desired homeomorphism. □

4. Linear Applications

In this section we demonstrate our main theorem.

Theorem 4.1. *For $n \geq 2$ there is a level preserving homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ isotopic to the identity such that the image of every element of $\mathcal{L}$ is linearly opaque in $\mathbb{R}^n$.*

Proof: We claim that the conditions of Theorem 3.2 are satisfied with the following set of data:

(1) $\{R_1, R_2, R_3, \ldots\}$ is any unbounded increasing sequence of positive numbers.

(2) $\{\delta_1, \delta_2, \delta_3, \ldots\}$ is any sequence such that $0 < \delta_k < R_{k+1} - R_k$.

(3) Λ_k is the subset of $\mathcal{G}^*([R_k, R_k + \delta_k]) \times \mathcal{G}^*([R_k, R_k + \delta_k])$ consisting of all elements (f, g) such that

$$\omega(im(f) \cup im(g)) \leq \beta$$

where β is some value such that $\frac{\pi}{2} < \beta < \pi$.

(4) $\Omega = \mathcal{L} \times \mathcal{L}$.

It will follow that the homeomorphism promised by Theorem 3.2 for this set of data is a linearly opaque homeomorphism with respect to $\mathcal{L}$ and is isotopic to the identity.

It should be clear that conditions (1), (2) and (3) of Theorem 3.2 are satisfied. Condition (4) will follow from Lemma 4.5 which is a result of the next three propositions.

Proposition 4.2. *Let $\{A, B\} \subset X \cap S_t$ and Y a set meeting S_t. Then $\theta_t(X, Y) \leq \pi - \frac{\alpha}{2}$ where $\alpha = \theta(A, B)$.*

Proof: Let P be the plane determined by the points A, O and B where O is the origin of $\mathbb{R}^n$. Let m be the line through O parallel to $\overline{AB}$ in P Then m meets S_t in two points, C and D. From the geometry in P it follows that $\theta_t(\{A, B\}, \{C, D\}) = \frac{\pi - \alpha}{2}$. Since C and D are antipodal points then it should be clear that $\theta_t(\{C, D\}, Y) \leq \frac{\pi}{2}$. Hence

$$\begin{aligned} \theta_t(X, Y) &\leq \theta_t(\{A, B\}, Y) \\ &\leq \theta_t(\{A, B\}, \{C, D\}) + \theta_t(\{C, D\}, Y) \\ &\leq \frac{\pi}{2} + \frac{\pi - \alpha}{2} = \pi - \frac{\alpha}{2} \end{aligned}$$

□

Proposition 4.3. *Suppose l is a line in $\mathbb{R}^n$ having distance r to the origin O and X is a set meeting S_R for $R \geq r$. Then $\theta_R(l,X) \leq \frac{\pi}{2} + \sin^{-1}\frac{r}{R}$.*

Proof: Let $\{A,B\} = l \cap S_R$ and $\alpha = \theta(A,B)$. Let P be the plane determined by A, B and O. Let l' be the line through O parallel to l in P. Choose $C \in l' \cap S_R$ nearest A. Then $\pi - \alpha = 2m\angle AOC = 2sin^{-1}\frac{r}{R}$. Therefore $\alpha = \pi - 2sin^{-1}\frac{r}{R}$. By Lemma 4.2, $\theta_R(l,X) \leq \pi - \frac{\alpha}{2} \leq \pi - (\frac{\pi}{2} - sin^{-1}\frac{r}{R}) = \frac{\pi}{2} + \sin^{-1}\frac{r}{R}$. □

Proposition 4.4. *Suppose l is a line in $\mathbb{R}^n$ having distance r to the origin O, $R' \geq R \geq r$ and X is a component of $l \cap S_{[R,R']}$. Then $\omega(X) \leq \sin^{-1}\frac{r}{R}$.*

Proof: From the geometry in the plane determined by l and O it should be clear that $\omega(X) + \cos^{-1}\frac{r}{R} \leq \frac{\pi}{2}$. Hence $\omega(X) \leq \frac{\pi}{2} - \cos^{-1}\frac{r}{R} = \sin^{-1}\frac{r}{R}$. □

Lemma 4.5. *Suppose l and l' are lines in $\mathbb{R}^n$ having distance r and r' to the origin O, respectively, and $R' \geq R \geq r \geq r'$. Then there are components X and Y of $l \cap S_{[R,R']}$ and $l' \cap S_{[R,R']}$, respectively such that*

$$\omega(X \cup Y) \leq \frac{\pi}{2} + 3\sin^{-1}\frac{r}{R}$$

Proof: From Proposition 4.3 it follows that there are components X and Y of $l \cap S_{[R,R']}$ and $l' \cap S_{[R,R']}$, respectively such that $\theta_R(X,Y) \leq \frac{\pi}{2} + \sin^{-1}\frac{r}{R}$. From Proposition 4.4, $\omega(X) \leq \sin^{-1}\frac{r}{R}$ and $\omega(Y) \leq \sin^{-1}\frac{r'}{R}$. Hence

$$\begin{aligned}\omega(X,Y) &\leq \omega(X) + \theta_R(X,Y) + \omega(Y)\\ &\leq \sin^{-1}\frac{r}{R} + (\frac{\pi}{2} + \sin^{-1}\frac{r}{R}) + \sin^{-1}\frac{r'}{R}\\ &\leq \frac{\pi}{2} + 3\sin^{-1}\frac{r}{R}\end{aligned}$$

□

Thus to conclude the proof of Theorem 4.1 we note that if l and l' are within a distance r to O, then for sufficiently large k, $\frac{\pi}{2} + 3\sin^{-1}\frac{r}{R_k} < \beta$. Therefore there is an $(f,g) \in \Lambda_k$ such that $im(f) \subset l$ and $im(g) \subset l'$. It follows that condition (4) of Theorem 3.2 is satisfied and we have proven our main theorem. □

5. Outward Applications

We will now demonstrate two types of generalizations to Theorem 4.1 which apply to classes of functions which are not necessarily linear.

Definition 5.1. Suppose $J \subset \mathbb{R}$. Then a map $f : J \to \mathbb{R}^n$ is *outward* if f is a proper embedding and there is a constant $0 \leq \kappa < \frac{\pi}{2}$ such that for any $t \in J$ there exists a constant $\delta > 0$ so that whenever $t - \delta < t' < t'' < t + \delta$ then the angle between $f(t')$ and $f(t'') - f(t')$ is less than or equal to κ. The value κ is called an *outward constant* for f.

In order to avoid ambiguity we define the angle between the origin O and any element of $\mathbb{R}^n$ to be 0.

Definition 5.2. A proper map $f : [0, \infty) \to \mathbb{R}^n$ is *eventually outward* if there exist a compact set C such that f is outward on each bounded subset of $[0, \infty) - f^{-1}(C)$. A map $f : \mathbb{R} \to \mathbb{R}^n$ is said to be *eventually outward* if both f^+ and f^- are eventually outward.

Recall that a map is proper if the preimage of any compact set is compact. Hence if $f : [0, \infty) \to \mathbb{R}^n$ is eventually outward, then $f([0, \infty))$ is unbounded.

Definition 5.3. A proper map $f : [0, \infty) = [0, \infty) \to \mathbb{R}^n$ is *absolutely eventually outward* if there exist a compact set C such that f is outward on $[0, \infty) - f^{-1}(C)$. A map $f : \mathbb{R} \to \mathbb{R}^n$ is said to be *absolutely eventually outward* if both f^+ and f^- are absolutely eventually outward.

Let $\mathcal{EO}$ denote the set of all eventually outward maps $f : \mathbb{R} \to \mathbb{R}^n$ and $\mathcal{AEO}$ denote the set of all absolutely eventually outward maps $f : \mathbb{R} \to \mathbb{R}^n$. Note that $\mathcal{AEO} \subset \mathcal{EO}$.

Definition 5.4. A map $f : [0, \infty) \to \mathbb{R}^n$ is *eventually radially increasing* if there is an $N \geq 0$ such that for any $t, t' \geq N$, $\|f(t')\| > \|f(t)\|$ whenever $t' > t$.

Proposition 5.5. *If $f : [0, \infty) \to \mathbb{R}^n$ is eventually outward, then f is eventually radially increasing.*

Proof: Choose C such that f is outward on $[0, \infty) - f^{-1}(C)$. Choose N so that $[N, \infty) \subset [0, \infty) - f^{-1}(C)$. Let $M > N$ and κ be an outward constant for f on $[N, M]$. Let δ be a Lebesgue number for a covering of $[N, M]$ consisting of neighborhoods satisfying the outward condition for κ. Suppose $N \leq t < t' \leq M$. Let $t = t_0 < t_1 < \ldots < t_m = t'$ so that $|t_i - t_{i+1}| < \delta$. Then,

$$\begin{aligned} \|f(t_{i+1})\|\|f(t_i))\| - \|f(t_i))\|^2 &\geq f(t_{i+1}) \cdot f(t_i) - f(t_i) \cdot f(t_i) \\ &= (f(t_{i+1}) - f(t_i)) \cdot f(t_i) \\ &= \|f(t_{i+1}) - f(t_i))\|\|f(t_i))\| \cos(\kappa) \end{aligned}$$

Dividing through by $\|f(t_i))\|$ and subtracting through by $\|f(t_i))\|$ we obtain

$$\|f(t_{i+1})\| \geq \|f(t_i))\| + \|f(t_{i+1}) - f(t_i))\| \cos(\kappa)$$

Noting the fact that f is an embedding on $[t, t']$ and $0 \leq \kappa < \frac{\pi}{2}$ we have

$$\|f(t_{i+1})\| > \|f(t_i))\|$$

Since the inequality holds for all $M > N$, then f is eventually radially increasing. □

Let $\mathcal{O}_\kappa([a, b])$ denote the subset of $\mathcal{G}([a, b])$ consisting of all outward functions which have outward constant κ. Let $\overline{\mathcal{O}}_\kappa([a, b])$ denote the closure of $\mathcal{O}_\kappa([a, b])$ in $\mathcal{G}([a, b])$.

Proposition 5.6. *$\overline{\mathcal{O}}_\kappa([a, b])$ is compact. Furthermore, if $f \in \overline{\mathcal{O}}_\kappa([a, b])$, then*

$$diam(f([a, b])) < \frac{b - a}{\cos \kappa}$$

Proof: Given $f \in \mathcal{O}_\kappa([a, b])$ and $[t, t'] \subset [a, b]$, let δ be a Lebesgue number for a covering of $[t, t']$ consisting of neighborhoods satisfying the outward condition for κ. Choose a partition of $[t, t']$ in the form $t = t_0 < t_1 < \ldots < t_m = t'$ so that $|t_i - t_{i-1}| < \delta$. Let θ_i denote the angle between $f(t_i) - f(t_{i-1})$ and $f(t_{i-1})$. (If $t_0 = 0$, let $\theta_1 = 0$.)

Then,

$$\begin{aligned}\|f(t') - f(t)\| &\leq \sum_{i=1}^{m} \|f(t_i) - f(t_{i-1})\| \\ &\leq \sum \frac{t_i - t_{i-1}}{\cos\theta_i} \\ &\leq \sum \frac{t_i - t_{i-1}}{\cos\kappa} \\ &= \frac{t' - t}{\cos\kappa}\end{aligned}$$

Thus

$$\frac{t' - t}{\cos\kappa} \geq \|f(t') - f(t)\| \tag{6}$$

Note that Equation (6) holds for all $f \in \mathcal{O}_\kappa([a,b])$ and all $[t,t'] \subset [a,b]$. It follows that

(1) Equation (6) holds for all $f \in \overline{\mathcal{O}}_\kappa([a,b])$ and all $[t,t'] \subset [a,b]$.
(2) $\overline{\mathcal{O}}_\kappa([a,b])$ is equicontinuous.
(3) $\overline{\mathcal{O}}_\kappa([a,b])$ is compact (Ascoli's Theorem).
(4) For all $f \in \overline{\mathcal{O}}_\kappa([a,b])$

$$diam(f([a,b])) < \frac{b-a}{\cos\kappa}$$

□

Definition 5.7. A function $f \in \mathcal{F}$ is said to have *split ends* if there exist $\epsilon > 0$ and a number N so that whenever $s, t \geq N$ then $dist(f^+(s), f^-(t)) > \epsilon$.

Theorem 5.8. *Suppose $\mathcal{M}$ and $\mathcal{M}'$ are subsets of $\mathcal{AEO}$ at least one of which consists of functions with split ends. Then there is a level preserving homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ isotopic to the identity such that the image of every element of $\mathcal{M}$ is $\mathcal{M}'$-opaque in $\mathbb{R}^n$.*

Proof: We claim that the conditions of Theorem 3.2 are satisfied with the following data:

(1) $\{R_1, R_2, R_3, \ldots\}$ is any unbounded increasing sequence of positive numbers.

(2) $\{\delta_1, \delta_2, \delta_3, \ldots\}$ is a positive sequence such that

$$\delta_k < min\{R_{k+1} - R_k, \frac{1}{2^{k+1}R_k}\}$$

(3) $\{\kappa_1, \kappa_2, \kappa_3, \ldots\}$ is a positive sequence such that $0 \leq \kappa_k < \frac{\pi}{2}$ and $\lim_{k\to\infty} \kappa_k = \frac{\pi}{2}$ and Λ_k is the subset of $\overline{\mathcal{O}}_{\kappa_k}([R_k, R_k+\delta_k]) \times \overline{\mathcal{O}}_{\kappa_k}([R_k, R_k+\delta_k])$ consisting of all elements (f, g) such that

$$\omega(im(f) \cup im(g)) \leq \pi(1 - \frac{1}{2^k R_k})$$

(4) $\Omega = \mathcal{M} \times \mathcal{M}'$

It should be clear that conditions (1) and (2) of Theorem 3.2 are satisfied. The compactness of Λ_k follows from Proposition 5.6 so condition (3) is also easily verified. Therefore it suffices to show that condition (4) is satisfied.

Lemma 5.9. *Suppose $f, g \in \mathcal{AEO}$ and f is a function with split ends. Then there is a number $j > 0$ such that whenever $k \geq j$ then there is a component X of $im(f) \cap S_{[R_k, R_k+\delta_k]}$ and a component Y of $im(g) \cap S_{[R_k, R_k+\delta_k]}$ such that,*

$$\omega(X \cup Y) \leq \pi(1 - \frac{1}{2^k R_k})$$

Proof: Let C be a compact set in $\mathbb{R}^n$ such that f^+, f^-, g^+ and g^- are outward on $[0, \infty) - f^{-1}(C)$ with outward constant κ. Let $\gamma > 0$ and choose N such that

$$\gamma < dist(f^+(s), f^-(t))$$

whenever $s, t \geq N$. Choose k so that $R_k \geq N$, $C \subset B_{R_k}$ and $(2^{k-1}\gamma - \pi)\cos\kappa > 1$. Let Y be a component of $im(g) \cap S_{[R_k, R_k+\delta_k]}$. Also let $A = im(f^+) \cap S_{R_k}$ and $B = im(f^-) \cap S_{R_k}$. Note that $\theta(A, B) \geq \frac{\gamma}{R_k}$. Thus by Proposition 4.2,

$$\begin{aligned}\theta_{R_k}(A \cup B, Y) &\leq \pi - \frac{\theta(A, B)}{2} \\ &\leq \pi - \frac{\gamma}{2R_k}\end{aligned}$$

Let X be the component of A or B of $im(f) \cap S_{[R_k, R_k+\delta_k]}$ such that

$$\theta_{R_k}(X, Y) \le \pi - \frac{\gamma}{2R_k}$$

Then by Proposition 5.6,

$$\begin{aligned} \omega(X) &\le \frac{\delta_k}{\cos\kappa} \\ &\le \frac{1}{2^{k+1}R_k \cos\kappa} \\ &\le \frac{2^{k-1}\gamma - \pi}{2^{k+1}R_k} \qquad \text{by choice of } k \\ &= \frac{\gamma}{4R_k} - \frac{\pi}{2^{k+1}R_k} \end{aligned}$$

Likewise

$$\omega(Y) \le \frac{\gamma}{4R_k} - \frac{\pi}{2^{k+1}R_k}$$

Therefore

$$\begin{aligned} \omega(X \cup Y) &\le \omega(X) + \theta_{R_k}(X, Y) + \omega(Y) \\ &\le (\frac{\gamma}{4R_k} - \frac{\pi}{2^{k+1}R_k}) + (\pi - \frac{\gamma}{2R_k}) + (\frac{\gamma}{4R_k} - \frac{\pi}{2^{k+1}R_k}) \\ &= \pi(1 - \frac{1}{2^k R_k}) \end{aligned}$$

□

It follows that condition (4) is satisfied and therefore Theorem 3.2 gives us the desired homeomorphism. □

Corollary 5.10. *For $n \ge 2$ there is a level preserving homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ isotopic to the identity such that the image of every element of $\mathcal{AEO}$ is linearly opaque in $\mathbb{R}^n$.*

Corollary 5.11. *For $n \ge 2$ there is a level preserving homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ isotopic to the identity such that the image of every element of $\mathcal{L}$ is $\mathcal{AEO}$-opaque.*

Theorem 5.12. *Suppose $\mathcal{M}$ and $\mathcal{M}'$ are subsets of $\mathcal{EO}$ at least one of which is a finite set of functions. Then there is a level preserving homeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ isotopic to the identity such that the image of every element of $\mathcal{M}$ is $\mathcal{M}'$-opaque in $\mathbb{R}^n$.*

Proof: Without loss of generality, assume that $\mathcal{M}$ is the finite set of functions. We claim that the conditions of Theorem 3.2 are satisfied for the following set of data:

(1) $\{R_1, R_2, R_3, \ldots\}$ is any unbounded increasing sequence of positive numbers such that the elements of $\mathcal{M}$ are outward away from $int(B_{R_1})$.

(2) Let $\{\kappa_1, \kappa_2, \kappa_3, \ldots\}$ be a positive sequence such that κ_k is an outward constant on $[R_k, R_{k+1}]$ for all $f \in \mathcal{M}$ and

$$\gamma_k = \min\{im(f) \cap (S_{R_k}) \mid f \in \mathcal{M}\}$$

Let $\{\delta_1, \delta_2, \delta_3, \ldots\}$ to be a sequence of positive numbers such that

$$\delta_k < \min\{R_{k+1} - R_k, \frac{\gamma_k \cos \kappa_k}{6R_k}\}$$

(3) Λ_k is the subset of $\overline{\mathcal{O}}_{\kappa_k}([R_k, R_k + \delta_k]) \times \overline{\mathcal{O}}_{\kappa_k}([R_k, R_k + \delta_k])$ consisting of all elements (f, g) such that

$$\omega(im(f) \cup im(g)) \leq \pi - \frac{\gamma_k}{6R_k}$$

(4) $\Omega = \mathcal{M} \times \mathcal{M}'$ or $\Omega = \mathcal{M}' \times \mathcal{M}$

Note that from the choice of R_1, $\gamma_k > 0$ for all k. Hence conditions (1) and (2) of Theorem 3.2 are satisfied for this set of data. Condition (3) follows from Proposition 5.6. To show that condition (4) is also satisfied let $(f, g) \in \Omega$. As in Lemma 5.9 it follows from Proposition 4.3 that there are components X of $im(f) \cap [R_k, R_k + \delta_k]$ and Y of $im(g) \cap [R_k, R_k + \delta_k]$ such that

$$\theta_{R_k}(X, Y) \leq \pi - \frac{\gamma_k}{2R_k}$$

Furthermore, by Proposition 5.6,

$$\omega(X) \leq \frac{\delta_k}{\cos \kappa} \leq \frac{\gamma_k}{6R_k}$$

Likewise

$$\omega(Y) \leq \frac{\gamma_k}{6R_k}$$

Therefore

$$\begin{aligned}\omega(X\cup Y) &\leq \omega(X)+\theta_{R_k}(X,Y)+\omega Y\\ &\leq \frac{\gamma_k}{6R_k}+(\pi-\frac{\gamma_k}{2R_k})+\frac{\gamma_k}{6R_k}\\ &= \pi-\frac{\gamma_k}{6R_k}\end{aligned}$$

□

References

[1] J.J Dijkstra, T.L. Goodsell and D.G. Wright, *On Compacta with Convex Projections*, Topology Appl., **94** (1999), 67-74.

[2] A.V. Kuzḿinykh, *Curves and Discontinua with Paradoxical Geometric Properties*, Amer. Math. Soc. Transl., (2) **176** (1996), 53-85.

Brigham Young University, Provo, Utah 84602
E-mail address: deniseh@math.byu.edu, wright@math.byu.edu