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MONOTONE HOMOGENEOUS CONTINUA, AN EXAMPLE

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Abstract. A metric continuum is said to be *monotone homogeneous* provided that, for every two points $p, q \in X$, there exists a monotone onto map $h : X \to X$ such that $h(p) = q$. In this paper we construct an example of a monotone homogeneous continuum X that does not satisfy Kelley's property. This answers a question by J. J. Charatonik.

INTRODUCTION

A *continuum* is a compact connected metric space. The set of positive integers is denoted by $\mathbb{N}$. A *map* is a continuous function. An onto map between continua $f : X \to Y$ is said to be *monotone* if $f^{-1}(p)$ is connected for each $p \in Y$.

A continuum X is said to be *monotone* (resp., *open, monotone open*) *homogeneous* provided that for each pair of points $p, q \in X$, there exists a monotone (resp., open, monotone and open) map $f : X \to X$ such that $f(p) = q$.

It is known ([7, (2.2) Chapter VIII]) that a map between continua $f : X \to Y$ is monotone if and only if $f^{-1}(B)$ is a subcontinuum of X for each subcontinuum B of Y.

A continuum X is said to have the property of Kelley at a point $p \in X$, provided that if A is a subcontinuum of X, p is a point of A and $\{p_n\}_{n=1}^{\infty}$ is a sequence in X such that $p_n \to p$, then there exists a sequence of subcontinua $\{A_n\}_{n=1}^{\infty}$ of X such that $A_n \to A$

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and $p_n \in A_n$ for each $n \in \mathbb{N}$. The continuum X has the property of Kelley if X has the property of Kelley at each point of X.

In [6], R. W. Wardle proved that homogeneous continua have the property of Kelley. In fact, J. J. Charatonik in [2] proved that open homeogeneous continua have the property of Kelley. In [5] H. Kato showed that there is a confluent homogeneous continuum without the property of Kelley.

Answering a question by J. J. Charatonik ([2, p. 380]), in this paper we construct a continuum X which is monotone homogeneous and it does not have the property of Kelley.

CONSTRUCTION OF THE EXAMPLE

Let C denote the usual Cantor ternary set contained in the unit closed interval $[0,1]$. Let $\mathcal{D} = \{D \subset [0,1] : D$ is a component of $[0,1] - C\}$.
A partition $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ of the set $\mathcal{D}$ is called a *dense partition* of $\mathcal{D}$ provided that $C = \mathrm{Bd}(\bigcup \mathcal{D}_1) = \mathrm{Bd}(\bigcup \mathcal{D}_2)$. The following fact is easy to prove.

(1) If $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathcal{D} = \mathcal{D}_3 \cup \mathcal{D}_4$ are dense partitions of $\mathcal{D}$, then there exists a homeomorphism $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$ and $\phi(\bigcup \mathcal{D}_1) = \bigcup \mathcal{D}_3$ (and then $\phi(C) = C$ and $\phi(\bigcup \mathcal{D}_2) = \bigcup \mathcal{D}_4$). Furthermore, if we fix elements $D_j \in \mathcal{D}_j$, $j = 1,2,3,4$, where all the elements of D_1 (resp., D_3) are on the left of all the elements of D_2 (resp., D_4), then the map ϕ can be chosen in such a way that $\phi(D_1) = D_3$ and $\phi(D_2) = D_4$.

From now on, we consider a fixed dense partition $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ of $\mathcal{D}$. For $j = 1,2$, put $U_j = \bigcup \mathcal{D}_j$. We ask that the set U_1 (and , then U_2) is symmetric with respect to the number $\frac{1}{2}$, and the component $D_0 = (\frac{1}{3}, \frac{2}{3}) \in \mathcal{D}_2$.

A point $p \in C$ is said to be a *left point of C* (resp., a *right point of C*) provided that $p = 0$ (resp., $p = 1$) or there exists an element $D \in \mathcal{D}$ such that $p = \sup D$ (resp., $p = \inf D$).

The following claim is easy to prove.

(2) If $p \in C$ is not a left (resp., right) point of C, then there exists a homeomorphism $\phi : [0,p] \to [0,1]$ (resp., $\phi : [p,1] \to [0,1]$)

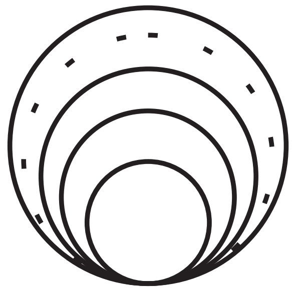

FIGURE 1

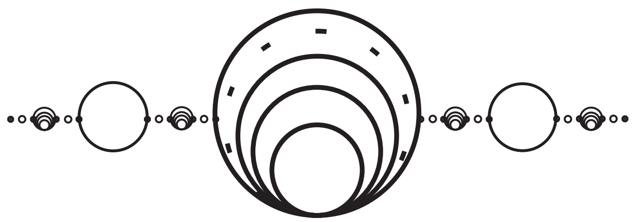

FIGURE 2

such that $\phi(p) = 1$ (resp., $\phi(p) = 0$), $\phi(U_1 \cap [0, p]) = U_1$ (resp., $\phi(U_1 \cap [p, 1]) = U_1$) and $\phi(C \cap [0, p]) = C$ (resp., $\phi(C \cap [p, 1]) = C$). Furthermore, if $D, E \in D_1$ and $D \subset [0, p]$ (resp., $D \subset [p, 1]$), then we may ask that $\phi(D) = E$.

For each $D \in \mathcal{D}$, let m_D be its middle point and let L_D be one half of its length. Let S_D be the circle in the Euclidean plane $\mathbb{R}^2$, with center at m_D and radius L_D.

For each $m \in \mathbb{N}$, let T_m be the circle in $\mathbb{R}^2$, with center at the point $(0, -\frac{1}{2^m})$ and radius $1 - \frac{1}{2^m}$. Let $T = \mathrm{cl}(\bigcup\{T_m : m \in \mathbb{N}\})$.

For each $D \in \mathcal{D}_2$, put $T_D = \{(m_D, 0) + L_D q \in \mathbb{R}^2 : q \in T\}$. Next, we construct a sequence of continua $X_1 \subset X_2 \subset ...$ in the following way: $X_1 = (C \times \{0\}) \cup (\bigcup\{S_D : D \in \mathcal{D}_1\}) \cup (\bigcup\{T_D : D \in \mathcal{D}_2\})$.

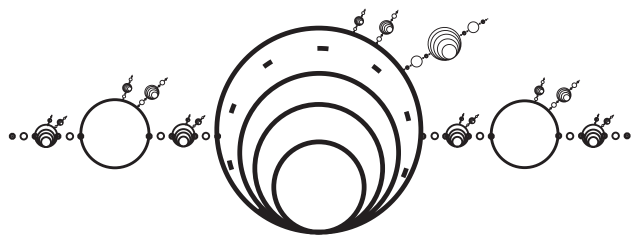

FIGURE 3

Now, suppose that X_n has been constructed. For each $m \in \mathbb{N}$, put

$$A_m = \{[(1,0) + \tfrac{1}{2^m} q] \cdot (\cos((\tfrac{\pi}{2})(1 - \tfrac{1}{2^m})) + i \sin((\tfrac{\pi}{2})(1 - \tfrac{1}{2^m}))) \in \mathbb{R}^2 : q \in X_n\},$$

where $\cdot$ is the product of the complex numbers.

And let

$$A_0 = \bigcup\{A_m : m \in \mathbb{N}\}.$$

Then define

$$X_{n+1} = X_n \cup (\bigcup\{\{(m_D, 0) + L_D a : a \in A_0\} \subset \mathbb{R}^2 : D \in \mathcal{D}\}).$$

This finishes the inductive construction.

Put

$$X_0 = \mathrm{cl}(\bigcup\{X_n : n \in \mathbb{N}\}).$$

For each circle G, contained in $\mathbb{R}^2$, let E_G be the sphere in the space $\mathbb{R}^3$ that contains the circle $G \times \{0\}$ as one of its maximal circles. For each $n \in \mathbb{N}$, let

$$Y_n = (X_n \times \{0\}) \cup (\bigcup\{E_G : G \text{ is a circle contained in } X_n\}).$$

Finally, define

$$X = (X_0 \times \{0\}) \cup (\bigcup\{E_G : G \text{ is a circle contained in } X_0\}).$$

Clearly, X is a continuum and X does not have the property of Kelley at the point $(\frac{1}{3}, 0, 0)$.

CONSTRUCTING HOMEOMORPHISMS FROM X ONTO X

Let $h : \mathbb{R}^3 \to \mathbb{R}^3$ be the homeomorphism given by $h(x, y, z) = (1-x, y, z)$. Since U_1 (and then U_2) is symmetric with respect to the number $\frac{1}{2}$, the map $h|Y_1 : Y_1 \to Y_1$ is a homeomophism. Applying h to the space X and making the appropriate adjustments on each of the spheres contained in Y_1, it is possible to extend the map $h|Y_1$ to a homeomorphism from X onto X. Thus there is a homeomorphism from X onto X that takes the point $(1, 0, 0)$ to the point $(0, 0, 0)$.

The following is clear from the way that X_0 was constructed.

(3) Let E be a component of $X - Y_1$, let p and q be the points in E such that $\{p\} = \mathrm{Bd}_X(E)$ and q is the farthest point from p in $\mathrm{cl}(E)$. Then there exists a homeomorphism $\phi : \mathrm{cl}(E) \to X$ such that $\phi(p) = (0, 0, 0)$ and $\phi(q) = (1, 0, 0)$.

The next claim follows from (2).

(4) If $p \in C$ is not a left (resp., right) point of C and J_p is the closure of the component of $X - \{(p, 0, 0)\}$ that contains $(0, 0, 0)$ (resp., $(1, 0, 0)$), then there exists a homeomorphism $\phi : J_p \to X$ such that $\phi(0, 0, 0) = (0, 0, 0)$ (resp., $\phi(p, 0, 0) = (0, 0, 0)$) and $\phi(p, 0, 0) = (1, 0, 0)$ (resp., $\phi(1, 0, 0) = (1, 0, 0)$). Furthermore, if $D = (d_1, d_2)$, $E = (e_1, e_2) \in \mathcal{D}_1$ and $D \subset [0, p]$ (resp., $D \subset [p, 1]$), then we may ask that $\phi(d_1, 0, 0) = (e_1, 0, 0)$.

Given a sphere E, contained in X, we make the following conventions:

-the *inside* of E (inside(E)) is the intersection of the solid sphere bounded by E with X,

-the sphere E is said to be *void* if the inside of E is E and E is not contained in the inside of any other sphere contained in X,

-the *north pole* of E (resp. the *south pole* of E) is the point $p = (x, y, z)$ in E for which y takes the largest (resp., the lowest) value,

-the sphere E is said to be *uncovered* if E is not contained in the inside of any other sphere contained in X.

A point p in a continuum Z is said to be an *end-point* of Z provided that there exists a local basis $\mathcal{B}$ of p in Z such that the boundary of each element in $\mathcal{B}$ is a one-point set.

Put $p_0 = (1, 0, 0)$ and $\theta = (0, 0, 0)$

The following claim helps to construct many homeomorphisms from X onto X.

(5) Let $n \in \mathbb{N}$ and let D be a component of $X - Y_n$. Then:
(a) there exists an uncovered sphere E, contained in X, and there exists a point $z \in E$ such that $\{z\} = \mathrm{Bd}_X(D) = \mathrm{cl}(D) \cap E$ and $E \subset Y_n - Y_{n-1}$ (put $Y_0 = \emptyset$),
(b) there exists a unique component F of $X-$ inside(E) such that $\theta \notin F$ and $F \cap Y_n \neq \emptyset$ and there exists a point $w \in E$ such that $\{w\} = \mathrm{Bd}_X(F) = \mathrm{cl}(F) \cap E$,
(c) $\mathrm{cl}(D)$ and $\mathrm{cl}(F)$ are homeomorphic to X, the continua cl $(D) \cap Y_{n+1}$ and $\mathrm{cl}(F) \cap Y_n$ are homeomorphic to Y_1, $F \cap Y_{n-1} = \emptyset$ and $\mathrm{diam}(D \cup E \cup F) < \frac{1}{2^{n-1}}$,
(d) there exists a unique end-point p (resp., q) of Y_n (resp., Y_{n+1}) such that $p \in F$ (resp., $q \in D$),
(e) if $v \in X - \{p\}$ and $r \in \mathrm{cl}(D) \cap Y_{n+1}$ is a cut point of X, then there exists a homeomorphism $\phi : X \to X$ such that $\phi(F) = D$, $\phi(v)$ is in the component of $X - \{r\}$ that contains θ, $\phi(p) = q$, $\phi(z) = w$, $\phi(u) = u$ for each $u \in X - (D \cup E \cup F)$ and $||u - \phi(u)|| < \frac{1}{2^{n-1}}$ for each $u \in X$.

Properties (a), (b), (c) and (d) are clear from the construction of X. From (4) it follows that there exists a homeomorphism $g : \mathrm{cl}_X(F) \to \mathrm{cl}_X(D)$ such that $g(v)$ is in the component of $\mathrm{cl}(D) - \{r\}$ that contains z, $g(p) = q$ and $g(w) = z$. Making an appropiate homeomorphism in the inside of E it is possible to extend g to a homeomorphism $\phi : X \to X$ with the mentioned properties.

By successive applications of (5) the following property follows easily.

(6) Let $p \in Y_n$. Then there exists a homeomorphism $\phi : X \to X$ such that $\phi(p) \in Y_1$. Furthermore, if p is an end-point of Y_n, then we can ask that $\phi(p) = p_0$.

THE CONTINUUM **X** IS MONOTONE HOMOGENEOUS

First, we show that, for each $p \in X$, there exists an onto monotone map that takes p to p_0. Let $p \in X$ and let D be a component of $X - Y_1$ such that $p \notin D$. Let q be the unique point in $\mathrm{Bd}_X(D)$. Then q is a cut point of X and $X - D$ is connected. Let $f : X \to \mathrm{cl}_X(D)$ be the map that shrinks the continuum $X - D$ to the point q. By

(3), there is a homeomorphism $g : \mathrm{cl}(D) \to X$ such that $g(q) = \theta$. And, we know that there is a homeomorphism $k : X \to X$ such that $k(\theta) = p_0$. Then $k \circ g \circ f : X \to X$ is a monotone onto map such that $(k \circ g \circ f)(p) = p_0$.

Now, take an end-point p of X. It is easy to show that $p \in (X - \bigcup\{Y_n : n \in \mathbb{N}\}) \cup \{q \in X : q$ is an end-point of Y_n, for some $n \in \mathbb{N}\}$. We will find a homeomorphism $f : X \to X$ such that $f(p_0) = p$. In the case that p is an end-point of Y_n for some $n \in \mathbb{N}$, the existence of f follows from (6). Then we may assume that $p \in X - \bigcup\{Y_n : n \in \mathbb{N}\}$. Then there exists a sequence of subsets $D_1, D_2, ...$ of X such that $\{p\} = \bigcap\{D_n : n \in \mathbb{N}\}$ and, for each $n \in \mathbb{N}$, D_n is the component of $X - Y_n$ that contains p. This implies that $D_1 \supset D_2 \supset D_3 \supset ...$

By (5), for each $n \in \mathbb{N}$, there exists an uncovered sphere E_n and a point $z_n \in E_n$ such that $\{z_n\} = \mathrm{Bd}_X(D_n) = \mathrm{cl}_X(D_n) \cap E_n$ and $E_n \subset Y_n - Y_{n-1}$. Let F_n be the unique component of $X-$ inside(E_n) such that $\theta \notin F_n$, $F_n \cap Y_{n-1} = \emptyset$ and $F_n \cap Y_n \neq \emptyset$. Let $w_n \in E$ be such that $\{w_n\} = \mathrm{Bd}_X(F_n) = \mathrm{cl}(F_n) \cap E_n$, Let p_n (resp., q_n) be the unique end-point of Y_n (resp., Y_{n+1}) such that $p_n \in F_n$ (resp., $q_n \in D_n$). We also know that $\mathrm{cl}(D_n)$ and $\mathrm{cl}(F_n)$ are homeomorphic to X, $\mathrm{cl}(D_n) \cap Y_{n+1}$ is homeomorphic to Y_1 and $\mathrm{diam}(D_n \cup E_n \cup F_n) < \frac{1}{2^{n-1}}$.

Since $X-$ inside(E_1) has only two components that intersects Y_1, one of them contains θ and the other one contains p_0, and θ and p_0 are the only end-points of Y_1, we conclude that $p_0 \in F_1$ and $p_1 = p_0$.

Given $n \in \mathbb{N}$, since $D_{n+1} \cup E_{n+1} \cup F_{n+1}$ is a connected subset of $X - Y_n$ and it contains p, we have that $D_{n+1} \cup E_{n+1} \cup F_{n+1} \subset D_n$. Then $E_{n+1} \subset D_n \cap Y_{n+1}$. Since $\mathrm{cl}(D_n) \cap Y_{n+1}$ is homeomorphic to Y_1, there exists a cut point r_n of X such that $r_n \in D_n \cap Y_{n+1}$, r_n separates q_n and θ in X and $E_{n+1} \cup F_{n+1}$ is in the component of $X - \{r_n\}$ that contains q_n.

Choose a sequence of cut points $\{c_n\}_{n=1}^{\infty}$ of X such that $c_n \in Y_1$ for each $n \in \mathbb{N}$ and $\{||c_n - p_0||\}_{n=1}^{\infty}$ is a strictly decreasing sequence and converges to 0.

We will define, inductively, a sequence of homeomorphisms $f_1, f_2, ...,$ from X onto X, with the following properties:

-$f_n(F_n) = D_n$, for each $n \in \mathbb{N}$,

-$(f_n \circ ... \circ f_1)(p_0) = q_n$, for each $n \in \mathbb{N}$,
-if $u \in X - (D_n \cup E_n \cup F_n)$, then $f_n(u) = u$,
-for each $n \in \mathbb{N}$ and for each $u \in X$, $||u - f_n(u)|| < \frac{1}{2^{n-1}}$,
-for each $n \in \mathbb{N}$, $(f_n \circ ... \circ f_1)(c_n)$ is in the component of $X - \{r_n\}$ that contains θ.

By (5), there exists a homeomorphism $f_1 : X \to X$ such that $f_1(F_1) = D_1$, $f_1(c_1)$ is in the component of $X - \{r_1\}$ that contains θ, $f_1(p_0) = f_1(p_1) = q_1$, $f_1(w_1) = z_1$, $f_1(u) = u$ for each $u \in X - (D_1 \cup E_1 \cup F_1)$ and $||u - f_1(u)|| < 1$ for each $u \in X$. Now, suppose that $f_1, ..., f_n$ have been defined. Since c_n is a cut point of X, the point $(f_n \circ ... \circ f_1)(c_{n+1})$ is a cut point of X. Thus $(f_n \circ ... \circ f_1)(c_{n+1}) \neq p_n$. By (5), there exists a homeomorphism $f_{n+1} : X \to X$ such that $f(F_{n+1}) = D_{n+1}$, $f_{n+1}((f_n \circ ... \circ f_1)(c_{n+1}))$ is in the component of $X - \{r_{n+1}\}$ that contains θ, $f_{n+1}(p_{n+1}) = q_{n+1}$, $f_{n+1}(w_{n+1}) = z_{n+1}$, $f_{n+1}(u) = u$ for each $u \in X - (D_{n+1} \cup E_{n+1} \cup F_{n+1})$ and $||u - f_{n+1}(u)|| < \frac{1}{2^n}$ for each $u \in X$.

Since $D_{n+1} \cup E_{n+1} \cup F_{n+1} \subset D_n$, p_{n+1} is an end-point of Y_{n+1} and it belongs to D_n, we have that $p_{n+1} = q_n$. Therefore, $(f_{n+1} \circ ... \circ f_1)(p_0) = q_{n+1}$.

This completes the inductive construction.

Thus the sequence of homeomorphisms $\{g_n\}_{n=1}^{\infty}$, defined by $g_n = f_n \circ ... \circ f_1$, uniformly converges to a continuous function $f : X \to X$. Since each g_n is onto, f is also onto. Next, we prove that f is one-to-one. Let $n \in \mathbb{N}$ and let K_n (resp., L_n) be the component of $X - \{c_n\}$ that contains θ (resp., p_0). Then $g_n(\{c_n\} \cup L_n)$ is a connected subset of X that contains $g_n(p_0) = q_n$ and $g_n(c_n)$ is in the component of $X - \{r_n\}$ that contains θ. This implies that there exists a point $x \in L_n$ such that $g_n(x) = r_n$. Since g_n is one-to-one, $r_n \notin g_n(K_n)$. Since $\theta = g_n(\theta) \in g_n(K_n)$ and $g_n(K_n)$ is connected, we have that $g_n(K_n)$ is contained in the component of $X - \{r_n\}$ that contains θ. By the choice of r_n, $g_n(K_n)$ and $E_{n+1} \cup F_{n+1}$ are in different components of $X - \{r_n\}$. Since $r_n \in Y_{n+1}$ and $D_{n+1} \cap Y_{n+1} = \emptyset$, we have that $D_{n+1} \cup \{z_{n+1}\}$ is a connected subset of $X - \{r_n\}$ that intersects E_{n+1}. This implies that $g_n(K_n) \cap D_{n+1} = \emptyset$. Thus $g_n(K_n) \cap (D_{n+1} \cup E_{n+1} \cup F_{n+1}) = \emptyset$. Hence, $f_{n+1}(g_n(x)) = g_n(x)$ for each $x \in K_n$. Since $D_{n+2} \cup E_{n+2} \cup F_{n+2} \subset D_{n+1}$, $f_{n+2}(f_{n+1}(g_n(x))) = g_n(x)$ for each $x \in K_n$. Similarly, $f_m(g_n(x)) = g_n(x)$ for each $x \in K_n$ and for each $m \geq n+1$. Thus, $g_m(x) = g_n(x)$ for each $x \in K_n$ and for each

$m \geq n$. Therefore, $f(x) = g_n(x)$ for each $x \in K_n$. We are ready to show that f is one-to-one. Let x, y be points in X such that $x \neq y$. If $x, y \notin \{p_0\}$, then there exists $n \in \mathbb{N}$ such that $x, y \in K_n$. Since g_n is one-to-one, we conclude that $f(x) \neq f(y)$. Now, suppose that $x \neq p_0$ and $y = p_0$, then there exists $n \in \mathbb{N}$ such that $x \in K_n$. Then $f(x) = g_n(x) \in g_n(K_n)$. Thus $f(x) \notin D_{n+1} \cup E_{n+1} \cup F_{n+1}$. Hence, $f(x) \notin \mathrm{cl}(D_{n+1})$. On the other hand, $f(y) = \lim g_m(p_0) = \lim q_m$. Since, for each $m \geq n+1$, $q_m \in D_m \subset \mathrm{cl}(D_{n+1})$, This implies that $f(y) \in \mathrm{cl}(D_{n+1})$. Therefore, $f(x) \neq f(y)$. We have proved that f is one-to-one. Hence, f is a homeomorphism.

Since $f(p_0) \in \mathrm{cl}(D_{n+1}) \subset D_n$ for each $n \in \mathbb{N}$, we have that $f(p_0) = p$.

Now, take a point $p \in X$ such that p is not an end-point of X. We will show that there exists a monotone onto map $f : X \to X$ such that $f(p_0) = p$. Since p is not an end-point of X, there exists $n \in \mathbb{N}$ such that $p \in Y_n$. By (6), we may assume that $p \in Y_1 - \{\theta, p_0\}$. We consider three cases:

Case 1. $p = (x_0, 0, 0) \in C \times \{(0, 0)\}$. Then x_0 is not a left point of C or x_0 is not a right point of C. We analize the subcase that x_0 is not a left point, the other one is similar.

Since $x_0 < 1$, we can choose a number $x_0 < x_1 < 1$ such that:
-if $x_0 = \inf D$ for some $D \in \mathcal{D}_1$, then $x_1 = \inf E$ for some $E \in \mathcal{D}_1$,
-if $x_0 = \inf D$ for some $D \in \mathcal{D}_2$, then $x_1 = \inf E$ for some $E \in \mathcal{D}_2$,
-if $x_0 \notin \mathrm{cl}(D)$ for any $D \in \mathcal{D}$, then $x_1 \notin \mathrm{cl}(E)$ for any $E \in \mathcal{D}$.

Put $p_1 = (x_1, 0, 0)$. For $i = 0, 1$, let A_i (resp., B_i) be the closure of the component of $X - \{(x_i, 0, 0)\}$ that contains θ (resp., p_0). Using (4) it is possible to prove that there exists a homeomorpism from B_0 to B_1 that takes p to p_1 and p_0 to p_0.

Note that $A_1 \cap B_0$ is a continuum that contains one (in fact, many) end-point q of X. Let $\phi : X \to X/(A_1 \cap B_0)$ be the map that shrinks the continuum $A_1 \cap B_0$ to a point. Then ϕ is monotone. Note that $X/(A_1 \cap B_0)$ is homeomorphic to the space $(A_0 \cup B_1)/\{p, p_1\}$ (the space obtained by identifying the points p and p_1 in the space $A_0 \cup B_1$) by a homeomorphism that sends $A_1 \cap B_0$ to the element $\{p, p_1\}$. Since $(A_0 \cup B_1)/\{p, p_1\}$ is homeomorphic to $A_0 \cup B_0 = X$, by a homeomorphism that sends the element $\{p, p_1\}$ to p, we conclude that there is a monotone onto map $g : X \to X$ such that $g(q) = p$. Since we know that it is possible to send p_0

to q by a homeomorphism from X onto X, we conclude that there exists the needed map f for this case.

Case 2. There exists an uncovered sphere E, contained in X, such that $E \subset Y_1$ and $p \in E$. Let $C_1, C_2, ...$ be the components of $X-$ inside(E) such that $C_n \cap Y_1 = \emptyset$ for each $n \in \mathbb{N}$ and diameter$(C_1) >$ diameter$(C_2) > ....$ Let C_0 be the component of $X - E$ that contains p_0. The case that $p \in \text{cl}(C_n)$, for some $n \geq 0$, can be reduced to Case 1. Thus we may assume that $p \notin \bigcup\{\text{ cl}(C_n) : n \geq 0\}$. Let $p_1 \in E$ be such that $\{p_1\} = E \cap \text{ cl}(C_0)$. Let γ be an arc contained in E such that the end-points of γ are p and p_1 and $\gamma \cap (\bigcup\{\text{cl}(C_n) : n \in \mathbb{N}\}) = \emptyset$. Let p_2 (resp., p_3) be the south (resp., north) pole of E. If $p \neq p_2$ (resp., $p \neq p_3$) we assume that $p_2 \notin \gamma$ (resp., $p_3 \notin \gamma$). Consider a map $g : (\bigcup\{\text{cl}(C_n) : n \geq 0\}) \cup \gamma \to X$ such that $g(C_0 \cup \gamma) = \{p\}$ and $g|\text{ cl}(C_{n+1})$ is a homeomorphism between $\text{cl}(C_{n+1})$ and $\text{cl}(C_n)$ for each $n \geq 0$. It is easy to show that g can be extended to an onto monotone map $f : X \to X$. Thus $f(p_0) = p$.

Case 3. There exists a sphere E_0, contained in X, such that $p \in E_0 \subset Y_1$, p is not the south pole of E_0 and there exists another sphere E such that E is contained in X, $E_0 \subset$ inside(E), $E \subset Y_1$ and E is uncovered.

Let q be the south pole of E. Let $E_0, E_1, E_2, ...$ be the spheres, contained in X, such that $E_n \neq E$ and E_n is contained in the inside of E for each $n \geq 0$. Fix a point $q_1 = (x_1, 0, 0) \in C \times \{(0,0)\}$ such that there exists an element $D \in \mathcal{D}_1$ with the property that $x_1 = \inf D$ and $E \subset (-\infty, x_1) \times \mathbb{R} \times \mathbb{R}$. Let A (resp., B) be the closure of the component of $X - \{q_1\}$ that contains θ (resp. p_0).

Let F be the void sphere, contained in X, such that $q_1 \in F \subset Y_1$. Notice that $F \subset B$ and there exists a monotone retraction $r : B \to F$. By (4), there exists a homeomorphism $f_1 : A \to X$ such that $f_1(q_1) = p_0$. By the Case 2, there is a monotone onto map $f_2 : X \to X$ such that $f_2(p_0) = q$. Let $f_3 : F \to E_0$ be a homeomorphism such that $f_3(q_1) = q$ and $f_3(r(p_0)) = p$. Let $f_4 : X \to \text{cl}(X - E_0)$ be a homeomorphism such that $f_4|\text{ cl}(X - (E_0 \cup E_1 \cup ...))$ is the identity map on $\text{cl}(X - (E_0 \cup E_1 \cup ...))$ and $f_4(E_{n-1}) = E_n$ for each $n \in \mathbb{N}$.

Define $f : X \to X$ by:

$$f(u) = \begin{cases} f_4(f_2(f_1(u))), & \text{if } u \in A \\ f_3(r(u)), & \text{if } u \in B \end{cases}$$

Since $f_4(f_2(f_1(q_1))) = f_4(f_2(p_0)) = f_4(q) = q$ and $f_3(r(q_1)) = f_3(q_1) = q$, we have that f is well defined and continuous. Since $f(A) = \text{cl}(X - E_0)$ and $f(B) = E_0$, we have that f is onto. Since $f(p_0) = f_3(r(p_0)) = p$, $f(p_0) = p$. Since $A \cap B = \{q_1\}$, $f(q_1) = q$, $f(A) \cap f(B) = \{q\}$ and each map $f|A$ and $f|B$ is monotone, it follows that f is monotone.

This completes the proof that X is monotone homogeneous.

In [3], J. J. Charatonik and W. J. Charatonik proved the following result:

THEOREM ([3, Theorem 4]). If Y is a compact metric monotone open homogeneous space such that each pair of points can be mapped by monotone open mappings from Y to Y into connected sets of arbitrarily small diameter, then Y is locally connected.

They gave an example ([3, Example 5]) that shows that in this theorem monotone open homogeneity can not be replaced by open homogeneity and they asked ([3, Question 6]) if monotone open homogeneity can be replaced by monotone homogeneity. Next, we answer this question in the negative by using the continuum X constructed in this paper.

Take any two points p, q in X. Choose a component B of $X - Y_1$ such that $p, q \notin B$. Let $z \in Y_1$ be such that $\text{Bd}(B) = \{z\}$. Let $f_1 : X \to \text{cl}(B)$ be the monotone retraction that sends $X - B$ to the point z. Let $f_2 : \text{cl}(B) \to X$ be a homeomorphism (see **(3)**). Thus $f = f_2 \circ f_1$ is a monotone onto map such that $f(p) = f(q)$. And, clearly, X is not locally connected at the point $(\frac{1}{3}, 0, 0)$.

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