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ALMOST CONTINUOUS IMAGES OF $\mathbb{R}$ AND ∞-ODS

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Abstract. We investigate conditions under which almost Peano continua contain dense arc components. These conditions involve the existence of almost continuous functions from the real line onto continua and excluding ∞-ods from continua. It is also shown that a topological space of cardinality less than or equal to continuum which has a dense path component is the almost continuous image of the real line.

1. Introduction

All terminology can be found in the Definitions and Terminology section of this paper. We consider conditions on an almost continuous image X of $[0,1]$ or $\mathbb{R}$ which guarantee the existence of a dense arcwise connected subset in X. These results will involve excluding ∞-ods from a continuum X. In particular, we prove:

Theorem 1. *Let X be an almost continuous image of $[0,1]$. If X is a decomposable continuum which contains no $\infty - od$, then X has a dense arc-component.*

Theorem 2. *Let X be an almost continuous image of $\mathbb{R}$. If X is a continuum which contains no $\infty - od$, then X has a dense arc-component.*

Theorem 1 is no longer true if we do not assume that X is decomposable since there is an indecomposable arc-like continuum

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which is the almost continuous image of $[0,1]$ but has no dense arc-component [2, Example 2]. Moreover, the assumption that X contain no ∞-od is also essential, [1, Example 7]. By Theorem 2, [2, Example 2] gives an example of a continuum which is the almost continuous image of $[0,1]$ but not $\mathbb{R}$.

It is not known which spaces, or even continua, are the almost continuous image of $\mathbb{R}$. We have a partial result in this direction.

Theorem 3. *If a space X has a dense path-component and $|X| \leq \mathfrak{c}$, then X is the almost continuous image of $\mathbb{R}$.*

An immediate corollary for continua is:

Corollary 4. *If a continuum X has a dense arc-component, then X is the almost continuous image of $\mathbb{R}$.*

I do not know if the converse of Theorem 3 or Corollary 4 is true or not.

We will use the following theorem due to Kellum [3] many times:

Theorem 5. *If X is a second countable T_1 space then X is the almost continuous image of a Peano continuum if and only if X is almost Peano.*

2. Definitions and Terminology

The symbol $\mathfrak{c}$ will denote the cardinality of the real line $\mathbb{R}$. The unit interval will be denoted by $[0,1]$. Let X and Y be sets and $A \subseteq X \times Y$. We denote the natural projection of A into X or Y by $\pi_X(A)$ or $\pi_Y(A)$, respectively.

A *continuum* is a nonempty compact connected metric space. For a set A we let $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ denote the closure and interior of A, respectively. Given two subcontinua A and B of a continuum X, we let $H(A,B)$ denote the standard Hausdorff distance, as defined in [4, 4.1], between A and B. A space is a *Peano continuum* provided that it is a locally connected, continuum. Two sets $A, B \subseteq X$ are said to be *mutally separated* provided that $\mathrm{cl}(A) \cap B = \emptyset$ and $\mathrm{cl}(B) \cap A = \emptyset$. A ∞-*od* is a continuum X such that there exist a continuum $S \subseteq X$ such that $X \setminus S$ has infinitely many components. We say a continuum X is *decomposable* provided that $X = A \cup B$ where A and B are proper subcontinua of X. If a continuum is not decomposable we say that it is *indecomposable*. The generalized open ball of radius $\epsilon > 0$ about a set A is denoted by $\mathrm{B}_\epsilon(A)$ and consists of all points which are less than distance ϵ from some point in A. When A has the form $\{x\}$ we write $\mathrm{B}_\epsilon(x)$ for $\mathrm{B}_\epsilon(\{x\})$.

A space is called a *path* if it is the continuous image of $[0,1]$. A space X is pathwise connected provided that any two points of X lie in some path contained in X. A *path component* of a space X is a maximal pathwise connected subset of X. A space is *almost Peano* provided that for any finite collection of nonempty open sets $\{U_i\}_{i=1}^n$ there is a path S such that $S \cap U_i \neq \emptyset$ for each $1 \leq i \leq n$. The Hahn-Mazurkiewicz Theorem [4, 8.14], implies that when X is Hausdorff we may replace the word path with the word Peano continuum in the defintion of almost Peano.

A function $f\colon X \to Y$ is an *almost continuous* function provided that every open set in $X \times Y$ containing f contains some continuous function $g\colon X \to Y$. We will denote the collection of almost continuous functions from a space X into a space Y by $\mathrm{Ac}(X,Y)$. We say a subset B of $X \times Y$ is a *blocking set* for $\mathrm{Ac}(X,Y)$ provided that B is closed, B has nonempty intersection with each continuous function, and B contains no set of the form $\{x\} \times Y$ where $x \in X$. It is easily checked that a function $f\colon X \to Y$ is almost continuous if and only if f has nonempty intersection with every blocking set for $\mathrm{Ac}(X,Y)$.

3. Proof of Theorem 1

We first prove a lemma which will help us identify ∞-ods.

Lemma 6. *Let X be a continuum and let Z and $\{F_n\}_{n=1}^\infty$ be subcontinua of X such that for every n we have:*

(i): *$F_i \cap F_j \subseteq Z$, for all $1 \leq i < j \leq n$ and*

(ii): *$Z \subsetneq F_n$ and $H(F_n, Z) < 1/n$.*

If $X = \bigcup_{n=1}^\infty F_n$, then X is an $\infty - od$.

Proof: It is enough for us to show that $F_n \setminus Z$ is open in $X \setminus Z$ for every $1 \leq n < \infty$. Notice that $F_n \setminus Z$ is nonempty by (ii) and $F_n \setminus Z$ is closed in $X \setminus Z$ by definition of the subspace topology.

Fix n. By way of contradiction assume that $F_n \setminus Z$ is not open $X \setminus Z$. Then, there is an $x \in F_n \setminus Z$ and a sequence of points $\{x_k\}_{k=1}^\infty$ in $X \setminus (F_n \cup Z)$ such that $\lim_{k\to\infty} x_k = x$. Notice that, by (i), $F_k \cap \{x_k\colon 1 \leq k < \infty\}$ is finite for every $1 \leq k < \infty$. So, by (ii), $\lim_{k\to\infty} x_k$ must be in Z, but $x \notin Z$. Thus, $F_n \setminus Z$ is open in $X \setminus Z$. □

Proof of Theorem 1: Assume that X is a decomposable continuum which is the almost continuous image of $[0,1]$ and has no dense arc-component. We show that X must contain an ∞-od. By Theorem 5, X is almost Peano.

Let $X = A \cup B$ where A and B are proper subcontinua; note that both A and B have non-empty interior.

By induction we construct a sequence $\{F_n\}_{n=1}^{\infty}$ of mutally disjoint Peano continua such that $F_n \cap (X \setminus A) \neq \emptyset \neq F_n \cap A$ for every n.

Since X is almost Peano, there is a Peano continuum P_1 with the property that $P_1 \cap (X \setminus A) \neq \emptyset \neq P_1 \cap \operatorname{int}(A)$. Let $F_1 = P_1$.

Suppose now that $n \geq 1$ and we have chosen $\{F_k\}_{k=1}^{n}$. We show how to pick F_{n+1}. For each $1 \leq k \leq n$ let D_k be the closure of the arc-component of X which contains F_k. By assumption, $D_k \neq X$ for each k. So there is an $\epsilon > 0$ such that for every D_k there is a x_k such that $\mathrm{B}_{\epsilon}(x_k) \cap D_k = \emptyset$. Since X is almost Peano, there is a Peano continuum P_{n+1} such that $P_{n+1} \cap \mathrm{B}_{\epsilon}(x_k) \neq \emptyset$ for every k and $P_{n+1} \cap (X \setminus A) \neq \emptyset \neq P_{n+1} \cap \operatorname{int}(A)$. Since $P_{n+1} \cap \mathrm{B}_{\epsilon}(x_k) \neq \emptyset$ for every k, it follows that P_{n+1} is not contained in any D_k. So, $P_{n+1} \cap F_k = \emptyset$ for every k. Also, $P_{n+1} \cap (X \setminus A) \neq \emptyset \neq P_{n+1} \cap A$. Thus, $F_{n+1} = P_{n+1}$ will have the desired properties.

We now construct the ∞-od. For every $1 \leq n < \infty$, A is a proper subcontinuum of $A \cup F_n$. There is, by boundary bumping [4], a continuum C_n such that $A \subsetneq C_n \subseteq A \cup F_n$ and $H(A, C_n) < 1/n$. Notice that for any $1 \leq i < j < \infty$, we have $C_i \cap C_j \subseteq A$ since $F_i \cap F_j = \emptyset$. By Lemma 6, $C = \bigcup_{n=1}^{\infty} C_n$ is an ∞-od. □

4. Proof of Theorem 2

Lemma 7. *If a continuum X is an almost continuous image of $\mathbb{R}$, then X is almost Peano. In particular, X is the almost continuous image of $[0,1]$.*

Proof: Let $f\colon \mathbb{R} \to X$ be an almost continuous map onto X. Let $\{U_k\}_{k=1}^{n}$ be a collection of nonempty open sets in X. Since f is an onto function, there is for every U_k an $x_k \in \mathbb{R}$ such that $f(x_k) \in U_k$. Let

$$V = (\mathbb{R} \times X) \setminus \bigcup \{\{x_k\} \times (X \setminus U_k) \colon 1 \leq k \leq n\}.$$

Notice that V is open and $f \subseteq V$. By almost continuity, there is a continuous function $g\colon \mathbb{R} \to X$ such that $g \subseteq V$. Let $J \subseteq \mathbb{R}$ be a compact interval such that $x_1, \ldots, x_n \in J$. Now $g[J]$ is a Peano continuum and, by our choice of V, we have $g[J] \cap U_k \neq \emptyset$ for each $1 \leq k \leq n$. Therefore, X is almost Peano. □

Lemma 8. *Suppose that $f\colon \mathbb{R} \to X$ is an almost continuous function onto a continuum X. If $f[[M, \infty)]$ is dense in X for every $M > 0$, then X has a dense arc-component.*

Proof: Let $\mathcal{B} = \{\mathrm{B}_n \colon 1 \leq n \in \infty\}$ be a countable base for X. Let $M_0 = 0$. Since $f[[1, \infty)]$ is dense in X, there exists an $M_1 \in \mathbb{R}$ with the property that $f[[1, M_1]] \cap \mathrm{B}_1 \neq \emptyset$. Since $f[[M_1 + 1, \infty)]$ is dense in X, there is an $M_2 \in \mathbb{R}$ such that $f[[M_1 + 1, M_2]] \cap \mathrm{B}_2 \neq \emptyset$. Continuing inductively we may find an increasing sequence $\{M_n\}_{n=0}^{\infty}$ of real numbers such that $f[[M_{n-1} + 1, M_n]] \cap B_n \neq \emptyset$. For each n pick $t_n \in [M_{n-1} + 1, M_n]]$ such that $f(t_n) \in \mathrm{B}_n$. Since $\{t_n \colon 1 \leq n < \infty\}$ is a discrete set of points all of which are distinct, the set $U \subseteq \mathbb{R} \times X$ defined by

$$U = (\mathbb{R} \times X) \setminus \left(\bigcup_{n=1}^{\infty} \{t_n\} \times (X \setminus \mathrm{B}_n) \right)$$

is open. Since $f \subseteq U$ and f is almost continuous, there is a continuous function $g\colon \mathbb{R} \to X$ such that $g \subseteq U$. Since $g \subseteq U$, we have $g(t_n) \in \mathrm{B}_n$ for every n. So $g[\mathbb{R}]$ is dense in X. Obviously, $g[\mathbb{R}]$ is arcwise connected so X has a dense arc-component. □

Proof of Theorem 2: Let $f\colon \mathbb{R} \to X$ be an almost continuous onto function. Assume that X contains no ∞-od. We show that X has a dense arc-component.

If X is decomposable, then Lemma 7 and Theorem 1 imply that X must have a dense arc-component.

We may now assume that X is indecomposable and nondegenerate. By way of contradiction, assume that X has no dense arc-component. By Lemma 8, it follows that for some $M > 0$, $f[[M, \infty)]$ is not dense in X.

Let $Z = \mathrm{cl}(f[[M, \infty)])$. Since $f[[M, \infty)]$ is connected [5, Theorem 1.7], Z is a continuum. Since $f[[M, \infty)]$ is not dense, Z is a proper subcontinuum of X and is nowhere dense in X [4, 5.20a & 11.17].

By induction we construct a sequence $\{W_n\}_{n=1}^{\infty}$ of subcontinua of X such that for every n:

(C1): $W_i \cap W_j \subseteq Z$ for every $1 \leq i < j \leq n$,
(C2): $Z \subsetneq W_n$, and
(C3): there is a $t_n \in \mathbb{R}$ and a continuous function $g_n\colon [t_n, \infty) \to X$ such that $W_n = g_n[[t_n, \infty)] \cup Z$.

Define $U \subseteq \mathbb{R} \times X$ to be the open set

$$U = ((-\infty, M) \times X) \cup \left(\bigcup_{n=0}^{\infty} [M + n, M + (n+1)) \times \mathrm{B}_{1/n+1}(Z) \right).$$

Notice that $f \subseteq U$. To construct W_1 pick a point $p_1 \in X \setminus Z$. Since $p_1 \notin Z$, there is a $t_1 \in (-\infty, M)$ such that $f(t_1) = p_1$. There is an $\epsilon_1 > 0$ such $\mathrm{B}_{\epsilon_1}(p_1) \cap Z = \emptyset$. Let $U_1 = U \setminus [\{t_1\} \times (X \setminus \mathrm{B}_{\epsilon_1}(p_1))]$. Notice that U_1 is open and $f \subseteq U_1$. Since f is almost continuous, there is a continuous $g_1\colon \mathbb{R} \to X$ such that $g_1 \subseteq U_1$. Let $W_1 = g_1[[t_1, \infty)] \cup Z$. Using the fact that $g_1 \subseteq U$ it is easy to verify that W_1 is a continuum. Clearly, $Z \subsetneq W_1$ since $f(t_1) \notin Z$. Thus, W_1 satisfies (C1), (C2), and (C3).

Assume that $n \geq 1$ and that $W_1, \ldots, W_n$ have been constructed so that (C1), (C2), and (C3) are satisfied. We construct W_{n+1} so that (C1), (C2), and (C3) are satisfied. For each $1 \leq k < n+1$ let A_k be the arc-component of X that contains $g_k[[t_k, \infty)]$. By assumption, $\mathrm{cl}(A_k) \neq X$ for each $1 \leq k < n+1$. Since $\mathrm{cl}(A_k)$ is a proper subcontinuum, it follows that $\mathrm{cl}(A_k) \cup Z$ is nowhere dense. So, $T = Z \cup \bigcup_{k=1}^{n} \mathrm{cl}(A_k)$ is closed and nowhere dense in X. Pick a point $p_{n+1} \in X \setminus T$. Since $p_{n+1} \notin Z$, there is a $t_{n+1} \in (-\infty, M)$ such that $f(t_{n+1}) = p_{n+1}$. There is an $\epsilon_{n+1} > 0$ such $\mathrm{B}_{\epsilon_{n+1}}(p_{n+1}) \cap Z = \emptyset$. Let $U_{n+1} = U \setminus \left[\{t_{n+1}\} \times (X \setminus \mathrm{B}_{\epsilon_{n+1}}(p_{n+1}))\right]$. Notice that U_{n+1} is open and $f \subseteq U_{n+1}$. Since f is almost continuous, there is a continuous $g_{n+1}\colon \mathbb{R} \to X$ such that $g_{n+1} \subseteq U_{n+1}$. Let $W_{n+1} = g_{n+1}[[t_{n+1}, \infty)] \cup Z$. Using the fact that $g_{n+1} \subseteq U$ it is easy to verify that W_{n+1} is a continuum. Clearly, $Z \subsetneq W_{n+1}$ since $f(t_{n+1}) \notin Z$. So (C2) and (C3) are satisfied. We show that (C1) is satisfied. By inductive assumption, it is enough for us to check that $W_{n+1} \cap W_k \subseteq Z$ for every $k < n+1$. Fix $k < n+1$. Since $f(t_{n+1}) \notin T \supseteq A_k$ and $g_{n+1}[[t_{n+1}, \infty)]$ is arcwise connected, it follows that $g_{n+1}[[t_{n+1}, \infty)] \cap A_k = \emptyset$. Since $g_k[[t_k, \infty)] \subseteq A_k$, we have $W_k \cap W_{n+1} \subseteq Z$. Thus, (C1) is satisfied completing the induction.

Since Z is a proper subcontinuum of W_n for each $1 \leq n < \infty$, we may find by boundary bumping [4] a subcontinuum F_n such that $Z \subsetneq F_n \subseteq W_n$ and $H(F_n, Z) < 1/n$. By (C1) $F_n \cap F_m \subseteq Z$, for all $m \neq n$. By Lemma 6, $\bigcup_{n=1}^{\infty} F_n$ is an ∞-od which contradicts our original assumption. □

5. Proof of Theorem 3

Let X be a space and $[a,b]$ be a compact interval in $\mathbb{R}$. Fix $p, q \in X$. We say a function $f\colon [a,b] \to X$ is *(p,q)-almost continuous* provided that for every open set $U \subseteq [a,b] \times X$ such that $f \subseteq U$ there is a continuous function $g\colon [a,b] \to X$ such that $g \subseteq U$ and $g(a) = p$ and $g(b) = q$. We say $B \subseteq [a,b] \times X$ is a *(p,q)-blocking set* provided that

- (a): B is closed,
- (b): B contains no set of the form $\{w\} \times X$ where $w \in [a,b]$,
- (c): $B \cap \{\langle a,p\rangle, \langle b,q\rangle\} = \emptyset$, and
- (d): $B \cap g \neq \emptyset$ for every continuous $g\colon [a,b] \to X$ such that $g(a) = p$ and $g(b) = q$.

We say a (p,q)-blocking set B is *irreducible* if no (p,q)-blocking set is properly contained in B. It is easily checked that if $f\colon [a,b] \to X$ is such that $f(a) = p$ and $f(b) = q$, then f is (p,q)-almost continuous if and only if f has nonempty intersection with every (p,q)-blocking set. Notice that by the definition of (p,q)-blocking set, if X is a single point, then there are no (p,q)-blocking sets and every function $f\colon [a,b] \to X$ is (p,q)-almost continuous. We will denote the collection of (p,q)-almost continuous functions from $[a,b]$ into X by $\mathrm{Ac}_{p,q}([a,b], X)$.

The next two lemmas are modifications of Lemma 1 and Lemma 2 of [3] for (p,q)-almost continuity; the proofs are essentially the same but are included for completeness.

Lemma 9. *If $p, q \in X$ and B is a (p,q)-blocking set for $\mathrm{Ac}_{p,q}([0,1], X)$, then B contains an irreducible (p,q)-blocking set for $\mathrm{Ac}_{p,q}([0,1], X)$.*

Proof: Let $\mathcal{B}$ be the collection of all (p,q)-blocking sets contained in B. Let $\mathcal{B}^* \subseteq \mathcal{B}$ be linearly ordered by inclusion. Suppose that

$g\colon [0,1] \to X$ is a continuous function such that $g(0) = p$ and $g(1) = q$. Notice that the graph of g is homeomorphic to $[0,1]$ and is compact. By assumption, $B^* \cap g \neq \emptyset$ for every $B^* \in \mathcal{B}^*$. So, $\{B^* \cap g\colon B^* \in \mathcal{B}^*\}$ is a collection of nonempty compact subsets of g linearly ordered by inclusion. Thus, $\bigcap\{B^* \cap g\colon B^* \in \mathcal{B}^*\} \neq \emptyset$. It follows that $C = \bigcap \mathcal{B}^*$ has nonempty intersection with g. Since g is arbitrary and $C \subseteq B$, it follows that C is a (p,q)-blocking set. The result now follows from the Zorn Lemma. □

Lemma 10. *If X is a path, $p, q \in X$ and B is an irreducible (p,q)-blocking set for* $\mathrm{Ac}([0,1], X)$*, then $\pi_{[0,1]}(B)$ is a nondegenerate interval.*

Proof: First note that $\pi_{[0,1]}(B)$ is not of the form $\{w\}$ where $w \in [0,1]$. Assume there is such a w. Clearly, $w \notin \{0,1\}$ since $B \cap \{\langle 0,p\rangle, \langle 1,q\rangle\} = \emptyset$. So, we may assume that $w \in (0,1)$. By path connectedness, there is for every $x \in X$ a continuous function $g\colon [0,1] \to X$ such that $g(0) = p$, $g(w) = x$, and $g(1) = q$. Thus, $\langle w,x\rangle \in B$. Since x is arbitrary, it follows that $\{w\} \times X \subseteq B$, a contradiction. Thus, $\pi_{[0,1]}(B)$ is not a point.

If we assume that $\pi_{[0,1]}(B)$ is not connected then, by compactness, there exist $a < b \in \pi_{[0,1]}(B)$ such that $(a,b) \cap \pi_{[0,1]}(B) = \emptyset$. Since B is irreducible, there exist continuous functions $g_a, g_b\colon [0,1] \to X$ such that $g_a(0) = g_b(0) = p$, $g_a(1) = g_b(1) = q$, $g_a|_{[b,1]} \cap B = \emptyset$, and $g_b|_{[0,a]} \cap B = \emptyset$. Again using path connectedness we may find a continuous $h\colon [0,1] \to X$ such that $h|_{[0,a]} = g_b|_{[0,a]}$, $h|_{[b,1]} = g_a|_{[b,1]}$, and $h \cap B = \emptyset$, a contradiction. □

Lemma 11. *Let X and Y be spaces and X be compact. If $B \subseteq X \times Y$ is closed and B contains no set of the form $\{x\} \times Y$ where $x \in X$, then there exist a finite collection $\{V_i\}_{i\in n}$ of nonempty open subsets of Y such that for any set $S \subseteq Y$, if $S \cap V_i \neq \emptyset$ for each $i \in n$, then $\{w\} \times S \not\subseteq B$ for every $w \in X$.*

Proof: Let $x \in X$. Since $\{x\} \times Y \not\subseteq B$ by assumption, there is a $y \in Y$ such that $\langle x,y\rangle \notin B$. Since B is closed, there exist open sets $U_x \subseteq X$ and $V_x \subseteq Y$ such that $B \cap (U_x \times V_x) = \emptyset$ and $\langle x,y\rangle \in (U_x \times V_x)$. Clearly, $V_x \neq \emptyset$. By compactness, there is a finite collection $\{U_{x_i}\}_{i\in n}$ such that $X = \bigcup_{i\in n} U_{x_i}$. Let S be a set such that $S \cap V_{x_i} \neq \emptyset$ for every $i \in n$. Let $w \in X$. There is an

$i \in n$ such that $w \in U_{x_i}$. By the way U_{x_i} and V_{x_i} were selected, we have $(\{w\} \times V_{x_i}) \cap B = \emptyset$. Since $S \cap V_{x_i} \neq \emptyset$, it follows that $\{w\} \times S \nsubseteq B$. Thus, $\{V_{x_i}\}_{i \in n}$ is the desired finite collection of nonempty open sets. □

Lemma 12. *If X is a second countable space and Y is a space with no more than $\mathfrak{c}$-many open sets, then $X \times Y$ contains no more than $\mathfrak{c}$-many open sets. In particular, if Y is a path, then there are at most $\mathfrak{c}$ many open subsets of $[0,1] \times Y$.*

Proof: Let $\mathcal{B}$ be a countable base for X. Let $\mathcal{T}$ denote the collection of open sets of Y. The collection $\mathcal{M} = \{B \times V : B \in \mathcal{B} \;\&\; V \in \mathcal{T}\}$ is a base for $X \times Y$. Clearly, $|\mathcal{M}| \leq \mathfrak{c}$. Let $U \subseteq X \times Y$ be open. The collection $\mathcal{B}_U = \{B \in \mathcal{B} : (\exists M \in \mathcal{M})(B = \pi_X(M) \;\&\; M \subseteq U)\}$ is countable. For each $B \in \mathcal{B}_U$ let $V_B = \bigcup\{\pi_Y(M) : M \in \mathcal{M} \;\&\; B = \pi_X(M) \;\&\; M \subseteq U\}$. Clearly, $B \times V_B \in \mathcal{M}$ and it is easily checked that $B \times V_B \subseteq U$. It is also the case that $U \subseteq \bigcup_{B \in \mathcal{B}_U} B \times V_B$. Thus, U is the countable union of elements from $\mathcal{M}$. Since U was arbitrary and $|\mathcal{M}| \leq \mathfrak{c}$, it follows that $X \times Y$ has at most $\mathfrak{c}$-many open sets.

To see the second statement it is enough to notice that $[0,1]$ is second countable and that any continuous image of $[0,1]$ has at most $\mathfrak{c}$-many open sets. □

Lemma 13. *If X has a dense pathwise connected subset D and $|X| \leq \mathfrak{c}$, then there is a collection $\mathcal{S}$ of compact subsets of $\mathbb{R} \times X$ such that*

(i) $|\mathcal{S}| \leq \mathfrak{c}$,
(ii) *if $f : \mathbb{R} \to X$ is such that $f \cap S \neq \emptyset$ for every $S \in \mathcal{S}$, then $f \in \mathrm{Ac}(\mathbb{R}, X)$, and*
(iii) *$\pi_{\mathbb{R}}(S)$ contains a nondegenerate interval for every $S \in \mathcal{S}$.*

Proof: We define $\mathcal{S}$. Let $\mathcal{F}$ denote the nonempty finite subsets of D. For each $F \in \mathcal{F}$ let P_F be a path in D such that $F \subseteq P_F$. Let $\mathcal{P} = \{P_F : F \in \mathcal{F}\}$. For each $F \in \mathcal{F}$ and $n \in \mathbb{Z}$ let $\mathcal{B}^n_F$ denote the subsets of $\mathbb{R} \times X$ which are irreducible (p,q)-blocking sets or irreducible (q,p)-blocking sets for $[n, n+1]$ and P_F where $n \in \mathbb{Z}$ and $\{p,q\} \subseteq F$. Let

$$\mathcal{S} = \bigcup\{\mathcal{B}^n_F : F \in \mathcal{F} \;\&\; n \in \mathbb{Z}\}.$$

By Lemma 10, we have (iii). Lemma 12 and $|\{\mathcal{B}^n_F : F \in \mathcal{F} \;\&\; n \in \mathbb{Z}\}| \leq \mathfrak{c}$ imply (i). We now show that (ii) holds.

Let $f\colon \mathbb{R} \to X$ be a function which has nonempty intersection with every member of $\mathcal{S}$. By way of contradiction, assume that f is not almost continuous. There is a blocking set $B \subseteq \mathbb{R} \times X$ for $\mathrm{Ac}(\mathbb{R}, X)$ such that $B \cap f = \emptyset$. We may find a function $j\colon \mathbb{Z} \to D$ such that $j \cap B = \emptyset$ since D is dense and such that $j(n) = f(n)$ whenever $f(n) \in D$.

Fix $n \in \mathbb{Z}$. Let $B_n = B \cap ([n, n+1] \times X)$. Since B_n is closed and contains no set of the form $\{w\} \times X$, there is, by Lemma 11, a finite collection $\{V_i\}_{i \in n}$ of nonempty open subsets of X such that for any set $S \subseteq X$, if $S \cap V_i \neq \emptyset$ for each $i \in n$, then $\{w\} \times S \not\subseteq B$ for every $w \in [n, n+1]$. Since D is dense in X, there is a finite set $F_1 \subseteq D$ such that $F_1 \cap V_i \neq \emptyset$ for every $i \in n$. Let $F = F_1 \cup \{j(n), j(n+1)\}$. By our choice of $\{V_i\}_{i \in n}$, we have $\{t\} \times P_F$ is not contained in B for every $t \in [n, n+1]$. So, we may find a function $k\colon [n, n+1] \to P_F$ such that $k(n) = j(n)$, $k(n+1) = j(n+1)$, and $k \cap B_n = \emptyset$. Consider the function $f^*_n\colon [n, n+1] \to P_F$ defined by

$$f^*_n(x) = \begin{cases} k(x) & \text{if } f(x) \notin P_F; \\ f(x) & \text{if } f(x) \in P_F. \end{cases} \tag{1}$$

Notice that $f^*_n \in \mathrm{Ac}_{j(n), j(n+1)}([n, n+1], P_F)$ since the range of f^*_n is contained in P_F and f^*_n has nonempty intersection with every member of $\mathcal{B}^n_F$. Since $f^*_n \cap B_n = \emptyset$, there is a continuous function $g_n\colon [n, n+1] \to P_F$ such that $g_n(n) = j(n)$, $g_n(n+1) = j(n+1)$, and $g_n \cap B_n = \emptyset$.

For each $n \in \mathbb{Z}$ let g_n be as constructed above. Now, $g = \bigcup_{n \in \mathbb{Z}} g_n$ is a continuous function such that $g \cap B$, a contradiction. Thus, f is almost continuous and (ii) holds. □

Proof of Theorem 3: Let D be a dense pathwise connected subset of X.

By Lemma 13, there is a collection $\mathcal{S}$ of compact subsets of $\mathbb{R} \times X$ such that

(i) $|\mathcal{S}| \leq \mathfrak{c}$,
(ii) if $f\colon \mathbb{R} \to X$ is such that $f \cap S \neq \emptyset$ for every $S \in \mathcal{S}$, then $f \in \mathrm{Ac}(\mathbb{R}, X)$, and
(iii) $\pi_{\mathbb{R}}(S)$ contains a nondegenerate interval for every $S \in \mathcal{S}$.

Let $\{S_\alpha\}_{\alpha\in\mathfrak{c}}$ be an enumeration of $\mathcal{S}$. Let $H_1, H_2 \subseteq \mathbb{R}$ be a partition of $\mathbb{R}$ such that $|H_1| = |H_2| = \mathfrak{c}$ and $|H_1 \cap I| = \mathfrak{c}$ for every nondegenerate interval $I \subseteq \mathbb{R}$. Using a standard diagonalization argument and (iii) we may by transfinite induction construct a function $f^*\colon H_1 \to X$ such that $f^* \cap S_\alpha \neq \emptyset$ for every $\alpha \in \mathfrak{c}$. By (ii), any extension of f^* to all of $\mathbb{R}$ will be almost continuous. Since $|X| \leq |H_2| = \mathfrak{c}$, we may extend f^* to an onto almost continuous function $f\colon \mathbb{R} \to X$. □

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