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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
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A CONSISTENT EXAMPLE OF A β-NORMAL, NON-NORMAL SPACE

LEW LUDWIG AND PAUL J. SZEPTYCKI

Introduction. In [AL], two new generalizations of normality were introduced. A space X is called *α-normal* if for any two disjoint closed subsets A and B of X there exist disjoint open subsets U and V of X such that $A \cap U$ is dense in A and $B \cap V$ is dense in B. A space X is called *β-normal* if for any two disjoint closed subsets A and B of X there exist open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and $\overline{U} \cap \overline{V} = \emptyset$. Clearly, normality implies β-normality and β-normality implies α-normality. In that article, many results about α-normality and β-normality were proved. It was shown that there exists in ZFC an α-normal space which is not β-normal. However, it was left open if there exists any spaces which are β-normal, but not normal (even consistently). It was also shown in [AL] that every regular, hereditarily separable space is α-normal. It turns out that more is needed to attain β-normal, since an example was given there of a regular, hereditarily separable space that is not β-normal. The aim of this article is to show that there exists a β-normal non-normal space assuming the existence of an S-space that is normal and right-separated of type ω_1. Examples of such spaces were constructed from CH and weaker axioms [T], [JKR]. See [R] for more on S-spaces.

The example. Collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of ω are said to be *separated* if there exists a set $X \subseteq \omega$ such that $a \subseteq^* X$ for each $a \in \mathcal{A}$ and $b \cap X =^* \emptyset$ for each $b \in \mathcal{B}$. Let $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ be an almost disjoint family of infinite subsets of ω such that $\mathcal{A}_0$ and $\mathcal{A}_1$ cannot be separated. Moreover, assume any countable subset of $\mathcal{A}$

can be separated from its complement. Luzin [L] has constructed such an example in ZFC.

Now consider the space $\Psi(\omega, \mathcal{A}) = \omega \cup \mathcal{A}$ with the usual Ψ-space topology (see [vD]). Then $\mathcal{A}_0$ and $\mathcal{A}_1$ are two disjoint closed subsets of $\Psi(\mathcal{A})$ which cannot be separated. Hence, $\Psi(\mathcal{A})$ is not normal, but it is pseudo-normal (i.e. any two disjoint closed sets, one of which is countable, can be separated).

Fix A, a normal S-space, right separated of type ω_1. Let $Z = A \times \{0\} \cup A \times \{1\} \cup \omega$ and fix a bijection $f : \Psi(\mathcal{A}) \to Z$ such that

$$f(\mathcal{A}_0) = A \times \{0\}$$
$$f(\mathcal{A}_1) = A \times \{1\}$$
$$f|\omega = id|\omega$$

The topology on Z is as follows: $U \subseteq Z$ is open in Z if and only if the set $\{b \in A : (b, i) \in U\}$ is open in A and for all $(a, i) \in U$ we have $f^{-1}((a, i)) \subseteq^* (U \cap \omega)$ (equivalently $f^{-1}(U)$ open in $\Psi(\mathcal{A})$) and points in ω are isolated. Observe that Z has the quotient topology on $\Psi(\mathcal{A}) \cup (A \times 2)$ obtained by identifying each $a \in \mathcal{A}$ to $f(a)$. Note that Z is not normal because then $\mathcal{A}_0$ and $\mathcal{A}_1$ could be separated.

Claim: Z is β-normal.

Proof: Let E and F be closed disjoint subsets of Z. Without loss of generality, assume $E, F \subseteq A \times \{0\} \cup A \times \{1\}$. For notational purposes, let

$$A_i = A \times \{i\}$$
$$E_i = E \cap A_i$$
$$F_i = F \cap A_i$$

where $i \in \{0, 1\}$. Moreover, since A_i is normal for $i \in \{0, 1\}$, there exists open disjoint sets G_i and H_i of A_i such that $E_i \subseteq G_i$, $F_i \subseteq H_i$, and $\mathrm{cl}_{A_i} G_i \cap \mathrm{cl}_{A_i} H_i = \emptyset$. The space Z is hereditarily separable, so there exists countable dense subsets D_{E_i} of E_i and D_{F_i} of F_i. Since A_i is normal and right separated in type-ω_1, hence locally countable, there exists countable open subsets U_{E_i} of A_i and U_{F_i} of A_i such that $D_{E_i} \subseteq U_{E_i}$, $D_{F_i} \subseteq U_{F_i}$, $\overline{U_{E_i}} \subseteq G_i$, and $\overline{U_{F_i}} \subseteq H_i$.

Since $f^{-1}(U_{E_i}) \subseteq \mathcal{A}_i$ and $f^{-1}(U_{F_i}) \subseteq \mathcal{A}_i$, are countable and $\Psi(\mathcal{A})$ is pseudo-normal, there exists clopen, pairwise disjoint

V_{E_i}, $V_{F_i} \subseteq \Psi(\mathcal{A})$ such that $V_{E_i} \cap \mathcal{A}_i = f^{-1}(U_{E_i})$ and $V_{F_i} \cap \mathcal{A}_i = f^{-1}(U_{F_i})$. Hence $f(V_{E_i})$ and $f(V_{F_i})$ are open in Z and

$$\overline{E_i \cap f(V_{E_i})} = E_i$$

$$\overline{F_i \cap f(V_{F_i})} = F_i$$

$$(f(V_{E_0}) \cup f(V_{E_1})) \cap (f(V_{F_0}) \cup f(V_{F_1})) \cap \omega = \emptyset.$$

It is straightforward to check that $\overline{f(V_{E_i})} \cap (A_0 \cup A_1) \subseteq \overline{G_i}$ and $\overline{f(V_{F_i})} \cap (A_0 \cup A_1) \subseteq \overline{H_i}$.

Finally, since $\overline{f(V_{E_i})} \cap A_i \subseteq \overline{G_i}$ and $\overline{f(V_{F_i})} \cap A_i \subseteq \overline{H_i}$ with $\overline{G_i} \cap \overline{H_i} = \emptyset$, we have

$$\overline{f(V_{E_0}) \cup f(V_{E_1})} \cap \overline{f(V_{F_0}) \cup f(V_{F_1})} = \emptyset.$$

That is, Z is β-normal.

Question 1. Does there exist in ZFC a β-normal, non-normal space?[1]

Remark. Towards an example in ZFC, we considered the above construction and found sufficient conditions to produce a β-normal, non-normal space. If there exists an uncountable λ and spaces X and Y such that

(1) X is normal, right separated of type λ with $hd(X) < \lambda$
(2) Y is not normal, scattered height 2, Y has λ non-isolated points and any two disjoint closed sets (one of which has size less than λ) can be separated

then there exists a β-normal non-normal space.

So, a positive answer to the following question would yield a positive answer to Question 1:

Question 2. Do there exist uncountable λ and spaces X and Y satisfying parts 1 and 2 of the above remark?

For $\lambda = \omega_1$ there do exist examples of spaces that satisfy part two of the remark. Most notably, the space $\Psi(\omega, \mathcal{A})$ of this paper has these properties. However, we know of no ZFC example described in part one of the remark:

[1] After the submission of this article, Eva Murtinova constructed a Tychonoff, β-normal, non-normal space in ZFC.

Question 3. Do there exist in ZFC an uncountable cardinal λ and a space X which is normal, right separated of type λ with $hd(X) < \lambda$?

Todorcevic has constructed many spaces in ZFC with the hereditary density less than the hereditary Lindelöf degree. For example, see Theorem 0.5 of [T]. One of these may give a positive answer to Question 2.

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Kenyon College, Gambier, OH 43022

York University, Toronto, Ontario M3J 1P3