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ON THE HYPERSPACES $\mathcal{C}_n(X)$ OF A CONTINUUM X, II

SERGIO MACÍAS

ABSTRACT. We continue our investigation of the hyperspaces of nonempty closed subsets with at most n components of a continuum X. We present more properties of these hyperspaces.

1. Introduction

In [13] we began the investigation of the hyperspaces of nonempty closed subsets of a continuum X having at most n components, denoted by $\mathcal{C}_n(X)$. Here we continue the study of such spaces. The paper is divided in several sections. In Section 2, we give the basic definitions needed for understanding the paper. In Section 3, we present some general properties of the hyperspaces $\mathcal{C}_n(X)$ of a continuum X, in particular it is shown that if X is a hereditarily indecomposable continuum and n is a positive integer then X is the only point at which $\mathcal{C}_n(X)$ is locally connected. We also study the arc components of $\mathcal{C}_n(X) \setminus \{X\}$, when X is an indecomposable continuum. In Section 4, we give results about retractions of the hyperspace 2^X onto $\mathcal{C}_n(X)$ and of $\mathcal{C}_n(X)$ onto $\mathcal{C}(X)$, when the continuum is locally connected. In Section 5, we present results of the hyperspaces of a graph X, in particular when X is the arc, $[0, 1]$,

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and the unit circle, $\mathcal{S}^1$. In Section 6, we prove that hereditarily indecomposable continua X have unique hyperspace $\mathcal{C}_n(X)$.

2. Definitions.

If (Z, d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior of A is denoted by $Int_Z(A)$, its boundary is denoted by $Bd_Z(A)$, and its closure is denoted by $\overline{A}$ or by $Cl_Z(A)$. Given a metric space Z, by a *deformation* we mean a map $h\colon Z \times [0,1] \to Z$ such that for each $z \in Z$, $h(z,1) = z$. Let $A = \{h(z,0) \mid z \in Z\}$. If the map $h_0\colon Z \to A$ given by $h_0(z) = h(z,0)$ is a retraction from Z onto A, then h is a *deformation retraction from Z onto A.* If h is a deformation retraction from Z onto A such that for each $z \in A$ and each $t \in [0,1]$, $h(z,t) = z$, then h is a *strong deformation retraction from Z onto A.* The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively.

A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z. A continuum is said to be *decomposable* provided it can be written as the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A continuum X is *hereditarily indecomposable* provided that each subcontinuum of X is indecomposable. An *arc* is any space homeomorphic to $[0,1]$. A *map* means a continuous function. A *Cantor fan* is any space homeomorphic to the cone over the Cantor set.

Given a continuum X and a point x in X, the *composant* of X containing x, is the union of all proper subcontinua of X containing x (see [11, p. 208]).

Given an onto map $f\colon X \to Y$ between continua, let $\mathcal{G}_f = \{f^{-1}(y) \mid y \in Y\}$. It is known that the quotient space $X/\mathcal{G}_f$ is a continuum (see [17, 3.21 and 3.10]).

A metric ρ for a continuum X is said to be *convex* provided that given two points x and y of X there exists a point z in X such that $\rho(x,z) = \dfrac{\rho(x,y)}{2} = \rho(z,y)$. It is known that every locally connected continuum admits a convex metric (see [2] and [14]).

A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end points.

Let X be a graph and let A be a subset of X. Let β a cardinal number. We say that A *is of order less than or equal to* β *in* X, written $\text{ord}(A, X) \leq \beta$, provided that for each open subset, U, of X containing A, there exists an open subset V of X such that $A \subset V \subset U$ and $Bd_X(V)$ has cardinality less that or equal to β. We say that A *is of order* β *in* X, written $\text{ord}(A, X) = \beta$, provided that $\text{ord}(A, X) \leq \beta$ and $\text{ord}(A, X) \not\leq \alpha$ for any cardinal number $\alpha < \beta$. A point x of a graph X is *a ramification point of* X if and only if $\text{ord}(\{x\}, X) \geq 3$. A point a of a graph X is a *end point of* X if and only if $\text{ord}(\{a\}, X) = 1$. For a given integer $m \geq 3$, a *simple* m*–od* is a graph having only one ramification point. The order in the graph of this ramification point is m, the ramification point is called the *vertex* of the simple m–od. We refer the reader to [17, Chapter IX] for more information about graphs.

Given a continuum X, we define its *hyperspaces* as the following sets:

$$
\begin{aligned}
2^X &= \{A \subset X \mid A \text{ is closed and nonempty}\} \\
\mathcal{C}(X) &= \{A \in 2^X \mid A \text{ is a connected}\} \\
\mathcal{C}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ components}\}, \ n \in \mathbb{N} \\
\mathcal{C}_\infty(X) &= \{A \in 2^X \mid A \text{ has finitely many components}\} \\
\mathcal{F}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}, \ n \in \mathbb{N}
\end{aligned}
$$

(1) $\quad \mathcal{F}(X) = \{A \in 2^X \mid A \text{ is finite}\}.$

We agree that $\mathcal{C}(X) = \mathcal{C}_1(X)$. Let us observe that for each $n \in \mathbb{N}$,

$$
\begin{gathered}
\mathcal{F}_n(X) \subset \mathcal{C}_n(X), \\
\mathcal{C}_n(X) \subset \mathcal{C}_{n+1}(X), \\
\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X), \\
\mathcal{C}_\infty(X) = \bigcup_{n=1}^{\infty} \mathcal{C}_n(X),
\end{gathered}
$$

and that

$$
\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X).
$$

On the other hand, it is known that 2^X is a metric space with the Hausdorff metric, $\mathcal{H}$, defined as follows:

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\},$$

(see [16, (0.1)]). Observe that the Hausdorff metric depends on the metric d of the continuum X, so we write $\mathcal{H}_d$ if we need to emphasize the metric of X. It is known that 2^X and $\mathcal{C}(X)$ are arcwise connected continua (see [16, (1.13)]), and for each $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are continua (for $\mathcal{C}_n(X)$ see [13, 3.1] and for $\mathcal{F}_n(X)$ see [4, p. 877]). Hence both $\mathcal{C}_\infty(X)$ and $\mathcal{F}(X)$ are connected subsets of 2^X. On the other hand, 2^X can be topologized with the *Vietoris Topology,* defined as follows: given a finite collection, $U_1, U_2, \ldots, U_m$, of open sets of X, we define

$$< U_1, \ldots, U_m >= \left\{ A \in 2^X \;|A \subset \bigcup_{k=1}^{m} U_k \text{ and } A \cap U_k \neq \emptyset \text{ for each } k \in \{1, \ldots, m\} \right\}.$$

It is known that the family of all subsets of 2^X of the form $< U_1, \ldots, U_m >$, as defined above, form a basis for a topology for 2^X (see [16, (0.11)]) called *Vietoris Topology,* and that the Vietoris topology and the topology induced by the Hausdorff metric coincide (see [16, (0.13)]). To simplify notation, $< U_1, \ldots, U_m >_n$ denotes the intersection of the open set $< U_1, \ldots, U_m >$, of the Vietoris Topology, with $\mathcal{C}_n(X)$.

An *order arc* in 2^X is an arc $\alpha\colon [0,1] \to 2^X$ such that if $0 \leq s < t \leq 1$ then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$.

Other definitions are given as required.

3. General Properties

3.1 Lemma. *Let $n \in \mathbb{N}$, and let X be a continuum. If $\mathcal{A}$ is a connected subset of $\mathcal{C}_n(X)$ then $\cup\mathcal{A} = \cup\{A \mid A \in \mathcal{A}\}$ has at most n components.*

Proof: Suppose the result is not true. Then there exists a connected subset $\mathcal{A}$ of $\mathcal{C}_n(X)$ such that $\cup\mathcal{A}$ has at least $n+1$ components. Thus, we can find $n+1$ pairwise separated subsets,

$C_1, \ldots, C_{n+1}$, of X such that $\cup\mathcal{A} = \bigcup_{j=1}^{n+1} C_j$. Let

$$\mathcal{B} = \left\{ A \in \mathcal{A} \;\middle|\; A \subset \bigcup_{j=1}^{n} C_j \right\}$$

and

$$\mathcal{D} = \{A \in \mathcal{A} \mid A \cap C_{n+1} \neq \emptyset\}.$$

Then $\mathcal{B}$ and $\mathcal{D}$ are separated subsets of $\mathcal{C}_n(X)$ and $\mathcal{A} = \mathcal{B} \cup \mathcal{D}$, a contradiction. Therefore, $\cup\mathcal{A}$ has at most n components.

□

3.2 Lemma. *Let $n \in \mathbb{N}$. If X is a continuum containing an open set with uncountably many components, then $\mathcal{C}_n(X)$ contains an open set with uncountably many components.*

Proof: Let U be an open subset of X with uncountably many components. Let $\Gamma =< U >_n$. Then Γ is an open subset of $\mathcal{C}_n(X)$, and $\cup\Gamma \subset U$. Since for each $x \in U$, $\{x\} \in \Gamma$, we have that $\cup\Gamma = U$. By 3.1, if Λ is a component of Γ then $\cup\Lambda$ has at most n components. Since U has uncountably many components, $\cup\Gamma = U$ and for each component Λ of Γ, $\cup\Lambda$ has at most n components, we have that Γ has uncountably many components.

□

The following result is easy to establish.

3.3 Lemma. *Let n be an integer greater than one, and let X be a continuum. Let A be a point of $\mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$, and suppose $A_1, \ldots, A_n$ are the components of A. Let $\varepsilon > 0$ be such that $\mathcal{V}^d_{2\varepsilon}(A_j) \cap \mathcal{V}^d_{2\varepsilon}(A_k) = \emptyset$ if and only if $j \neq k$ and $j, k \in \{1, \ldots, n\}$. Let B a point of $\mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X)$, and suppose $B_1, \ldots, B_n$ are the components of B. Then, $\mathcal{H}(A, B) < \varepsilon$ if and only if $\mathcal{H}^2(\{A_1, \ldots, A_n\}, \{B_1, \ldots, B_n\}) < \varepsilon$, where $\mathcal{H}^2$ is the Hausdorff metric on $\mathcal{F}_n(\mathcal{C}(X))$ induced by $\mathcal{H}$.*

As a consequence of Lemma 3.3, we have the following result.

3.4 Theorem. *Let n be an integer greater than one. If X is a continuum, then the function $f_n \colon \mathcal{C}_n(X) \setminus \mathcal{C}_{n-1}(X) \to \mathcal{F}_n(\mathcal{C}(X))$ given by*

$$f_n(A) = \{K \mid K \text{ is a component of } A\}$$

is an embedding.

3.5 Corollary. *If X is a continuum such that $\mathcal{C}(X)$ is of finite dimension then for each $n \geq 2$,* $\dim(\mathcal{C}_n(X)) \leq n(\dim(\mathcal{C}(X)))$.

Proof: Let $n \geq 2$. Then $\dim(\mathcal{F}_n(\mathcal{C}(X))) \leq n(\dim(\mathcal{C}(X)))$ (see [5, Lemma 3.1]). On the other hand, $\dim(\mathcal{C}_n(X)) \leq \dim(\mathcal{F}_n(\mathcal{C}(X)))$ (see [7, Corollary 1, p. 32). Therefore $\dim(\mathcal{C}_n(X)) \leq n(\dim(\mathcal{C}(X)))$.

□

A continuum X is said to have *Kelley's property* provided that given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a, b \in X$, $d(a, b) < \delta$, and $a \in A \in \mathcal{C}(X)$ then there exists $B \in \mathcal{C}(X)$ such that $b \in B$ and $\mathcal{H}(A, B) < \varepsilon$. The number δ is called a *Kelley's number* for the given ε.

3.6 Lemma. *Let $n \in \mathbb{N}$, and let X be a continuum having Kelley's property. If $\mathcal{W}$ is a subcontinuum of $\mathcal{C}_n(X)$ not containing X and having nonempty interior, then $\cup\mathcal{W} \in \mathcal{C}_n(X)$ and $Int_X(\cup\mathcal{W}) \neq \emptyset$.*

Proof: By [13, 7.2], $\cup\mathcal{W} \in \mathcal{C}_n(X)$. Let $A \in Int_{\mathcal{C}_n(X)}(\mathcal{W})$. Let $A_1, \dots, A_k$ be the components of A, then $1 \leq k \leq n$. Let $\varepsilon > 0$ be given such that $\mathcal{V}^{\mathcal{H}}_{\varepsilon}(A) \cap \mathcal{C}_n(X) \subset \mathcal{W}$, and that $\mathcal{V}^d_{\varepsilon}(A_j) \cap \mathcal{V}^d_{\varepsilon}(A_\ell) = \emptyset$ if and only if $j \neq \ell$, $j, \ell \in \{1, \dots, k\}$. Let $\delta > 0$ be a Kelley's number for the given ε. Clearly $A \subset \cup\mathcal{W}$. For each $j \in \{1, \dots, k\}$, let $a_j \in A_j$. For every $j \in \{1, \dots, k\}$, let $x_j \in \mathcal{V}^d_{\delta}(a_j)$. Since X has Kelley's property, there exists a subcontinuum B_j of X such that $x_j \in B_j$ and $\mathcal{H}(A_j, B_j) < \varepsilon$ for each $j \in \{1, \dots, k\}$. Let $B = \bigcup_{j=1}^{k} B_j$, then $B \in \mathcal{C}_n(X)$. Since $\mathcal{V}^d_{\varepsilon}(A) = \bigcup_{j=1}^{k} \mathcal{V}^d_{\varepsilon}(A_j)$, $\mathcal{V}^d_{\varepsilon}(B) = \bigcup_{j=1}^{k} \mathcal{V}^d_{\varepsilon}(B_j)$, and $\mathcal{H}(A_j, B_j) < \varepsilon$ for each $j \in \{1, \dots, k\}$, we have that $\mathcal{H}(A, B) < \varepsilon$. Hence $B \in \mathcal{W}$, and $B \subset \cup\mathcal{W}$. Therefore, $\bigcup_{j=1}^{k} \mathcal{V}^d_{\varepsilon}(a_j) \subset \cup\mathcal{W}$, and $Int_X(\cup\mathcal{W}) \neq \emptyset$.

□

3.7 Theorem. *Let $n \in \mathbb{N}$. If X is an indecomposable continuum having Kelley's property, then X is the only point at which $\mathcal{C}_n(X)$ is locally connected.*

Proof: Using order arcs and [16, (1.8)], it is easy to see that for each $\varepsilon > 0$, $\mathcal{V}_\varepsilon^{\mathcal{H}}(X) \cap \mathcal{C}_n(X)$ is arcwise connected. Suppose that $\mathcal{C}_n(X)$ is locally connected at the point $A \neq X$. Since $\mathcal{C}_n(X)$ is locally connected at A, there exists a subcontinuum $\mathcal{W}$ of $\mathcal{C}_n(X)$ such that $A \in Int_{\mathcal{C}_n(X)}(\mathcal{W})$ and $\cup\mathcal{W} \neq X$. By 3.6, $\cup\mathcal{W} \in \mathcal{C}_n(X)$ and $Int_X(\cup\mathcal{W}) \neq \emptyset$. Since $\cup\mathcal{W}$ has finitely many components and $Int_X(\cup\mathcal{W}) \neq \emptyset$, it follows that at least one of the components of $\cup\mathcal{W}$ has nonempty interior, which is impossible because $\cup\mathcal{W} \neq X$ and X is an indecomposable continuum (see [11, Theorem 2, p. 207]).

□

As a consequence of the previous Theorem, we have the following result.

3.8 Theorem. *Let $n \in \mathbb{N}$. If X is a hereditarily indecomposable continuum, then X is the only point at which $\mathcal{C}_n(X)$ is locally connected.*

Proof: Since hereditarily indecomposable continua have Kelley's property (see [16, (16.27)]), the result follows from 3.7.

□

Given a continuum X, a positive integer n and a composant κ of X, $\mathcal{C}_n(\kappa)$, denotes the set $\mathcal{C}_n(\kappa) = \{A \in \mathcal{C}_n(X) \mid A \subset \kappa\}$.

3.9 Theorem. *Let n be an integer greater than one. If κ is a composant of an indecomposable continuum X, then $\mathcal{C}_n(\kappa)$ is an arc component of $\mathcal{C}_n(X) \setminus \{X\}$.*

Proof: It is known that $\mathcal{C}_n(X) \setminus \{X\}$ is not arcwise connected (see [13, 6.3]). First, observe that $\mathcal{C}_n(\kappa)$ is arcwise connected. To see this, let $A \in \mathcal{C}_n(\kappa)$. Then, since A has finitely many components and X is indecomposable, it is easy to show that there exists a proper subcontinuum, B, of X containing A. Thus, there exists an order arc from A to B (see [16, (1.8)]). Since $\mathcal{C}(\kappa)$ is arcwise connected (see [16, (1.52.1)], we have that $\mathcal{C}_n(\kappa)$ is arcwise connected.

Let $\mathcal{A}$ be the arc component of $\mathcal{C}_n(X) \setminus \{X\}$ containing $\mathcal{C}_n(\kappa)$, and suppose there exists $B \in \mathcal{A} \setminus \mathcal{C}_n(\kappa)$. Let $A \in \mathcal{C}(\kappa)$. Since

A and B belong to $\mathcal{A}$, there is an arc $\alpha\colon [0,1] \to \mathcal{A}$ such that $\alpha(0) = A$ and $\alpha(1) = B$. Let $\beta\colon [0,1] \to \mathcal{C}_n(X)$ be given by $\beta(t) = \cup\alpha([0,t])$. Then β is well defined (see [13, 7.2]), β is an order arc and $\beta(0) = \alpha(0) = A$. Hence, $\beta(t) \in \mathcal{C}(X)$ for each $t \in [0,1]$ (see [16, (1.11)]), and $B \subset \beta(1)$. Since B is not contained in κ and $B \subset \beta(1)$, we have that $\beta(1)$ is a subcontinuum of X intersecting two different composants of it. Thus, $\beta(1) = X$. Let $t_0 = \min\{t \in [0,1] \mid \beta(t) = X\}$. Then $\beta(t_0) = X$ and $t_0 > 0$. Observe that if $0 \leq t < t_0$, then $\beta(t)$ is a nowhere dense subset of X. Note that for each $0 < t < t_0$, $X = \beta(t_0) = \beta(t) \cup (\cup\alpha\,([t,t_0]))$. Since $\beta(t)$ is nowhere dense in X, we have that $X = \cup\alpha\,([t,t_0])$ for each $t < t_0$. By continuity, $\alpha(t_0) = X$, a contradiction. Therefore $\mathcal{C}_n(\kappa) = \mathcal{A}$.

□

3.10 Theorem. *Let n be an integer greater than one, and let X be an indecomposable continuum. If $\mathcal{A}$ is an arc component of $\mathcal{C}_n(X)\setminus\{X\}$, which is not of the form $\mathcal{C}_n(\kappa)$, where κ is a composant of X, then there exist finitely many composants $\kappa_1,\ldots,\kappa_\ell$, of X and ℓ positive integers $m_1,\ldots,m_\ell$, such that there exists a one–to–one map from $\prod_{j=1}^{\ell}\mathcal{C}_{m_j}(\kappa_\ell)$ (with the "max" metric ρ_1) onto $\mathcal{A}$.*

Proof: Let A_0 be a point of $\mathcal{A}$, and let $\kappa_1,\ldots,\kappa_\ell$ be the composants of X which intersect A_0. Since $\mathcal{A}$ is not of the form $\mathcal{C}_n(\kappa)$, $\ell > 1$.

First, we show that every element of $\mathcal{A}$ intersects each κ_j for each $j \in \{1,\ldots,\ell\}$. To see this, suppose that there is a point B of $\mathcal{A}$ such that $B \cap \kappa_j = \emptyset$, for some $j \in \{1,\ldots,\ell\}$. Since A_0 and B belong to $\mathcal{A}$, there exists an arc $\alpha\colon [0,1] \to \mathcal{A}$ such that $\alpha(0) = A_0$ and $\alpha(1) = B$. Let $\beta\colon [0,1] \to \mathcal{C}_n(X)$ be given by $\beta(t) = \cup\alpha([0,t])$. Then β is well defined (see [13, 7.2]), β is an order arc, $\beta(0) = \alpha(0) = A_0$, and $\beta(1) = \cup\alpha([0,1])$ is an element of $\mathcal{C}_n(X)$ intersecting κ_j and containing B. On the other hand, the map $\gamma\colon [0,1] \to \mathcal{C}_n(X)$ given by $\gamma(t) = \cup\alpha([1-t,1])$ is also well defined (see [13, 7.2]), γ is an order arc, $\gamma(0) = \alpha(1) = B$ and $\gamma(1) = \cup\alpha([0,1]) = \beta(1)$. Thus, there exists an order arc in $\mathcal{C}_n(X)$ from B to $\beta(1)$, $B \cap \kappa_k = \emptyset$ and $\beta(1) \cap \kappa_j \neq \emptyset$, this contradicts ([16, (1.8)]) if $\cup\alpha([0,1])$ is a proper subset of X. Otherwise, a similar argument to the one given in 3.9, shows that there exists $t_0 \in [0,1]$

such that $\alpha(t_0) = X$, which is also a contradiction. Therefore, every element of $\mathcal{A}$ intersects each κ_j, $j \in \{1, \dots, \ell\}$. A similar argument proves that if $A \in \mathcal{A}$ and $A \cap \kappa \neq \emptyset$, for some composant κ of X, then $\kappa \in \{\kappa_1, \dots, \kappa_\ell\}$.

For each $j \in \{1, \dots, \ell\}$, let

$$m_j = \max\{\text{number of components of } A \text{ contained in } \kappa_j \mid A \in \mathcal{A}\}.$$

Let us observe that $\sum_{j=1}^{\ell} m_j = n$. Let $f\colon \prod_{j=1}^{\ell} \mathcal{C}_{m_j}(\kappa_j) \to \mathcal{A}$ given by $f(A_1, \dots, A_\ell) = \bigcup_{j=1}^{\ell} A_j$. Let us see first that f is well defined. Clearly $f(A_1, \dots, A_\ell) \in \mathcal{C}_n(X)$. On the other hand, for each $j \in \{1, \dots, \ell\}$, A_j and $A_0 \cap \kappa_j$ both belong to $\mathcal{C}_{m_j}(\kappa_j)$. Since $\mathcal{C}_{m_j}(\kappa_j)$ is arcwise connected, there is an arc $\alpha_j\colon [0,1] \to \mathcal{C}_{m_j}(\kappa_j)$ such that $\alpha_j(0) = A_0 \cap \kappa_j$ and $\alpha_j(1) = A_j$. Let $\alpha\colon [0,1] \to \mathcal{C}_n(X) \setminus \{X\}$ be given by $\alpha(t) = \bigcup_{j=1}^{\ell} \alpha_j(t)$. Then, α is a path joining $\alpha(0) = \bigcup_{j=1}^{\ell} \alpha_j(0) = \bigcup_{j=1}^{\ell} (A_0 \cap \kappa_j) = A_0 \cap \bigcup_{j=1}^{\ell} \kappa_j = A_0$ and $\alpha(1) = \bigcup_{j=1}^{\ell} \alpha_j(1) = \bigcup_{j=1}^{\ell} A_j$. Therefore $\bigcup_{j=1}^{\ell} A_j \in \mathcal{A}$.

If A is a point of $\mathcal{A}$, then $(A \cap \kappa_1, \dots, A \cap \kappa_\ell)$ is an element of $\prod_{j=1}^{\ell} \mathcal{C}_{m_j}(\kappa_j)$ such that $f((A \cap \kappa_1, \dots, A \cap \kappa_\ell)) = \bigcup_{j=1}^{\ell} (A \cap \kappa_j) = A$. Thus, f is onto.

If $(A_1, \dots, A_\ell)$ and $(B_1, \dots, B_\ell)$ are two distinct points of $\prod_{j=1}^{\ell} \mathcal{C}_{m_j}(\kappa_j)$, then $A_{j_0} \neq B_{j_0}$ for some $j_0 \in \{1, \dots, \ell\}$. Hence $\bigcup_{j=1}^{\ell} A_j \neq \bigcup_{j=1}^{\ell} B_j$, being a disjoint union. Therefore f is one–to–one.

To see that f is continuous, let ε be given positive number. Let $(A_1, \ldots, A_\ell)$ and $(B_1, \ldots, B_\ell)$ be two points of $\prod_{j=1}^{\ell} \mathcal{C}_{m_j}(\kappa_j)$ such that $\rho_1\left((A_1, \ldots, A_\ell), (B_1, \ldots, B_\ell)\right) < \dfrac{\varepsilon}{2}$. Then, for each $j \in \{1, \ldots, \ell\}$, $\mathcal{H}(A_j, B_j) < \frac{\varepsilon}{2}$. Hence, for every $j \in \{1, \ldots, \ell\}$, $A_j \subset \mathcal{V}^d_{\frac{\varepsilon}{2}}(B_j) \subset \mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{r=1}^{\ell} B_r\right)$ and $B_j \subset \mathcal{V}^d_{\frac{\varepsilon}{2}}(A_j) \subset \mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{r=1}^{\ell} A_r\right)$. Thus, $\bigcup_{j=1}^{\ell} A_j \subset \mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{j=1}^{\ell} B_j\right)$ and $\bigcup_{j=1}^{\ell} B_j \subset \mathcal{V}^d_{\frac{\varepsilon}{2}}\left(\bigcup_{j=1}^{\ell} A_j\right)$. Therefore, $\mathcal{H}\left(\bigcup_{j=1}^{\ell} A_j, \bigcup_{j=1}^{\ell} B_j\right) \leq \dfrac{\varepsilon}{2} < \varepsilon$.

□

3.11 Theorem. *Let n be an integer greater than one. Let X be an indecomposable continuum. If $\mathcal{A}$ is an arc component of $\mathcal{C}_n(X) \setminus \{X\}$, whose elements intersect at least two composants of X, then, for any arc $\alpha\colon [0,1] \to \mathcal{A}$, $\cup\alpha([0,1])$ is not connected.*

Proof: Let $\alpha\colon [0,1] \to \mathcal{A}$ be an arc and suppose $\cup\alpha([0,1])$ is connected. Observe that $\alpha(0)$ is a nonconnected subset of $\cup\alpha([0,1])$ intersecting at least two composants of X. Hence $\cup\alpha([0,1]) = X$. An argument similar to the one given in the proof of 3.9, shows that there exists a point t_0 in $[0,1]$ such that $\alpha(t_0) = X$, which is not possible. Therefore, $\cup\alpha([0,1])$ is not connected.

□

4. Retractions

First we consider maps between some of the hyperspaces.

4.1 Theorem. *Let $n \in \mathbb{N}$. Then for any continuum X, there exists a map of 2^X onto $\mathcal{C}_n(X)$.*

Proof: If X is locally connected then 2^X and $\mathcal{C}_n(X)$ are locally connected continua (see [20, Théorème II et Théorème II_m]), hence the result follows from [17, 8.19].

Suppose that X is not locally connected. Then there exists a map, f, of 2^X onto the Cantor fan (see [16, (1.39)]). On the other hand, there exists a map, g, of the Cantor fan onto $\mathcal{C}_n(X)$ (see [10, Remark on p. 29 and Theorem 2.7]). Thus $g \circ f$ is a map of 2^X onto $\mathcal{C}_n(X)$.

□

4.2 Theorem. *Let n be an integer greater that one. If X is a continuum containing an open set with uncountably many components, then there exists a map of $\mathcal{C}_n(X)$ onto $\mathcal{C}(X)$.*

Proof: Since X contains an open subset with uncountably many components, $\mathcal{C}_n(X)$ contains an open subset with uncountably many components (see 3.2). Thus there exists a map, f of $\mathcal{C}_n(X)$ onto the Cantor fan (see [1, Theorem II]). On the other hand, there exists a map, g, of the Cantor fan onto $\mathcal{C}(X)$ (see [10, Theorem 2.7]). Hence, $g \circ f$ is a map of $\mathcal{C}_n(X)$ onto $\mathcal{C}(X)$.

□

Let us observe that if X is a locally connected continuum, then for each $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ is locally connected continuum (see [20, Théorème II_m]), so, by [17, 8.19], we have that if X is a locally connected continuum and $n > 1$, then there exists an onto map from $\mathcal{C}_n(X)$ to $\mathcal{C}(X)$. We have the following

Question. If X is a nonlocally connected continuum, then is $\mathcal{C}(X)$ a continuous image of $\mathcal{C}_n(X)$ for each $n > 1$?

The following result is a modification of [6, Lemma 2.1].

4.3 Lemma. *Let $n \in \mathbb{N}$ and let X be a nonlocally connected continuum. Let p be a point of X at which X is not connected im kleinen, and let $\mathcal{A} \subset \mathcal{C}_n(X)$ such that $\{p\} \in \mathcal{A}$. Then there does not exist a deformation, h, of 2^X onto $\mathcal{A}$ such that for each $t \in [0,1]$, $h(\{p\}, t) = \{p\}$.*

Proof: Since X is not connected im kleinen at p, there exists a neighborhood U of p and a sequence, $\{K_j\}_{j=1}^{\infty}$, of components of $\overline{U}$ converging to a continuum $K \subset \overline{U}$ such that $p \in K$ and such that for each $j \in \mathbb{N}$, $K_j \cap K = \emptyset$ (see [18, (12.1), p. 18]). Let $\{p_j\}_{j=1}^{\infty}$ be a sequence in $\overline{U}$ converging to p and such that $p_j \in K_j$ for each $j \in \mathbb{N}$. Suppose there exists a deformation $h\colon 2^X \times [0,1] \to 2^X$ from 2^X onto $\mathcal{A}$ such that for each $t \in [0,1]$, $h(\{p\}, t) = \{p\}$. Let

h' be the segment homotopy associated with h (see [16, (16.3)]), given by $h'(A,t) = \cup\{h(A,s) \mid 0 \leq s \leq t\}$. Observe that for each $t \in [0,1]$, $h'(\{p\},t) = \{p\}$ and for each $A \in 2^X$, $h'(A,0) = h(A,0) \in \mathcal{C}_n(X)$. Let $\varepsilon > 0$ be such that $\mathcal{V}^d_\varepsilon(p) \subset U$. Then there exists $\delta > 0$ such that if $A \in 2^X$ and $\mathcal{H}(A,\{p\}) < \delta$ then for each $t \in [0,1]$, $\mathcal{H}(h'(A,t),h'(\{p\},t)) = \mathcal{H}(h'(A,t),\{p\}) < \varepsilon$. For each $j \in \mathbb{N}$, let $A_j = \{p\} \cup \{p_\ell\}_{\ell=j}^\infty$. Hence, there exists $j_0 \in \mathbb{N}$ such that if $j \geq j_0$ then $\mathcal{H}(A_j,\{p\}) < \delta$. Choose $j \geq j_0$. Then, $\{h'(A_j,t) \mid t \in [0,1]\}$ is a connected subset of $\mathcal{V}^{\mathcal{H}}_\varepsilon(\{p\})$ such that $h'(A_j,0) \in \mathcal{C}_n(X)$. Thus $\cup\{h'(A_j,t) \mid t \in [0,1]\}$ is a closed subset of X having at most n components (see [16, (1.25) and (16.5)]). Since $A_j \subset \cup\{h'(A_j,t) \mid t \in [0,1]\} \subset \mathcal{V}^d_\varepsilon(p) \subset U$ and since $A_j \cap K_\ell \neq \emptyset$ for each $\ell \geq j$, we obtain a contradiction. We conclude that there does not exist a deformation h of 2^X onto $\mathcal{A}$ such that for each $t \in [0,1]$, $h(\{p\},t) = \{p\}$.

□

4.4 Corollary. *Let $n \in \mathbb{N}$. If X is a nonlocally connected continuum, then there does not exist a strong deformation of 2^X onto $\mathcal{F}_n(X)$.*

Question. Let $n \in \mathbb{N}$. If X is a nonlocally connected continuum, then does there exist a retraction or deformation retraction of 2^X onto $\mathcal{F}_n(X)$?

The proof of the following result is essentially the same as the one given in [6, Lemma 2.1].

4.5 Lemma. *Let n be an integer greater that one. Let X be a nonlocally connected continuum, let $p \in X$ such that X is not connected im kleinen at p, and let $\mathcal{A} \subset \mathcal{C}(X)$ such that $\{p\} \in \mathcal{A}$. Then there does not exist a deformation h of $\mathcal{C}_n(X)$ onto $\mathcal{A}$ such that for each $t \in [0,1]$, $h(\{p\},t) = \{p\}$.*

4.6 Corollary. *Let n be an integer greater that one. If X is a nonlocally connected continuum, then there does not exist a strong deformation retraction of $\mathcal{C}_n(X)$ onto $\mathcal{F}_1(X)$.*

If X is a locally connected continuum with a convex metric ρ, let $K_\rho\colon [0,\infty) \times 2^X \to 2^X$ be given by

$$K_\rho(t,A) = \{x \in X \mid \rho(x,y) \leq t \text{ for some } y \in A\}.$$

4.7 Lemma [16, (0.65.3)(a)]**.** *If X is a locally connected continuum with a convex metric ρ then for any two points, x and y, of X, there exists an arc γ with x and y as its end points such that γ is isometric to* $[0, \rho(x, y)]$.

As a consequence of 4.7, we have that if X is a locally connected continuum with a convex metric ρ, then $K_\rho(t, A)$ has at most as many components as A for each $t \in [0, \infty)$. Also observe that if $t \geq \mathrm{diam}(X)$, then $K_\rho(t, A) = X$, for each $A \in 2^X$.

4.8 Theorem [16, (0.65.3)(f)]**.** *If X is a locally connected continuum with a convex metric ρ, then K_ρ is continuous.*

In the following result we define a map generalizing the map α_ρ given in the proof of [16, (6.12)], and the proof of its properties is a modification of the given in that result.

4.9 Theorem. *Let $n \in \mathbb{N}$ and let X be a locally connected continuum with a convex metric ρ. Then the function $\alpha_\rho^n \colon 2^X \to \mathbb{R}$ given by*

$$\alpha_\rho^n(A) = \inf\{t \geq 0 \mid K_\rho(t, A) \in \mathcal{C}_n(X)\}$$

satisfies the following:

(a) *$K_\rho(\alpha_\rho^n(A), A)$ belongs to $\mathcal{C}_n(X)$ for each $A \in 2^X$.*
(b) *If A and B belong to 2^X, $t \geq 0$, $K_\rho(t, A) \in \mathcal{C}_n(X)$ and $\mathcal{H}_\rho(A, B) \leq \eta$, then $K_\rho(t + \eta, B) \in \mathcal{C}_n(X)$.*
(c) *$\alpha_\rho^n(A)$ is continuous.*

Proof: To see $K_\rho(\alpha_\rho^n(A), A)$ belongs to $\mathcal{C}_n(X)$ for each $A \in 2^X$, observe that, by definition of α_ρ^n, there exists a decreasing sequence, $\{t_j\}_{j=1}^\infty$, of real numbers converging to $\alpha_\rho^n(A)$ such that $K_\rho(t_j, A)$ belongs to $\mathcal{C}_n(X)$ for each $j \in \mathbb{N}$. By the continuity of K_ρ, we have that $\lim\limits_{j\to\infty} K_\rho(t_j, A) = K_\rho(\alpha_\rho^n(A), A)$. Since $\mathcal{C}_n(X)$ is closed in 2^X, we have that $K_\rho(\alpha_\rho^n(A), A) \in \mathcal{C}_n(X)$.

Now, suppose A and B belong to 2^X, $t \geq 0$, $K_\rho(t, A) \in \mathcal{C}_n(X)$ and $\mathcal{H}_\rho(A, B) \leq \eta$. Observe first that $A \subset K_\rho(t, A) \subset K_\rho(t + \eta, B)$. Let $x \in K_\rho(t + \eta, B)$, then there exists $b \in B$ such that $\rho(x, b) \leq t + \eta$. Since $b \in B$ and $\mathcal{H}_\rho(A, B) \leq \eta$, there exists $a \in A$ such that $\rho(a, b) \leq \eta$. Since ρ is a convex metric, there exist arcs γ_1 and γ_2 such that γ_1 has x and b as its end points, γ_2 has b and a as its end points, γ_1 is isometric to $[0, \rho(x, b)]$ and γ_2

is isometric to $[0, \rho(b, a)]$ (see 4.7). Let $G_x = \gamma_1 \cup \gamma_2$, then G_x is a connected subset of $K_\rho(t + \eta, B)$ containing x and intersecting $K_\rho(t, A)$. Hence, $K_\rho(t + \eta, B)$ has at most as many components as $K_\rho(t, A)$. Therefore, $K_\rho(t + \eta, B) \in \mathcal{C}_n(X)$.

To show α^n_ρ is continuous, let $\eta \geq 0$. Let A and B two elements of 2^X such that $\mathcal{H}_\rho(A, B) \leq \eta$. By (a), $K_\rho(\alpha^n_\rho(A), A)$ belongs to $\mathcal{C}_n(X)$. Hence, by (b), $K_\rho(\alpha^n_\rho(A) + \eta, B)$ also belongs to $\mathcal{C}_n(X)$. By definition of $\alpha^n_\rho(B)$, we have that $\alpha^n_\rho(B) \leq \alpha^n_\rho(A) + \eta$. Interchanging the roles of A and B, in the above argument, we obtain $\alpha^n_\rho(A) \leq \alpha^n_\rho(B) + \eta$. Therefore $|\alpha^n_\rho(A) - \alpha^n_\rho(B)| \leq \eta$. Thus, α^n_ρ is continuous.

□

4.10 Theorem. *Let $n \in \mathbb{N}$ and let X be a continuum. Then $\mathcal{C}_n(X)$ is a strong deformation retract of 2^X if and only if X is locally connected.*

Proof: If X is not locally connected then there does not exist a strong deformation retraction from 2^X onto $\mathcal{C}_n(X)$ (see 4.3).

If X is locally connected then the map $h\colon 2^X \times [0, 1] \to 2^X$ given by

$$h(A, t) = K_\rho\left((1 - t)\alpha^n_\rho(A), A\right)$$

is a strong deformation retraction from 2^X onto $\mathcal{C}_n(X)$ (see 4.8 and 4.9).

□

4.11 Theorem. *Let n be an integer greater that one and let X be a continuum. Then $\mathcal{C}(X)$ is a strong deformation retract of $\mathcal{C}_n(X)$ if and only if X is locally connected.*

Proof: If X is not locally connected, then there does not exist a strong deformation retraction from $\mathcal{C}_n(X)$ onto $\mathcal{C}(X)$ (see 4.5).

If X is locally connected then the map $h\colon \mathcal{C}_n(X) \times [0, 1] \to \mathcal{C}_n(X)$ given by

$$h(A, t) = K_\rho\left((1 - t)\alpha^1_\rho(A), A\right)$$

is a strong deformation retraction from $\mathcal{C}_n(X)$ onto $\mathcal{C}(X)$.

□

4.12 Lemma. *Let $n \in \mathbb{N}$. If X is a locally connected continuum, then $\mathcal{F}_1(X)$ is a retract of $\mathcal{C}_n(X)$ if and only if X is an absolute retract.*

Proof: Since X is a locally connected continuum, $\mathcal{C}_n(X)$ is an absolute retract (see [20, Théorème II$_m$]). If $\mathcal{F}_1(X)$ is a retract of $\mathcal{C}_n(X)$ then $\mathcal{F}_1(X)$ is an absolute retract (see [3, (2.2), p. 86]). Since X is homeomorphic to $\mathcal{F}_1(X)$, X is an absolute retract.

If X is an absolute retract then clearly $\mathcal{F}_1(X)$ is a retract of $\mathcal{C}_n(X)$.

□

4.13 Theorem. *Let $n \in \mathbb{N}$. If X is a locally connected continuum, then the following are equivalent:*

(1) *X is an absolute retract.*
(2) *$\mathcal{F}_1(X)$ is a retract of $\mathcal{C}_n(X)$.*
(3) *$\mathcal{F}_1(X)$ is a deformation retract of $\mathcal{C}_n(X)$.*
(4) *$\mathcal{F}_1(X)$ is a strong deformation retract of $\mathcal{C}_n(X)$.*

Proof: Clearly (4) implies (3), and (3) implies (2). By 4.12, (2) implies (1). Suppose now, X is an absolute retract. Then $\mathcal{C}_n(X)$ is an absolute retract (see [20, Théorème II$_m$]). Hence, there exists a retraction $r_n\colon 2^X \to \mathcal{C}_n(X)$. By [G, 2.5], there exists a strong deformation retraction $G\colon 2^X \times [0,1] \to 2^X$, from 2^X onto $\mathcal{F}_1(X)$. Thus, $r_n \circ G|_{\mathcal{C}_n(X)}$ is a strong deformation retraction from $\mathcal{C}_n(X)$ onto $\mathcal{F}_1(X)$. Therefore, (1) implies (4).

□

4.14 Theorem. *Let $n \in \mathbb{N}$ and let X be a locally connected continuum. Then, the following hold:*

(1) *$\mathcal{C}_n(X)$ is a retract of 2^X.*
(2) *$\mathcal{C}_n(X)$ is a deformation retract of 2^X.*
(3) *$\mathcal{C}_n(X)$ is a strong deformation retract of 2^X.*

Proof: By [20, Théorème II$_m$] (1) holds. Since 2^X is contractible (see [16, (16.18)], by [19, 32E.4] (1) and (2) are equivalent. By 4.10, (3) is equivalent to the fact that X is locally connected.

□

4.15 Theorem. *Let n be an integer greater than one, and let X be a locally connected continuum. Then, the following hold:*

(1) *$\mathcal{C}(X)$ is a retract of $\mathcal{C}_n(X)$.*
(2) *$\mathcal{C}(X)$ is a deformation retract of $\mathcal{C}_n(X)$.*
(3) *$\mathcal{C}(X)$ is a strong deformation retract of $\mathcal{C}_n(X)$.*

Proof: By [20, Théorème II_m] (1) holds. Since locally connected continua have Kelley's property, we have that $\mathcal{C}_n(X)$ is contractible (see [13, 3.8]). Thus, by [19, 32E.4] (1) and (2) are equivalent. By 4.11, (3) is equivalent to the fact that X is locally connected.

□

5. Graphs

We begin characterizing graphs as the class of locally connected continua X having finite dimensional hyperspaces $\mathcal{C}_n(X)$.

5.1 Theorem. *A locally connected continuum X is a graph if and only if for each $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ is of finite dimension.*

Proof: If X is a graph then $\dim(\mathcal{C}(X)) < \infty$ (see [10, Lemma 5.2]). Hence, given $n \in \mathbb{N}$, by 3.5, we have that $\dim(\mathcal{C}_n(X)) \leq n(\dim(\mathcal{C}(X)) < \infty$.

If X is a locally connected continuum such that for each $n \in \mathbb{N}$, $\dim(\mathcal{C}_n(X)) < \infty$, then we have that $\dim(\mathcal{C}(X)) < \infty$. Thus, X is a graph (see [10, Lemma 5.2]).

□

5.2 Theorem. *Let n be a positive integer greater than one. Then $\mathcal{C}_n([0,1]) \setminus \mathcal{C}_{n-1}([0,1])$ is embeddable in $\mathbb{R}^{2n}$.*

Proof: Given $A \in \mathcal{C}_n([0,1]) \setminus \mathcal{C}_{n-1}([0,1])$, without loss of generality, we may assume that $[a_1, a_1'], \ldots, [a_n, a_n']$ are the components of A and $a_1 \leq a_1' < a_2 \leq a_2' < \ldots < a_n \leq a_n'$. Define $\xi\colon \mathcal{C}_n([0,1]) \setminus \mathcal{C}_{n-1}([0,1]) \to \mathbb{R}^{2n}$ by $\xi(A) = (a_1, a_1', \ldots, a_n, a_n')$. Clearly ξ is a one–to–one function. To see its continuity, let $\varepsilon > 0$ and let $B = \bigcup_{j=1}^{n} [b_j, b_j']$ be a point of $\mathcal{C}_n([0,1]) \setminus \mathcal{C}_{n-1}([0,1])$ such that $\mathcal{H}(A, B) < \varepsilon$. Then, for each $j \in \{1, \ldots, n\}$, $|a_j - b_j| < \varepsilon$ and $|a_j' - b_j'| < \varepsilon$. Hence $||\xi(A) - \xi(B)|| = \max\{|a_j - b_j|, |a_j' - b_j'| \mid j \in \{1, \ldots, n\}\} < \varepsilon$. Thus, ξ is continuous. Let us observe that we also have that $\xi\left(\mathcal{V}_\varepsilon^{\mathcal{H}}(A) \cap \mathcal{C}_n([0,1]) \setminus \mathcal{C}_{n-1}([0,1])\right) = \mathcal{V}_\varepsilon^{\mathbb{R}^{2n}}(\xi(A)) \cap \xi\left(\mathcal{C}_n([0,1]) \setminus \mathcal{C}_{n-1}([0,1])\right)$. Therefore, ξ is an embedding.

□

As a consequence of 5.2, [7, Corollary 1, p. 32], and [13, 3.5], we have the following result.

5.3 Corollary. $\dim(\mathcal{C}_n([0,1])) = 2n$.

5.4 Theorem. *Let $n \in \mathbb{N}$, and let $\mathcal{E}_n = \{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \mid 0 \leq x_1 \leq, \ldots, \leq x_{2n} \leq 1\}$. Then the function $f\colon \mathcal{E}_n \to \mathcal{C}_n([0,1])$ given by $f((x_1, \ldots, x_{2n})) = \bigcup_{j=1}^{n} [x_{2j-1}, x_{2j}]$ is continuous and onto. Furthermore, $\mathcal{E}_n/\mathcal{G}_f$ is homeomorphic to $\mathcal{C}_n([0,1])$.*

Proof: Let $\varepsilon > 0$ be given. If $(x_1, \ldots, x_{2n})$ and $(x'_1, \ldots, x'_{2n})$ belong to $\mathcal{E}_n$ and $D((x_1, \ldots, x_{2n}), (x'_1, \ldots, x'_{2n})) = \max\{|x_j - x'_j| \mid j \in \{1, \ldots, 2n\}\} < \varepsilon$, then we have that

$$\mathcal{H}(f((x_1, \ldots, x_{2n})), f((x'_1, \ldots, x'_{2n}))) \leq \max\{|x_j - x'_j| \mid j \in \{1, \ldots, 2n\}\} < \varepsilon.$$

Thus, f is continuous.

If $A = \bigcup_{j=1}^{k} [x_j, y_j]$, where $k \leq n$, then $(x_1, y_1, \ldots, x_k, y_k, y_k, \ldots, y_k)$ belongs to $\mathcal{E}_n$ and $f((x_1, y_1, \ldots, x_k, y_k, y_k, \ldots, y_k)) = A$. Then, f is onto. The fact that $\mathcal{E}_n/\mathcal{G}_f$ is homeomorphic to $\mathcal{C}_n([0,1])$ follows from the Transgression Lemma (see [17, 3.22]).

□

Let $\mathcal{S}^1$ denote the unit circle in the plane. If $\theta \in \mathcal{S}^1$ and $t \in [0,1)$, then let $A(\theta, t)$ be the arc (possibly degenerate) contained in $\mathcal{S}^1$ having θ as mid point and length $2\pi t$. For $t = 1$, $A(\theta, 1)$ denotes $\mathcal{S}^1$. The following result is easily established.

5.5 Lemma. *Let $\varepsilon > 0$ be given. If θ and φ belong to $\mathcal{S}^1$, t and s belong to $[0,1]$ and satisfy that $||\theta - \varphi|| < \varepsilon$ and $|t - s| < \frac{\varepsilon}{2\pi}$ then $\mathcal{H}(A(\theta, t), A(\varphi, s)) < \varepsilon$.*

Let $n \in \mathbb{N}$. If $n > 1$, let $T^n = \underbrace{\mathcal{S}^1 \times \cdots \times \mathcal{S}^1}_{n \text{ times}}$. If $n = 1$, then $T^1 = \mathcal{S}^1$.

5.6 Theorem. *Let $n \in \mathbb{N}$. If $f\colon T^n \times [0,1]^n \to \mathcal{C}_n(\mathcal{S}^1)$ is a function given by $f(\theta_1, \ldots, \theta_n, t_1, \ldots, t_n) = \bigcup_{j=1}^{n} A(\theta_j, t_j)$, then f is continuous, onto, $(T^n \times [0,1]^n)/\mathcal{G}_f$ is homeomorphic to $\mathcal{C}_n(\mathcal{S}^1)$, and $\dim(\mathcal{C}_n(\mathcal{S}^1)) = 2n$.*

Proof: Let $\varepsilon > 0$ be given. If $(\theta_1, \dots, \theta_n, t_1, \dots, t_n)$ and $(\varphi_1, \dots, \varphi_n, s_1, \dots, s_n)$ belong to $T^n \times [0,1]^n$ and $D((\theta_1, \dots, \theta_n, t_1, \dots, t_n),\ (\varphi_1, \dots, \varphi_n, s_1, \dots, s_n)) < \frac{\varepsilon}{2\pi}$, then for each $j \in \{1, \dots, n\}$, we have that $|\theta_j - \varphi_j| < \frac{\varepsilon}{2\pi} < \varepsilon$ and $|t_j - s_j| < \frac{\varepsilon}{2\pi}$. By 5.5, it follows that $\bigcup_{j=1}^{n} A(\theta_j, t_j) \subset \mathcal{V}^d_\varepsilon\left(\bigcup_{j=1}^{n} A(\varphi_j, s_j)\right)$ and $\bigcup_{j=1}^{n} A(\varphi_j, s_j) \subset \mathcal{V}^d_\varepsilon\left(\bigcup_{j=1}^{n} A(\theta_j, t_j)\right)$. Thus, $\mathcal{H}\left(\bigcup_{j=1}^{n} A(\theta_j, t_j), \bigcup_{j=1}^{n} A(\varphi_j, s_j)\right) < \varepsilon$. Therefore, f is continuous.

Let A be an element of $\mathcal{C}_n(\mathcal{S}^1)$, and suppose $A_1, \dots, A_k$ be the components of A, with $k \leq n$. For each $j \in \{1, \dots, k\}$, let θ_j be the midpoint of A_j and let t_j be the length of A_j. Then $\left(\theta_1, \dots, \theta_k, \underbrace{\theta_k, \dots, \theta_k}_{n-k \text{ times}}, \frac{t_1}{2\pi}, \dots, \frac{t_k}{2\pi}, \underbrace{\frac{t_k}{2\pi}, \dots, \frac{t_k}{2\pi}}_{n-k \text{ times}}\right)$ is an element of $T^n \times [0,1]^n$ whose image under f is A. Thus f is onto.

The fact that $(T^n \times [0,1]^n)/\mathcal{G}_f$ is homeomorphic to $\mathcal{C}_n(\mathcal{S}^1)$ follows from the Transgression Lemma (see [17, 3.22]). By 3.5, [7, Corollary 1, p. 32], and [13, 3.5], we have that $\dim(\mathcal{C}_n(\mathcal{S}^1)) = 2n$.

□

5.7 Theorem. *Let n be an integer greater than one. If X is a continuum such that $\mathcal{C}_n(X)$ is homeomorphic to either $\mathcal{C}_n([0,1])$ or $\mathcal{C}_n(\mathcal{S}^1)$ then X is homeomorphic to either $[0,1]$ or $\mathcal{S}^1$.*

Proof: Suppose that $\mathcal{C}_n(X)$ is homeomorphic to $\mathcal{C}_n([0,1])$, the other case is similar. Since $\mathcal{C}_n(X)$ is homeomorphic to $\mathcal{C}_n([0,1])$, X is a locally connected continuum (see [13, 3.2]). Since X is locally connected, and $\dim(C_n(X)) = 2n$ (see 5.3), it is easy to see that X does not contain simple 3–ods. (If X contains a 3–od, then, using an argument similar to the one given in the proof of 3.5 of [13], we can construct a k–cell in $\mathcal{C}_n(X)$ with $k > 2n$.) Thus, X is either homeomorphic to $[0,1]$ or $\mathcal{S}^1$ (see [17, 8.40(b)]).

□

Recently, A. Illanes showed that $\mathcal{C}_2([0,1])$ is not homeomorphic to $\mathcal{C}_2(\mathcal{S}^1)$ (see [8, Lemma 2]) and R. Schori proved that $\mathcal{C}_2([0,1])$ is homeomorphic to $[0,1]^4$ (see [8, Lemma 1]).

Question. Let n be an integer greater than two. Is $\mathcal{C}_n([0,1])$ homeomorphic to $\mathcal{C}_n(\mathcal{S}^1)$?

6. Hereditarily indecomposable continua

In [12], it was shown that hereditarily indecomposable continua have unique hyperspace 2^X, in this section we show the corresponding result for the hyperspaces $\mathcal{C}_n(X)$.

6.1 Theorem. *Let $n \in \mathbb{N}$, and let X be a hereditarily indecomposable continuum. If Y is a continuum such that $\mathcal{C}_n(Y)$ is homeomorphic to $\mathcal{C}_n(X)$ then, X and Y are homeomorphic.*

Proof: Let $h\colon \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ be a homeomorphism. We consider first the case $n = 1$. Since X is hereditarily indecomposable, $\mathcal{C}(X)$ is uniquely arcwise connected (see [16, (1.61)]), hence $\mathcal{C}(Y)$ is uniquely arcwise connected. Then Y is hereditarily indecomposable (see [16, (1.61)]). Thus, Y is homeomorphic to X (see [16, (0.60)]).

Suppose now that $n \geq 2$. Since X is hereditarily indecomposable, X is the only point at which $\mathcal{C}_n(X)$ is locally connected (see 3.5), then $h(X) = Y$. Let us show that $h(\mathcal{C}(X)) \subset \mathcal{C}(Y)$. To see this, let $A \in \mathcal{C}(X) \setminus \mathcal{F}_n(X)$, then $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected (see [13, 6.9]). Thus, $\mathcal{C}_n(Y) \setminus \{h(A)\}$ is not arcwise connected, thus, $h(A) \in \mathcal{C}(Y)$ (see [13, 6.2]). Since $\mathcal{C}(Y)$ is closed in $\mathcal{C}_n(Y)$ and $\mathcal{F}_1(X) \subset Cl_{\mathcal{C}_n(X)}(\mathcal{C}(X) \setminus \mathcal{F}_1(X))$, we have that $h(\mathcal{C}(X)) \subset \mathcal{C}(Y)$.

Now we prove that $h(\mathcal{F}_1(X)) \subset \mathcal{F}_1(Y)$. To this end, suppose there exists a point $\{x\}$ in $\mathcal{F}_1(X)$ such that $h(\{x\}) \in \mathcal{C}(Y) \setminus \mathcal{F}_1(Y)$. Then, there exists an order arc $\alpha\colon [0,1] \to \mathcal{C}_2(Y)$ such that $\alpha(0) \in \mathcal{F}_2(Y)$, $\alpha(1) = h(\{x\})$, and $\alpha([0,1)) \subset \mathcal{C}_2(Y) \setminus \mathcal{C}(Y)$ (see [15, (2.2)]). Hence, $h^{-1} \circ \alpha\colon [0,1] \to \mathcal{C}_n(X)$ is a one to one map such that $h^{-1} \circ \alpha(1) = \{x\}$ and $h^{-1} \circ \alpha([0,1)) \subset \mathcal{C}_n(X) \setminus \mathcal{C}(X)$. This contradicts the fact that in 2^X, singletons are not arcwise accessible from $2^X \setminus \mathcal{C}(X)$ (see [15, (3.4)]). Therefore, $h(\mathcal{F}_1(X)) \subset \mathcal{F}_1(Y)$.

Let $Y' \in \mathcal{C}(Y)$ be such that $\mathcal{F}_1(Y') = h(\mathcal{F}_1(X))$. Note that $\mathcal{C}(Y') \subset \mathcal{C}(Y)$ and $h^{-1}(\mathcal{C}(Y'))$ is an arcwise connected subcontinuum of $\mathcal{C}_n(X)$. Thus, $\mathcal{C}(X) \cap h^{-1}(\mathcal{C}(Y'))$ is arcwise connected (see [15, (5.2)]), and $\mathcal{F}_1(X) \subset \mathcal{C}(X) \cap h^{-1}(\mathcal{C}(Y'))$. We claim that $h^{-1}(Y') \in \mathcal{C}(X)$. Suppose, to the contrary, that $h^{-1}(Y') \in \mathcal{C}_n(X) \setminus \mathcal{C}(X)$. Let x_1 and x_2 be two points in different composants of X. Let $\beta_1, \beta_2 \colon [0,1] \to h^{-1}(\mathcal{C}(Y'))$ be two one to one maps such that $\beta_j(0) = \{x_j\}$ and $\beta_j(1) = h^{-1}(Y')$, $j \in \{1,2\}$. By [15, (3.4)], we have that $\beta_j([0,1]) \cap \mathcal{C}(X) \neq \{\{x_j\}\}$, $j \in \{1,2\}$. Hence, $(\beta_1([0,1]) \cup \beta_2([0,1])) \cap \mathcal{C}(X)$ contains an arc from $\{x_1\}$ to $\{x_2\}$ (see [15, (5.2)]). Since X is indecomposable and x_1 and x_2 are in different composants of X, we have that $X = \cup[(\beta_1([0,1]) \cap \beta_2([0,1])) \cap \mathcal{C}(X)]$ (see [16, (1.51)]). Then, $X \in (\beta_1([0,1]) \cup \beta_2([0,1])) \cap \mathcal{C}(X)$ (see [16, (1.50)]). Thus, $X \in h^{-1}(\mathcal{C}(Y'))$. It follows that $Y = h(X) \in \mathcal{C}(Y')$ and $Y = Y'$, a contradiction. Therefore $h^{-1}(Y') \in \mathcal{C}(X)$.

We claim that $h^{-1}(Y') = X$. To see this, let x_1' and x_2' be two points in different composants of X. Since $\mathcal{C}(X) \cap h^{-1}(\mathcal{C}(Y'))$ is arcwise connected, there exist two one to one maps $\alpha_1, \alpha_2 \colon [0,1] \to \mathcal{C}(X) \cap h^{-1}(\mathcal{C}(Y'))$ such that $\alpha_j(0) = \{x_j'\}$ and $\alpha_j(1) = h^{-1}(Y')$, $j \in \{1,2\}$. Then, $\alpha_1([0,1]) \cup \alpha_2([0,1])$ contains an arc from $\{x_1'\}$ to $\{x_2'\}$. Since X is indecomposable, we have that $X = \cup(\alpha_1([0,1]) \cap \alpha_2([0,1]))$ (see [16, (1.51)]). Thus, $X \in \alpha_1([0,1]) \cup \alpha_2([0,1])$ (see [16, (1.50)]). Then, $h(X) \in \mathcal{C}(Y')$. Therefore, since $h(X) = Y$, $Y \in \mathcal{C}(Y')$. Hence, $Y' = Y$. Therefore, by definition of Y', $h(\mathcal{F}_1(X)) = \mathcal{F}_1(Y)$.

□

The author thanks the referee for his (her) suggestions, which improve the paper.

ADDED. By a small modification of the proof of 4.3, we obtain the following: *Let $n \in \mathbb{N}$ and let X be a nonlocally connected continuum. Let p be a point of X at which X is not connected im kleinen, and let $\mathcal{A} \subset \mathcal{C}_n(X)$ such that $\{p\} \in \mathcal{A}$. If $m > n$, then there does not exist a deformation, h, of $\mathcal{C}_m(X)$ onto $\mathcal{A}$ such that for each $t \in [0,1]$, $h(\{p\}, t) = \{p\}$.* As a consequence of this, we also obtain: *Let $n \in \mathbb{N}$. If X is a nonlocally connected continuum and $m > n$, then there does not exist a strong deformation of $\mathcal{C}_m(X)$ onto $\mathcal{F}_n(X)$.* With a small modification of the proof of 76.2 of [9] we obtain the following: *Let $n > 1$ and X be a continuum. If $\mathcal{F}_1(X)$ is a deformation retract of 2^X ($\mathcal{C}_n(X)$, respectively), then $\mathcal{F}_1(X)$ is a deformation retract of $\mathcal{C}_n(X)$ ($\mathcal{C}(X)$, respectively).*

Professor Gerardo Acosta has told the author that there is interest in the following question: Let X and Y be continua and let n and m be positive integers; if $\mathcal{C}_n(X)$ is homeomorphic to $\mathcal{C}_m(Y)$ then is $n = m$? In connection with the question, the author realizes that Theorem 6.1 can be stated as follows (with the same proof):

6.2 Theorem. *Let n and m be positive integers, $m > 1$, and let X be a hereditarily indecomposable continuum. It Y is a continuum such that $\mathcal{C}_m(Y)$ is homeomorphic to $\mathcal{C}_n(X)$, then Y is homeomorphic to X.*

The author thanks Professor Acosta for informing him of the question.

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Instituto de Matemáticas Circuito Exterior Ciudad Universitaria México, D.F., C.P. 04510 México
E-mail address: macias@servidor.unam.mx