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A CONTINUOUS DECOMPOSITION OF THE PLANE INTO ACYCLIC CONTINUA EACH OF WHICH CONTAINS AN ARC

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Abstract. Since 1950 several decompositions of the plane into pseudo-arcs have been described. We construct a continuous decomposition of the plane into nondegenerate acyclic continua none of which is a pseudo-arc. In fact each decomposition element contains a straight line segment.

1. Introduction

In the early 1950's R. D. Anderson [A1952] constructed a continuous decomposition of the plane into nondegenerate acyclic continua and announced [A1950] that there exists a continuous decomposition of the plane into pseudo-arcs. A pseudo-arc can be characterized as a chainable, hereditarily indecomposable continuum. In his dissertation [B1958], M. Brown describes a construction which when applied to the plane results in a decomposition consisting of a single point and concentric hereditarily indecomposable continua each of which separates the plane. W. Lewis and J. J. Walsh [L1978] used a very geometrical construction to continuously decompose the plane into pseudo-arcs. Recently in [P1998] and in [K2000] Whitney maps have been used to refine continuous decompositions into continuous decompositions made up of pseudo-arcs. For example, in [K2000] M. Brown's decomposition of the plane is modified and

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then refined using Whitney maps to obtain a new continuous decomposition of the plane into pseudo-arcs.

In 1976, Michel Smith [S1976] using results from E. Dyer [D1953] and W. S. Mahavier [M1967] proved that any continuous decomposition of the plane into acyclic continua must have elements that are indecomposable. In [M1989] L. Mohler and L. G. Oversteegen constructed an example to answer in the negative an old question of B. Knaster; namely, whether every continuously irreducible continuum must contain an hereditarily indecomposable tranche. Similar questions about continuous decompositions of the plane naturally arise. This paper addresses the general question of how "nice" the decomposition elements must be; for example, must there always exist a decomposition element that is hereditarily indecomposable? Here we answer this question in the negative by using the techniques of Lewis and Walsh [L1978] to construct a continuous decomposition of plane into chainable continua each of which contains an arc. In fact the geometric nature of these techniques actually allows us to say that each of the decomposition elements contains a straight line segment of a given length. Specifically we will prove the following theorem.

Theorem 1.1. *Given an $\varepsilon > 0$ there exists a continuous decomposition of the plane into chainable continua so that each element of the decomposition contains a straight line segment with length larger than ε.*

Many other questions about continuous decompositions of the plane naturally occur.

Question 1.2. *Given a continuous decomposition G of the plane into nondegenerate acyclic continua, must it be the case that given any $\varepsilon > 0$ there exists an nondegenerate indecomposable continuum C and a $g \in G$ such that $\mathrm{diam}(C) < \varepsilon$ and $C \subset g$?*

Question 1.3. *What can be said about the elements of a continuous decomposition of the plane into acyclic continua where all the the elements are homeomorphic?*

Question 1.4 (Lewis)**.** *Is there a continuous decomposition of the plane into acyclic continua none of which are chainable* [L1996]*?*

2. Our continuous decomposition

Our strategy in describing our decomposition will be to define a sequence $\{P_n\}_{n=1}^{\infty}$ of partitions of the plane into cells with non-overlapping interiors so that the conditions of the following lemma from [L1978] are satisfied. Also see [S1995]. Before stating our version of the lemma we introduce the following notation. If P is a collection of sets, then P^* denotes the union of members of P. If p is a set, then $\mathrm{st}^1(p, P) = \{p' \in P : p' \cap p \neq \emptyset\}$ and inductively $\mathrm{st}^{i+1}(p, P) = \mathrm{st}^1(\mathrm{st}^i(p, P)^*, P)$. We abbreviate $\mathrm{st}^1(p, P)$ by $\mathrm{st}(p, P)$. By $\mathrm{N}_\varepsilon(x)$ we mean the set of points in the plane which are less than an ε distance from some point in x.

Lemma 2.1 (Lewis and Walsh)**.** *Let X be locally compact and $\{P_n\}_{n=1}^{\infty}$ be a sequence so that for each $n \in \mathbb{Z}^+$:*

(1) *The collection P_n is a locally finite family of non-empty closed subsets of X with $P_n^* = X$, with the elements of P_n having pairwise disjoint interiors, and with $\mathrm{Cl}(\mathrm{Int}(p_n)) = p_n$ for each $p_n \in P_n$.*
(2) *For each $p_n \in P_n$, $\mathrm{st}^3(p_n, P_{n+1})^* \subset \mathrm{st}(p_n, P_n)^*$.*
(3) *There is a positive number L such that for each pair $p_n, p'_n \in P_n$ with $p_n \cap p'_n \neq \emptyset$, we have that $p_n \subset \mathrm{N}_{L/2^n}(p'_n)$.*
(4) *There is a positive number K such that for each $p_{n+1} \in P_{n+1}$, there is a $p_n \in P_n$ with $p_{n+1} \cap p_n \neq \emptyset$ and $p_n \subset \mathrm{N}_{K/2^{n+1}}(p_{n+1})$.*

Let G be defined by $g \in G$ if $g = \cap_{n=1}^{\infty} \mathrm{st}(p_n, P_n)^*$ where $\cap_{n=1}^{\infty} p_n \neq \emptyset$; then G is a continuous decomposition of X.

In the following construction we follow [L1978]. See also [S1994]. The sequence, $\{P_n\}_{n=1}^{\infty}$, is defined inductively. Assuming we have already constructed $\{P_i\}_{i=1}^{n}$, we start stage $n+1$ of the induction given $\widehat{R}_{n+1}$, a division of the plane into either congruent vertical or congruent horizontal strips. Instead of defining the cells P_{n+1} directly, we first define cells Q_{n+1} which are much simpler to describe. The cells of P_{n+1} are the images of the cells of Q_{n+1} under a homeomorphism H_{n+1} that is also described inductively. There are three positive rational numbers a_{n+1}, b_{n+1}, and c_{n+1} that constrain the construction at stage $n+1$. Given a vertical (resp. horizontal) division, $\widehat{R}_{n+1}$, we refine the division by dividing each strip into finer strips, each with the width a_{n+1} and infinite length. We denote the new division by R_{n+1}. The construction alternates between working with horizontal strips and vertical strips starting with vertical strips when $n = 1$. Thus when $(n+1)$ is odd the strips of R_{n+1} run vertically with width a_{n+1}.

We partition the plane into a collection of cells Q_{n+1} with non-overlapping interiors by partitioning each strip into cells. The cells $q_{n+1} \in Q_{n+1}$ that partition a vertical strip are exactly as in Figure 1 and are called vertical cells. All the cells $q_{n+1} \in Q_{n+1}$ are identical. The vertical cell q_{n+1} has the width of a_{n+1}; has the height of exactly $b_{n+1} + c_{n+1}$; and has the thickness; i.e., vertical transverse thickness, of b_{n+1}. Thus the top boundary of a vertical cell is simply the vertical displacement of the bottom boundary by the constant b_{n+1}. Note that in our construction c_{n+1} is always much larger than b_{n+1}. Each vertical cell is symmetrical about a vertical line; the two identical halves being referred to as *chevrons*. Each chevron is also vertically symmetrical and consists of two *half-chevrons*. Thus each vertical cell consists of four congruent half-chevrons. Each half-chevron is a parallelogram. We refer to the short sides of this parallelogram as the *side boundaries* of the half-chevron. The cells that partition horizontal strips are like those described above except that they are rotated a quarter turn clockwise. They are called horizontal cells.

Once we have the collection Q_{n+1} defined, a homeomorphism $h_{n+1} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined so that h_{n+1}^{-1} "straightens" the boundaries between the cells. When $(n+1)$ is odd we define h_{n+1} so it maps vertical lines onto themselves by translation so that the preimage of any given vertical cell $q_{n+1} \in Q_{n+1}$ is a rectangle with width a_{n+1} and height b_{n+1}. Note that h_{n+1} restricted to the side boundaries of cells is the identity. The set P_{n+1} is defined to be $\{H_{n+1}(q_{n+1}) : q_{n+1} \in Q_{n+1}\}$, where $H_{n+1} = H_n \circ h_n = h_1 \circ \cdots \circ h_n$. When $(n+1)$ is even h_{n+1} maps each horizontal line onto itself by translation and for each $q_{n+1} \in Q_{n+1}$, we have that $h_{n+1}^{-1}(q_{n+1})$ is a rectangle of width b_{n+1} and height a_{n+1}. To finish stage $n+1$ we use $\{h_{n+1}^{-1}(q_{n+1}) : q_{n+1} \in Q_{n+1}\}$ to define $\widehat{R}_{n+2}$, a horizontal (resp. vertical) division of $\mathbb{R}^2$ with strips defined by horizontal (resp. vertical) lines placed at a distance of b_{n+1} apart.

To completely define our construction then we need only to define how to choose a_1, b_1, and c_1, and h_1 and how to choose a_{n+1}, b_{n+1},

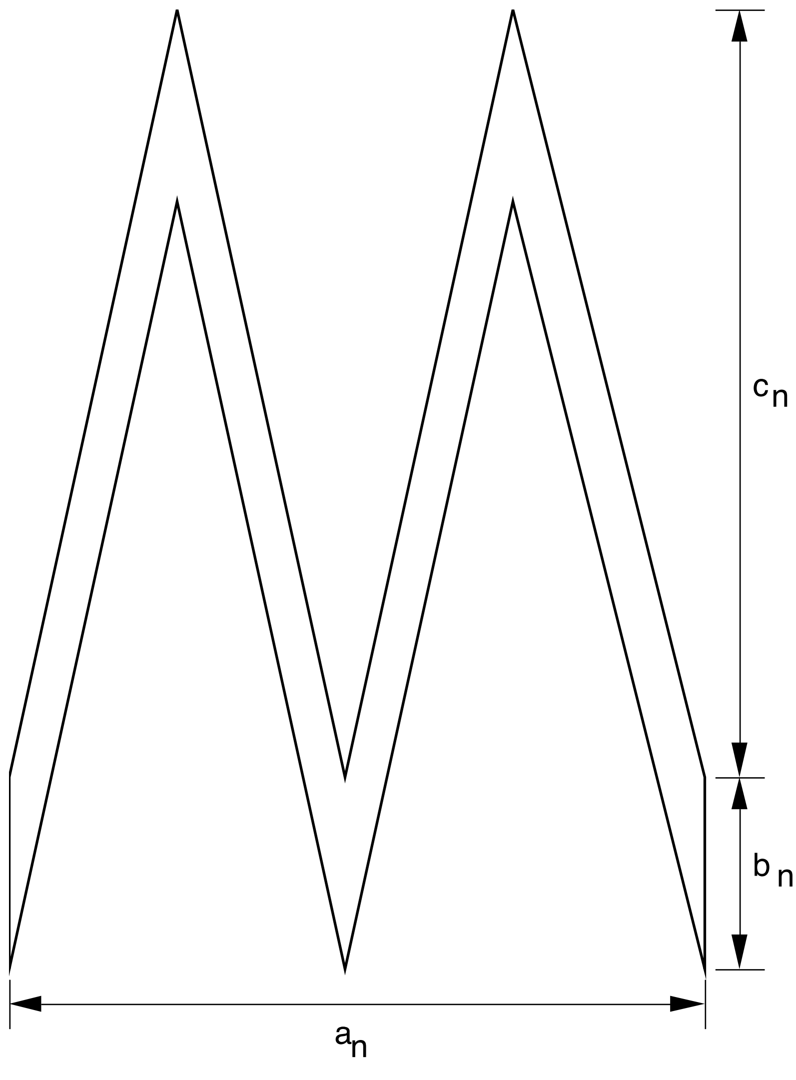

FIGURE 1. A Cell in Q_n.

and c_{n+1}, and h_{n+1} when we have completed stage n. We can begin, for example, by setting $\delta_1 = 1/4$, $c_1 = 4$, $a_1 = \delta_1/2$, $b_1 = a_1/2$, and $H_1 = Id$, where Id is the identity on $\mathbb{R}^2$. Define h_1 as described above so that h_1^{-1} "straightens" the boundaries of the cells $q_1 \in Q_1$ defined by a_1, b_1, and c_1.

At stage $n+1$ we proceed as follows:

(1) Pick $\delta_{n+1} > 0$ so that
$$|x - x'| < \delta_{n+1} \implies |H_{n+1}(x) - H_{n+1}(x')| < 1/2^{n+1}$$
for all $x, x' \in \mathbb{R}^2$ where $H_{n+1}(x) = H_n \circ h_n$.

(2) Set $c_{n+1} = (3/4)a_n$.

(3) Pick $a_{n+1} < \delta_{n+1}/2$ and so that a_{n+1} divides evenly into $(b_n/4)$.

(4) Set $b_{n+1} = a_{n+1}/2$.

(5) Define h_{n+1} as described above so that h_{n+1}^{-1} "straightens" the boundaries of the cells $q_{n+1} \in Q_{n+1}$ defined by a_{n+1}, b_{n+1}, and c_{n+1}.

The fact that Condition 1 of Lemma 2.1 holds follows immediately. To prove that Condition 2 holds it suffices to show that if $p_n \in P_n$, then
$$\mathrm{st}^3(H_{n+1}^{-1}(p_n), H_{n+1}^{-1}(P_{n+1}))^* \subset \mathrm{st}(H_{n+1}^{-1}(p_n), H_{n+1}^{-1}(P_n))^*.$$
This follows from the fact that $3a_{n+1} < b_n$ and $c_{n+1} + 3b_{n+1} < a_n$. We turn our attention to Condition 3. If $q_{n+1}, q'_{n+1} \in Q_{n+1}$ and $q_{n+1} \cap q'_{n+1} \neq \emptyset$, then $q_{n+1} \subset \mathrm{N}_\varepsilon(q'_{n+1})$ where $\varepsilon = a_{n+1} + b_{n+1}$. Since $\varepsilon < \delta_{n+1}$ Condition 3 holds. To show that Condition 4 holds consider $q_{n+1} \in Q_{n+1}$. Since $c_{n+1}/2 > \frac{1}{4}a_n$ we know that there is a $q_n \in Q_n$ so that $q_n \cap h_n(q_{n+1}) \neq \emptyset$ and so that $q_n \subset \mathrm{N}_\varepsilon(h_n(q_{n+1}))$ where $\varepsilon = a_n + b_n$ but $\varepsilon < \delta_n$ and so $H_n(q_n) \subset \mathrm{N}_{K/2^{n+1}}(H_{n+1}(q_{n+1}))$ where $K = 2$. Thus Condition 4 holds.

Therefore by Lemma 2.1 our decomposition is continuous. We will denote this decomposition by G. As in [L1978] each element of G is chainable.

3. Proof of Theorem 1.1

We extend the notion of half-chevron of a cell to that of the *half-chevron* of $\mathrm{st}(q_n, Q_n)^*$. A *half-chevron* of $\mathrm{st}(q_n, Q_n)^*$ is defined to be a parallelogram that consists of three cell half-chevrons taken from the cells in $\mathrm{st}(q_n, Q_n)$. Thus $\mathrm{st}(q_n, Q_n)^*$ contains 12 half-chevrons. The *side boundaries* of a half-chevron of $\mathrm{st}(q_n, Q_n)^*$ are the short sides of the parallelogram, which are vertical when n is odd and horizontal when n is even.

We say that $\mathrm{st}(q_{n+1}, Q_{n+1})^*$ *crosses* a half-chevron of $\mathrm{st}(q_n, Q_n)^*$, say q_n^p, when $\mathrm{st}(h_n(q_{n+1}), h_n(Q_{n+1}))^*$ intersects q_n^p and extends beyond both the side boundaries of q_n^p by at least $3b_{n+1}$. This definition ensures that if $\mathrm{st}(q_{n+1}, Q_{n+1})^*$ crosses a half-chevron q_n^p of $\mathrm{st}(q_n, Q_n)^*$ then for any half-chevron q_{n+1}^p of $\mathrm{st}(q_{n+1}, Q_{n+1})$ we have that $h_n(q_{n+1}^p) \cap q_n^p$ is a parallelogram that intersects both side boundaries of q_n^p.

Now we let $g \in G$; i.e., $g = \cap_{n=1}^{\infty} \mathrm{st}(p_n, P_n)$ where $\cap_{n=1}^{\infty} p_n \neq \emptyset$. For each $n \in \mathbb{Z}^+$ we let $q_n = H_n^{-1}(p_n)$ and $Q_n = H_n^{-1}(P_n)$. We will now show that g must contain an arc A.

First we observe that for every $n \in \mathbb{Z}^+$ there exists at least one half-chevron of $\mathrm{st}(q_n, Q_n)^*$ that is crossed by $\mathrm{st}(q_{n+1}, Q_{n+1})$. This follows from the fact that $(c_{n+1} - 3b_{n+1})/2 > a_n/4$. For each $n \in \mathbb{Z}^+$ we denote by q_n^p one such half-chevron.

Next observe that if $R \subset q_{n+1}^p$ is a parallelogram that intersects both side boundaries of q_{n+1}^p, then $h_n(R) \cap q_n^p = R'$ is a parallelogram that intersects both side boundaries of q_n^p. This follows from the definition of h_n and the fact that $h_n(R)$ must extend at least $3b_{n+1}$ beyond each side boundary of q_n^p.

For each given $n \in \mathbb{Z}^+$ we will denote the parallelogram q_n^p by R_n^n. For $k \in \{1, 2, ..., n\}$ we define R_{n+1}^k inductively as $R_{n+1}^k = h_k(R_{n+1}^{k+1}) \cap q_k^p$. Thus R_{n+1}^1 is a parallelogram that intersects both side boundaries of $R_1^1 = q_1^p$. Since $R_{n+1}^n \subset R_n^n$ we have that in general

$$R_{n+1}^k = h_k(R_{n+1}^{k+1}) \cap q_k^p \subset h_k(R_n^{k+1}) \cap q_k^p = R_n^k$$

for all $k \in \{1, 2, ..., n\}$ and in particular that

$$R_1^1 \supset R_2^1 \supset \cdots \supset R_n^1 \supset \cdots.$$

Finally observe that because the lengths of the side boundaries of q_n^p are $3b_n$, the lengths of the short sides of the parallelograms R_n^1 approach zero as n increases. Thus we have that $A = \cap_{n=1}^{\infty} R_n^1$ is a line segment that is contained in g and since A intersects both side boundaries of q_1^p it has length greater than $c_1 - 3b_1$, which can be made arbitrarily large.

The proof of Theorem 1.1 is complete.

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