Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Spring 2000
Pages 285-292

ON TRANSITIVE OPERATORS

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ABSTRACT. Let X be a topological space and $Y \subset X$. The map $f : Y \to X$ is said to be *transitive* if there exists an element $x_0 \in Y$ for which the forward orbit $O_+f(x_0) = \{f^n(x_0)|n = 1, 2, \dots\}$ is a dense subset of X. It is proved in this paper that the shift operators are transitive.

1. Introduction

Let X be a topological space and $Y \subset X$. The map $f : Y \to X$ is said to be *transitive* if there exists an element $x_0 \in Y$ for which the forward orbit $O_+f(x_0) = \{f^n(x_0)|n = 1, 2, \cdots\}$ is a dense subset of X. We will call such a point x_0 a *cyclic element* of f. It is well known ([1], ch. 18) that if X is complete space without isolated points and f is continuous function then the set of all cyclic elements of f is G_δ dense subset of X.

Note that transitivity is a topological property:

Proposition 1.1. *Suppose that $f : X \to X$ is transitive and let $g : X \to Y$ be a homeomorphism. Then $h = g \circ f \circ g^{-1}$ is also transitive.*

This paper contains sufficient conditions and some examples for two types of transitive operators.

The first part is devoted to shift operators in products of spaces. For the product $\mathbf{X} = \prod_{n=1}^{\infty} X_n$ the left shift $l : \mathbf{X} \to \mathbf{X}$ is defined by the formula $l(\mathbf{x}) = (x(2), x(3), \dots, x(n+1), \dots)$ where for $\mathbf{x} \in \mathbf{X}$ we have $\mathbf{x} = (x(1), x(2), \dots, x(n), \dots)$. To define a generalized shift operator we need the notion of *the shift map*:

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The injective map $\nu : \mathbf{N} \to \mathbf{N}$ of integers is said to be a shift map whenever the following condition holds:

$$(\mathbf{S}) \qquad \bigcap_{n=1}^{\infty} \nu^n(\mathbf{N}) = \emptyset$$

Furthermore, for every integer n, let X_n be a topological space and let $g_n : Y_{\nu(n)} \to X_n$ be a map which is defined on a dense separable subset $Y_{\nu(n)}$ of $X_{\nu(n)}$ and for which we have $g(Y_{\nu(n)}) = X_n$. In the sequel we let $\mathbf{Y} = \prod_{n=1}^{\infty} Y_n$

Definition 1.2. The map $g : \mathbf{Y} \to \mathbf{X}$ defined by the formula

$$g_\nu(\mathbf{x}) = g(\mathbf{x}) = (g_1(x(\nu(1))), g_2(x(\nu(2))), \ldots, g_n(x(\nu(n))), \cdots)$$

is called a generalized shift operator (shortly G-shift operator, or just a shift operator).

2. Transitivity of the shift operators

Denote for an integer $n \in \mathbf{N}$ by $\mathbf{I}_n$ the set $\mathbf{I}(n) = \{1, 2, \ldots, n\}$. In proving the main theorem we will use the following two lemmas.

Lemma 2.1. *For every k there exists an integer $n_k \in \mathbf{N}$ such that for $n \geq n_k$ we have $\mathbf{I}(k) \cap \nu^n(\mathbf{N}) = \emptyset$.*

Proof: We have $\emptyset = \bigcap_{n=1}^{\infty} \nu^n(\mathbf{N})$. The map ν is injective, hence $\emptyset = \mathbf{I}(k) \cap \bigcap_{n=1}^{\infty} \nu^n(\mathbf{N}) = \bigcap_{n=1}^{\infty} (\nu^n(\mathbf{N}) \cap \mathbf{I}(k))$. Because the set $\mathbf{I}(k)$ is finite, it follows from the above that $\nu^{n_k}(\mathbf{N}) \cap \mathbf{I}(k) = \emptyset$ for some $n_k \in \mathbf{N}$. From the inclusion $\nu^{n+1}(\mathbf{N}) \subset \nu^n(\mathbf{N})$ we obtain that $\mathbf{I}(k) \cap \nu^n(\mathbf{N}) = \emptyset$ for $n \geq n_k$. Evidently sequence $\{n_k\}$ may be choose to be increasing.

Lemma 2.2. *If ν is a shift function then there exists a sequence $p_1 < p_2 < \cdots$ of integers such that if $k \neq l$ then $\nu^{p_k}(\mathbf{I}(k)) \cap \nu^{p_l}(\mathbf{I}(l)) = \emptyset$.*

Proof: We may put for example $p_k = n_1 + \cdots + n_k$. Thus for $k < l$ we have $\emptyset = \mathbf{I}(k) \cap \nu^{n_k}(\mathbf{N}) = \nu^{p_k}(\mathbf{I}(k) \cap \nu^{n_k}(\mathbf{N}))$. Thus for the injective function ν^{p_k} we obtain $\nu^{p_k}(\mathbf{I}(k)) \cap \nu^{p_k+n_k}(\mathbf{N}) =$

$\emptyset$. Furthermore evidently $p_l \geq p_k + n_k$ and $\nu^{p_l}(\mathbf{I}(l)) \subset \nu^{p_l}(\mathbf{N}) \subset \nu^{p_k+n_k}(\mathbf{N})$, so $\nu^{p_k}(\mathbf{I}(k)) \cap \nu^{p_l}(\mathbf{I}(l)) = \emptyset$.

Now we can prove the main result of the paper.

Theorem 2.3. *The operator g is transitive.*

Proof: Let for A_n, $n = 1, 2, \ldots$ be a dense subset of $Y_n \subset X_n$. For every integer n let us denote with z_n some fixed point of $Y_n \subset X_n$ and then consider a countable subset

$$\mathbf{A}_n = A_1 \times A_2 \times \cdots \times A_n \times \{z_{n+1}\} \times \{z_{n+2}\} \times \cdots$$

of $\mathbf{X}$. Let $\mathbf{A} = \bigcup_{n=1}^{\infty} \mathbf{A}_n$. We will show that $\mathbf{A}$ is dense in $\mathbf{X}$, i.e. that $\mathbf{A} \cap V \neq \emptyset$ for an arbitrary non - empty open set $V \subset \mathbf{X}$. We may of course assume that $V = U_1 \times \cdots \times U_n \times X_{n+1} \times X_{n+2} \times \cdots$ where for $i = 1, \cdots, n$ U_i is an open subset of X_i. Let $a_i \in A_i \cap U_i$ for every $i = 1, \cdots, n$. Clearly for $\mathbf{a} = (a_1, \cdots, a_n, z_{n+1}, \cdots)$ we have $\mathbf{a} \in \mathbf{A} \cap V$.

We are going to construct in the sequel an element $\mathbf{c} \in \mathbf{X}$ whose forward orbit $O_+(g)(\mathbf{c})$ is a dense set in $\mathbf{X}$. For this purpose, let us define for each $\mathbf{a} \in \mathbf{A}$ the *weight* $w(\mathbf{a})$ of $\mathbf{a}$ by means of the formula $w(\mathbf{a}) = \max\{i | \mathbf{a}(i) \neq z_i\}$. It is easy to verify that the set $\mathbf{A}$ is countable and moreover that it can be written as a sequence $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \cdots\}$ such that the condition $w(\mathbf{a}_n) \leq n$ holds for every $n \in \mathbf{N}$. So we can write $\mathbf{a}_n = (a_{n1}, \ldots, a_{nn}, z_{n+1}, \ldots)$.

Furthermore let

$$M = \bigcup_{k=1}^{\infty} (\mathbf{I}(k) \times \{p_k\})$$

and let ν^{∞} be the function $\nu^{\infty} : M \to \mathbf{N}$ defined by $\nu^{\infty}(i, p_k) = \nu^{p_k}(i)$ where p_k is as in Lemma 2.2. Clearly ν^{∞} is injective and $\nu^{\infty}(\mathbf{I}_k \times \{p_k\}) \cap \nu^{\infty}(\mathbf{I}_l \times \{p_l\}) = \emptyset$ for $k \neq l$ since $\nu^{\infty}(\mathbf{I}(k)) = \nu^{p_k}(\mathbf{I}(k))$ for every $k \in \mathbf{N}$.

Now we can point out the construction of the element $\mathbf{c} \in \mathbf{X}$:

(a) $\mathbf{c}(j) = z_j \in Y_j$ if $j \notin \nu^{\infty}(M)$.

(b) if $j \in \nu^{\infty}(M)$ then $j = \nu^{p_k}(i)$ for some k and $i \in \mathbf{I}(k)$. Choose now $c(j) \in h^{-1}(a_{ki})$ where $h = g_i \circ g_{\nu(i)} \circ \cdots \circ g_{\nu^{p_k-1}(i)}$.

It remains to show that the forward orbit $O_+(g)(\mathbf{c}) = \{g^n(\mathbf{c}) | n =\in \mathbf{N}\}$ is a dense set. Let as above V be an open subset of $\mathbf{X}$ and

$V = U_1 \times \cdots \times U_n \times X_{n+1} \times \cdots$. Then for some $k \geq n$ one should have $\mathbf{a}_k = (a_{k1}, \ldots, a_{kk}, z_{k+1}, \ldots) \in \mathbf{A} \cap V$ where $a_{ki} \in U_i$. It follows now by the construction of the element $\mathbf{c}$ that $g^{p_k}(\mathbf{c})$ has the form

$$g^{p_k}(\mathbf{c}) = (a_{k1}, a_{k2}, \ldots, a_{kk}, \omega_{k+1}, \ldots)$$

where ω_n for $n \geq k+1$ is some (arbitrary) element of X_n. Clearly $g^{p_k}(\mathbf{c}) \in V$, which finishes the proof.

Denote with X the set of all irrational numbers in the interval $[0, 1]$. Every element $x \in X$ can be written as a continued fraction: $x = [0; x_1, x_2, \cdots]$.

Example 2.4. *The function $f : X \to X$ defined by the equation*

$$f([0; x_1, x_2, x_3, \cdots]) = [0; x_2, x_3, \cdots]$$

is transitive.

Proof: The space X is homeomorphic to the product $\mathbf{N}^{\mathbf{N}}$ under the function $g : \mathbf{N}^{\mathbf{N}} \to X$; $g(x_1, x_2, \cdots) = [0; x_1, x_2, \cdots]$ [2]. Example 2.4 follows now by Proposition 1.1.

Example 2.5. *Let K be the Cantor set and for $x \in K$ let $x = 0, x_1x_2\cdots$ be the expression of x in ternary number system. The function $f : K \to K$; $f(x) = 0, x_2x_3\cdots$ is transitive.*

The spaces K and $\{0, 2\}^{\mathbf{N}}$ are homeomorphic, moreover the equality $f(x_1, x_2, \ldots) = 0, x_1x_2\ldots$ give the desired homeomeorphism [2].

A rich class of examples may be obtained by usung results in [3]. It is described in [3] the class of infinite - dimensional linear topological spaces which are homeomorphic to $\mathbf{R}^{\mathbf{N}}$. The following statement obtains from the Anderson - Kadec Theorem ([3], ch. VI):

Example 2.6. *For every infinite - dimension separable Frechet space X there exists a transitive operator $f : X \to X$ which is conjugate of the left shift in $\mathbf{R}^{\mathbf{N}}$.*

3. Linear transitive operators

This part contains some examples of transitive operators in Frechet spaces as well in different spaces which are homeomorphic to products.

It contains also the following some examples of linear transitive operators which are similar to the shifts. In particular we consider shifts relatively given basis in given linear topological space. Our tool in this part is based on the following sufficient condition concerning tranzitivity of the linear operators:

Theorem 3.1. *Let $L : X \to X$ be a linear operator defined on the Frechet space X. Suppose that the following conditions are fulfilled:*

(a) $\bigcup\limits_{n=1}^{\infty} L^{-n}(0)$ is a dense set in X.

(b) There exists a continuous right inverse B of L such that $B^n(x) \to \mathbf{0}$ for every $x \in X$, where $\mathbf{0} \in X$ is the zero element of X.

Then L is transitive.

Proof: Let ϱ be a invariant metric in X (i.e. $\varrho(x, y) = \varrho(x - y, 0))$ which is compatible with the topology in X. Denote $M = \bigcup\limits_{n=1}^{\infty} L^{-n}(0)$ and for $x \in M$ let $\deg(x) = \min\{n \in \mathbf{N} | L^n(x) = 0\}$ be the *degree* of x. Let $A = \{a_1, a_2, \dots, a_n, \dots\}$ be a countable dense subset of M - note that A is dense in X.

Now we choose by induction a sequence $k_1 < k_2 < \cdots < k_n < \dots$ of integers which satisfies the following conditions:

(i) $\varrho(\mathbf{0}, B^{k_n - k_m}(a_n)) \leq \frac{1}{2^n}$ for $m \leq n$ and

(ii) $k_{n+1} - k_n > \deg a_n$ for $n \in \mathbf{N}$.

Let us set $k_1 = \deg a_1$ and suppose that the integers k_m are defined for $m < n$. Because $\lim\limits_{l \to \infty} B^l(a_n) = \mathbf{0}$ there exists an integer p_n such that $\varrho(B^l(a_n), \mathbf{0}) < \frac{1}{2^n}$ whenever $l > p_n$. Now we can take for example $k_n = p_n + \deg a_n + k_{n-1}$. Note that it follows from the choice of p_n that $\varrho(B^{k_n}(a_n), \mathbf{0}) < \frac{1}{2^n}$. To continue on put $b_n = B^{k_n}(a_n)$ and $c_n = b_1 + \cdots + b_n$. We are going to prove that there exists $c = \lim\limits_{n \to \infty} c_n$ and that the forward orbit $O_+L(c)$ of c is a dense set in X.

Note for this purpose that for $m > n$ we have $\varrho(c_m, c_n) = \varrho(c_m - c_n, \mathbf{0}) = \varrho(b_{n+1} + \cdots + b_m, \mathbf{0})$. Hence $\varrho(c_m, c_n) \leq \varrho(b_{n+1}, \mathbf{0}) + \cdots \varrho(b_m, 0) \leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m} < \frac{1}{2^n}$ which means that the sequence $\{c_n\}$ is convergent.

In the same way one can prove also that for every n the sequence $\{A^{k_n}(c_m)\}$ is convergent when $m \to \infty$. Indeed for $m > l$ we have

$$\varrho(A^{k_n}(c_m), A^{k_n}(a_l)) = \varrho(A^{k_n}(a_m - a_l), \mathbf{0}) = \varrho(\sum_{i=l+1}^{m} B^{k_i - k_n}(a_i), \mathbf{0})$$

Thus $\varrho(A^{k_n}(c_m), A^{k_n}(a_l)) \leq \sum\limits_{i=l+1}^{m} \varrho(B^{k_i-k_n}(a_i), \mathbf{0}) \leq \sum\limits_{i=l+1}^{m} \frac{1}{2^i} < \frac{1}{2^l}$ which shows that $\{A^{k_n}(c_m)\}$ is a Cauchy sequence.

It remains to prove that $O_+L(c)$ is a dense set in X. Take for this purpose an arbitrary $x \in X$ and $\varepsilon > 0$. Because A is a dense set, there exists $n \in \mathbf{N}$ such that $\varrho(x, a_n) < \varepsilon$. Now for $m > n$ we have $A^{k_n}(c_m) = \sum\limits_{i=1}^{m} A^{k_n}(b_i)$ and it follows from (ii) that $A^{k_n}(c_m) = a_n + \sum\limits_{i=n+1}^{m} B^{k_i-k_n}(a_i) = a_n + b$, where $b = \sum\limits_{i=n+1}^{m} B^{k_i-k_n}(a_i)$. Hence $\varrho(A^{k_n}, x) = \varrho(a_n + b, x) = \varrho(a_n - x + b, \mathbf{0}) = \varrho(a_n - x, -b) \leq \varrho(a_n - x, \mathbf{0}) + \varrho(\mathbf{0}, -b) = \varrho(a_n, x) + \varrho(b, \mathbf{0})$. It follows now from (i) that $\varrho(A^{k_n}(c_m), x) \leq \varepsilon + \sum\limits_{i=n+1}^{m} \varrho(B^{k_i-k_n}(a_i), \mathbf{0}) \leq \varepsilon + \sum\limits_{i=n+1}^{m} \frac{1}{2^i} < \varepsilon + \frac{1}{2^n}$. In the inequality $\varrho(A^{k_n}(c_m), x) < \varepsilon + \frac{1}{2^n}$ we let $m \to \infty$ to obtain that $\varrho(A^{k_n}(c), x) < \varepsilon + \frac{1}{2^n}$.

Keeping in mind that in the above considerations the integer n may be choosen arbitrarilly large we obtain that the forward orbit $O_+L(c)$ is a dense subset of X.

Theorem 3.1 gives a sufficient condition only. The example below shows that it is not necessary. In Example 3.2 we consider the space $C^\infty(\mathbf{R})$ of all smooth real functions with the standard topology generated for example by the metric:

$$\varrho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}$$

where for $f \in C^\infty(\mathbf{R})$ we have $\|f\|_n = \sup\{|f^{(i)}(x)| | i \leq n; \quad x \in [-n, n]\}$.

Example 3.2. *There exists a function $f \in C^\infty(\mathbf{R})$ for which the sequence $\{g_n | n \in \mathbf{N}\}$ where $g_n(x) = f(x+n)$; $x \in \mathbf{R}$ forms a dense subset of C^∞.*

Note that this example states that the operator $L : C^{\infty}(\mathbf{R}) \to C^{\infty}(\mathbf{R})$; $L(f)(x) = f(x+1)$ is transitive and in the same time it is obvious that the conditions (a) and (b) from Theorem 3.1 do not hold.

Proof: Let $\{a(m,n)\}$ be a sequence of integers which has the following property:

$$|a(m,n) - a(p,q)| > 2(m+n+p+q)$$

for $(m,n) \neq (p,q)$.

Furthermore denote the intervals $[a(m,n)-n, a(m,n)+n]$ with $\Delta(m,n)$. Note that $\Delta = \bigcup\limits_{m,n} \Delta(m,n)$ is a closed subset of the real line because $\{\Delta(m,n)\}_{m,n}$ is a discret system of intervals.

Now let $\mathcal{P} = \{p_1, p_2, \dots\}$ be a countable dense set in $C^{\infty}(\mathbf{R})$ and denote by $p_{m,n}$ the restriction of the function p_m over the interval $[-n,n]$. We define the function $\bar{f} : \Delta \to \mathbf{R}$ by setting $\bar{f}(x) = p_{mn}(x - a(m,n))$ for $x \in \Delta(m,n)$. Evidently $\bar{f} \in C^{\infty}(\Delta)$, and as it is well known, $\bar{f}$ can be extended to a smooth function f over the real line $\mathbf{R}$. We are going to prove that f is the desired function.

For that purpose let us choice $h \in C^{\infty}$, $\varepsilon > 0$ and an integer n. To complete the proof it is sufficient to find a functon g_m for which the inequality $\|g_m - h\|_n < \varepsilon$ holds. Clearly one may suppose in additional that $\sum\limits_{i=n}^{\infty} 2^{-i} < \frac{\varepsilon}{2}$. Because of density of $\mathcal{P}$, one can find a function $p_m \in \mathcal{P}$ for which $\|p_m - h\|_n < \frac{\varepsilon}{2}$. Actually we would have $\|p_m - h\|_j < \frac{\varepsilon}{2}$ for $j \leq n$, since $\|\cdot\|_j \leq \|\cdot\|_n$. Note furthermore that for $x \in [-n,n]$ we have

$$g_{mn}(x) = f(x + a(m,n)) = p_{mn}(x + a(m,n) - a(m,n)) = p_m(x)$$

That's why

$$\varrho(g_{a(m,n)}, h) \leq \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|p_{mn} - h\|_i}{1 + \|p_{mn} - h\|_i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i}$$

Because of the choice of n we have

$$\varrho(g_{a(m,n)}, h) \leq \frac{\varepsilon}{2} \sum_{i=1}^{n} \frac{1}{2} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} < \varepsilon$$

which completes the proof.

Theorem 3.1 is applicable in different situations; below we point out some examples.

Example 3.3. [4] *The operator* $\frac{d}{dx} = L : C^\infty \to C^\infty$ *is transitive.*

It is easy to verify that L satisfies the conditions (a) and (b) of Theorem 3.1. More precisely, the set $\bigcup\limits_{n=1}^{\infty} L^{-n}(0)$ contains all polynomials and one can put $B(f)(x) = \int\limits_0^x f(t)dt$.

Similarly as an easy consequence from Theorem 3.1 one can obtain that every linear differential operator with constant coefficients is transitive [5] as well as for different types of partial differential operators [4].

A weighted shift operator L on (complex) Hilbert space H is an operator that maps each vector in some orthonormal basis $\{e_n\}$ into a scalar multiple of the next vector: $L(e_n) = w_n e_{n+1}$ [6]. It follows from Theorem 3.1 that:

Example 3.4. *Let* L *be a weighted shift operator. If*

$$\lim_{n\to\infty} \sup_k |w_k w_{k+1} \cdots w_{k+n-1}| = \infty$$

then L *is transitive.*

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