Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Spring 2000
Pages 309-320

TIMES OF DIFFUSION IN HAMILTONIAN SYSTEMS*

ENRICO VALDINOCI†

Abstract. In the context of Arnold diffusion, we consider several Hamiltonian systems close to integrability and we show that the times of diffusion are polynomial in the inverse of the splitting.

After the pioneering work of Poincaré, the problem of the stability of nearly-integrable Hamiltonian systems has been widely studied. Very roughly, the problem is whether a very small perturbation of an "integrable" (i.e., completely stable) system can produce an appreciable instability of the action variables. This question naturally arises in models related with celestial mechanics, in which the action variables represent the length of the semiaxes of the ellipses on which the planets run or the inclination of a planet's axis. The study of this problem has now interest in many fields of mathematics, such as ergodic theory, PDEs and differential geometry; the development of the theory made use of techniques based on functional analysis, renormalization group, statistical physics and field theory. A crucial rôle were also played by some arithmetic conditions. Moreover, the machinery developed to address this problem has become a strong tool of applied mathematics with the advent

*This report presents the results contained in the paper *UpperBounds on Arnold Diffusion Time via Mather Theory*, written in collaboration with Professor Ugo Bessi and Professor Luigi Chierchia, submitted to Journal de Mathématiques Pures et Appliquées.

† It is a pleasure to thank Prof. Ugo Bessi and Prof. Luigi Chierchia for the opportunity of working with them on the problem presented in this note.

of computer assisted proofs and of constructive methods to asses the reliability of numerical experiments.

The example in [A] first showed the possibility of an order one drift of the action variables in perturbations of integrable Hamiltonian systems. This phenomenon, known as Arnold diffusion, happens in systems with more than two degrees of freedom; two degree of freedom systems are stable to perturbation, since the chaotic regions are necessarily sandwiched between KAM tori[1]. Indeed, if $n = 2$, we can reduce the four-dimensional phase space to a three-dimensional energy surface, since one of the action can be recovered by energy conservation. From the KAM results, it is known that this space is densely covered by two-dimensional tori and this provide a *topological obstruction* to diffusion, since the trajectories trapped in-between these tori can only experience a small variation of the actions, for any time and for any initial datum.

If the number of degrees of freedom is $n > 2$, the orbits are not enclosed anymore by the KAM tori, so that a drift of order one in the actions may occur even in systems arbitrarily close to integrability. In this situation, the dynamics can be very intricate, due to the coexistence of stable and chaotic motions: indeed the KAM results yield *metric stability*, i.e. perpetual stability for the majority of initial data, whereas *topological instability* can be produced by a drift in the actions.

The KAM tori and their stable and unstable manifolds are the building blocks to construct drifting orbit (see, for example, [CV] and references therein). Roughly, this instability is due to the transverse intersection between stable and unstable manifolds of different KAM tori. The stable and unstable manifolds of the KAM tori are usually called whiskers. The so called problem of the splitting consists in proving that the whiskers intersect transversally and in giving a quantitative estimate of this transversality. This is usually a very delicate matter, and the estimate relies on averaging techniques or graph theory and make use of the arithmetic properties of the frequencies. For a construction of the whiskered tori see, for example, [Gr], [T] and [V]; for the problem of the splitting see

[1]Here, KAM means that these tori are constructed by a Kolmogorov-Arnold-Moser technique (see, for instance, [M]).

[DG], [DGJS] and [GGM]. For a tutorial on the main ideas in KAM theory see [Ll].

Heuristically, one may think that the splitting is the "angle" formed by the intersection of the whiskers: a rigorous definition involves a quantitative statement on the nondegeneracy of a minimum of the Melnikov function (see, for instance, Proposition 1 of [BCV]).

The geometry of our system is then given by a chain of whiskered tori; the splitting measures the transversality of the intersection of the approaching and departing whiskers. Thus, by elementary considerations, one sees that the distance between the first and the last torus of the chain is given by the product between the splitting and the number of the tori in the chain.

By a topological argument, from this geometry it is possible to prove the existence of orbits of diffusion in a very general setting. However, it is not easy to deduce only from this geometry a good bound on the times in which such diffusion takes place: for instance, the proof presented in [CG] and [CV] would lead to time estimates that are much worse than exponential.

The scope of this note is to consider several examples for which the existence of the mechanism of Arnold diffusion has been proven, and to show that the **diffusion time** is of the order of an inverse power of the splitting. The exposition of this note will be quite informal: we will not enter here in detailed proofs of our statements, and we will just highlight some of the ideas involved. For full details we refer to [BCV].

The problem of the time of stability in this context goes back to [N]. Several estimates on the speed of Arnold diffusion have been recently considered also by [Be] and [Cr]. Related results have been announced by [Bo] and [BB].

Here, we consider the following families of Hamiltonians:

- **A priori unstable system** (see [CG]):

$$\mathcal{H} = |I|^2 + p^2 + \cos q - 1 + \varepsilon f(\phi, q)\,. \tag{1}$$

- **Isochronous system** (see [G]):

$$\mathcal{H} = \omega \cdot I + p^2 + \cos q - 1 + \varepsilon f(\phi, q)\,. \tag{2}$$

- **Linear and quadratic degenerate systems** (see [B]):

$$\mathcal{H} = \varepsilon\omega \cdot I + p^2 + \varepsilon^d(\cos q - 1) + \varepsilon^{d'} f(\phi, q) \qquad \text{and} \tag{3}$$

$$\mathcal{H} = \varepsilon|I|^2 + p^2 + \varepsilon^d(\cos q - 1) + \varepsilon^{d'} f(\phi, q)\,. \tag{4}$$

Here and in the sequel, $(I, \phi) \in \mathbf{R}^{n-1} \times \mathbf{T}^{n-1}$ and $(p, q) \in \mathbf{R} \times \mathbf{T}$ will be canonically conjugated (action-angle) variables. The constants d and d' above are assumed[2] to verify $1 \le d \le 2$ and $d' > 3 + d/2$, f is a suitable nondegenerate perturbation, ε is a small (positive) parameter. We will also assume suitable Diophantine condition on the frequency ω. All these assumptions are quite natural, and they are made in order to apply the results of [B], [CG], [G] and [GGM].

We notice that all the above mentioned Hamiltonians are obtained by coupling pendula, rotators and oscillators. They are simplified models for some problems arising in celestial mechanics, and the example of [A] is included in (1) as a particular case. The result addressed in this note is the following:

Theorem 1. *Under suitable nondegeneracy conditions, all the systems above have orbits exhibiting a drift of order one (i.e. independent of ε) in the action variables, for which the time of drift is of the order of an inverse power of the splitting. The energy of these orbits is controlled independently of ε.*

More precisely, since it is known (see [B], [CG]) that the splitting is of the order of a power of ε in cases 1–4, a quantitative version of the previous result is the following:

Theorem 2. *Under suitable nondegeneracy conditions, the systems introduced above have an orbit $\Big(I(t), \phi(t), p(t), q(t)\Big)$ satisfying*

$$|I(T) - I(0)| \ge C_1 \qquad \text{with} \qquad 0 \le T \le \frac{C_2}{\varepsilon^{C_3}}\,,$$

where the C_i's are suitable constants, independent of ε.

The detailed proof of this result is in [BCV]: the proof is a combination of the known KAM results and a variational technique. The orbit of diffusion corresponds to a local minimum of an opportune functional. Roughly speaking, this functional is constructed

[2] If d and d' do not fall in the range mentioned here, exponentially long times of stability have been proven by [B] for all initial conditions.

by adding together suitable action functionals. Such variational tool is similar in spirit to Mather theory (see [Ma]).

Other systems could be also considered, by using the "abstract" version of the result discussed here, contained in Proposition 1 of [BCV]. Indeed, the techniques discussed here do not rely much on the particular form of the Hamiltonians, but mainly on the geometric structure and on the quantitative relations among the parameters. Grossly, we have:

Let $\mathcal{H} = h(I;\varepsilon)+\mathcal{P}(p,q;\varepsilon)+\mathcal{F}(\phi,q)$ be a Hamiltonian system, in which h is convex, $\mathcal{P}$ is "pendulum-like" and $\mathcal{F}$ is a small perturbation. Assume that the "usual" KAM structure (whiskered tori and splitting) holds. Let σ be the splitting. Let θ be the time needed to travel on the whisker from the Poincaré section to a neighborhood of the torus where the KAM normal form holds. Then the time of diffusion can be bounded by

$$\left(\frac{1}{\sigma^C}+\theta\right)\cdot\frac{1}{\sigma},$$

for a suitable constant C.

In this way, it is also possible to consider some "a-priori stable" stable systems as

$$\mathcal{H} = \omega\cdot I + p^2 + \varepsilon(\cos q - 1) + \mu(\cos q - 1) f(\phi)\,. \tag{5}$$

Here, we assume $n = 2$ and $\omega = (1, \frac{1+\sqrt{5}}{2})$; f is an appropriate analytic function and $\mu = \varepsilon^p$, for a suitable $p > 1$: this system has been studied in [DGJS] and [DG], where an exponential upper and lower bound on the splitting is given. In this case, the proof we will present here still gives times of diffusion that are polynomial in the splitting and hence exponential in the perturbation:

Theorem 3. *System (5) has an orbit $\Big(I(t),\phi(t),p(t),q(t)\Big)$ satisfying*

$$|I(T)-I(0)|\geq C_1 \qquad \text{with} \qquad 0\leq T\leq C_2 e^{C_3/\varepsilon^{C_4}}\,,$$

where the C_i's are suitable constants, independent of ε.

The main steps of the proof of Theorems (1), (2) and (3) are the following:

The Lagrangian formulation. First of all, we transform our Hamiltonian systems into Lagrangian ones. This is done in order to apply in the sequel a variational technique close to Mather theory, which has a more convenient setting in Lagrangian formulation. We will denote the corresponding Lagrangian by $\mathcal{L}$. Since in some of the examples above we do not have strict uniform convexity of the unperturbed Hamiltonian, it is convenient to add a small kinetic energy, depending on a parameter $\kappa > 0$, vanishing as κ goes to zero. The results will be independent of κ, hence we can pass $\kappa \to 0$ and obtain our result by a limit argument.

The KAM results. The constructive results of [B], [CG], [GGM] and [V] are set into a variational form. In particular, we consider the preservation of a family of KAM tori $\mathcal{T}_1, \dots, \mathcal{T}_N$ with stable whiskers $W_1^s, \dots, W_N^s$ and unstable whiskers $W_1^u, \dots, W_N^u$ with a Diophantine rotation frequency. These whiskered tori lie on the energy surface $\{\mathcal{H} = E\}$. Near the tori, the Hamiltonian can be put into a "normal form", in which the orbits have a very explicit and simple form, namely a linear ergodic flow on the torus and a hyperbolic motion on the whiskers.

The difference between the action of a point in $\mathcal{T}_1$ from the action of a point in $\mathcal{T}_N$ is of order one.

Since these manifold are Lagrangian, there exist smooth functions Φ_i^s, depending on the angles, such that each stable whisker W_i^s is described by the graph of $c_i + \nabla\Phi_i^s$, and an analogous statement holds for W_i^u.

Also, it is convenient[3] to "smooth out" the above constants c_i: therefore we consider a smooth bump function S_i, with domain in the space of the angles, supported in a neighborhood of the point where the splitting is evaluated, and such that $\nabla S_i = c_{i+1} - c_i$. Thus, we define $\tilde{c}_i(\phi, q) := c_i + \nabla S_i(\phi, q)$.

Here and in the following, it is also convenient to replace the space of the angles $\mathbf{T}^n$ by its cover $\mathbf{T}^{n-1} \times \mathbf{R}$ (i.e. we look at the cover of the q variable), and consider Poincaré sections $P_i := \{q = \pi + 2i\pi\}$ for $i = 0, \dots, N-1$.

[3]This is done in order to have a smooth Lagrangian in the argument described in Figure 3.

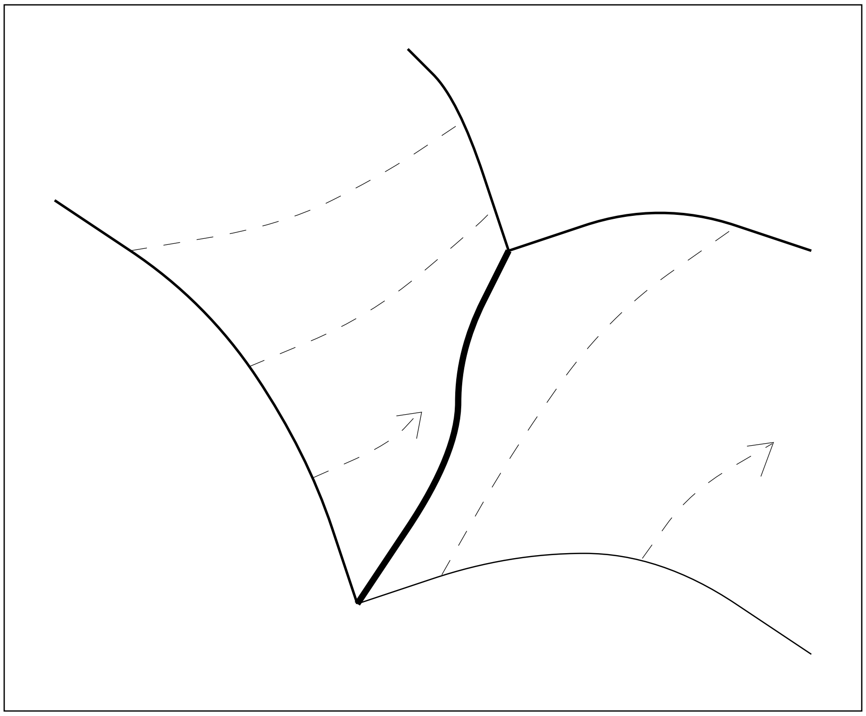

Figure 1: symbolic representation of a KAM torus and its whiskers.
The sides should be thought as identified.

For any $i = 1, \ldots, N$, we consider the natural projection of $\mathcal{T}_i$ on the space of the angles $\mathbf{T}^{n-1} \times \mathbf{R}$. We denote this projection by T_i and we fix a point $z_i \in T_i$. By periodicity, we may and do assume the $q-$coordinate of T_i to be close to $2i\pi$. The position of the tori T_i with respect to the Poincaré sections P_i is described in Figure 2.

Mather theory. We now introduce the variational principle that will provide the desired orbit of diffusion.

The basic idea is that much information on the system is coded inside the action functional: we then select suitable orbits in the intricate phase portrait of our system by requiring of them to satisfy global variational properties instead of the local ones that every orbit satisfies. This technique has a long tradition, since it goes

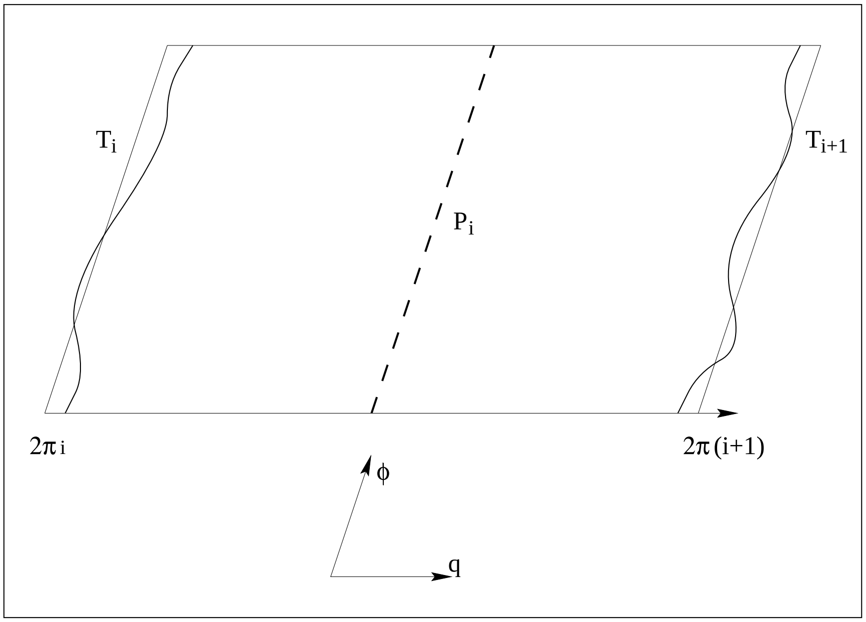

Figure 2: position of the tori T_i with respect to the Poincaré sections P_i in the space of the angles.

back to Morse and Hedlund and it has been extensively applied by Mather, Bangert and Mañé. Following [Ma], we introduce the following function: for $X_1, X_2 \in \mathbf{T}^n$,

$$h_i^t(X_1, X_2) := \min \int_0^t \mathcal{L}(X(\tau), X'(\tau)) - \tilde{c}_i(X(\tau)) \cdot X'(\tau) + E\, dt\,,$$

where the minimum is taken over all the absolutely continuous curves $X : [0,t] \longrightarrow \mathbf{T}^n$ satisfying $X(0) = X_1$, $X(t) = X_2$. The existence of such minimum is given by a Theorem of Tonelli. Also we define $h_i^\infty := \liminf_{t\to\infty} h_i^t$, that can be roughly considered as the minimal action of the orbits connecting X_1 and X_2 in an infinite time. Mather theory assures that this limit is finite, and it provides a connection between the derivative of h_i^∞ and the stable and unstable manifolds. Indeed, if z_i lies on the KAM torus T_i, we have that $\partial_x h_i^\infty(x, z_i)$ coincides (up to a sign) with the derivative of the function Φ_i^s that parameterizes the stable manifold. A similar result holds in the unstable case.

The variational argument. We consider suitable functionals built by the functions h_i^t and h_i^∞. The functional involving h_i^∞ will heuristically represent the action of a chain of pseudo-orbits connecting the KAM tori, while the functional involving h_i^t will be the key tool towards the existence of the drifting orbit. In detail: for any $i = 1, \dots, N$, we select a point z_i on the torus T_i, and define

$$F(\vec{x}) := h_1^\infty(z_1, x_1) + h_2^\infty(x_1, z_2) + h_2^\infty(z_2, x_2) + \cdots +$$

$$+ h_{N-1}^\infty(z_{N-1}, x_{N-1}) + h_N^\infty(x_{N-1}, z_N)\,,$$

where $\vec{x} := (x_1, \dots, x_{N-1})$, and x_i lie on the Poincaré section P_i, near the point of evaluation of the splitting.

From the relations between h_i^∞ and the whiskers, and making use of the results on the splitting, it follows that F has a nondegenerate minimum.

We also define

$$G(\vec{x}, \vec{t}) := h_1^\infty(z_1, x_1) + h_2^{t_1}(x_1, x_2) + h_3^{t_2}(x_2, x_3) + \cdots +$$

$$+ h_{N-1}^{t_{N-2}}(x_{N-2}, x_{N-1}) + h_N^\infty(x_{N-1}, z_N)\,,$$

where $\vec{t} := (t_1, \dots, t_{N-2})$, with $t_i \geq 0$. A local minimum of G corresponds to an orbit starting near the first torus and arriving near the last one in a time $T := t_1 + \cdots + t_{N-2}$.

Indeed, if $(\vec{x}, \vec{t})$ is a local minimum for G, we consider the orbit $\Big(\phi(t), q(t)\Big)$ obtained by gluing the orbit realizing $h_1^\infty(z_1, x_1)$ for times $t \in (-\infty, 0]$, the orbit realizing $h_2^{t_1}(x_1, x_2)$ for $t \in (0, t_1]$, the orbit realizing $h_3^{t_2}(x_2, x_3)$ for $t \in (t_1, t_1 + t_2]$, etc.

We have that such orbit must satisfy the Euler-Lagrange equations in

$$(0, t_1) \cup (t_1, t_1 + t_2) \cup \cdots \cup (t_1 + \cdots + t_{N-3}, t_1 + \cdots + t_{N-2})\,,$$

since, on these intervals, it minimizes the action functional. The Euler-Lagrange equations are also satisfied in $t_1 + \cdots + t_i$: this is a consequence of the fact that the orbit above must cross the Poincaré section with positive speed and of a standard variational argument (see Figure 3).

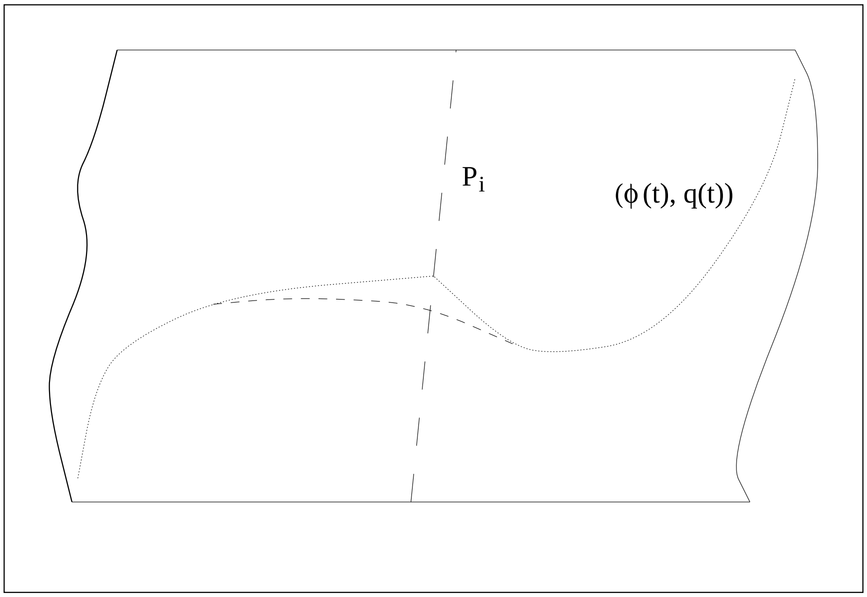

Figure 3: a small variation of the orbit.

The fact that the speed of the crossing of the Poincaré section at time $t_1+\cdots+t_i$ is positive can be proven by a comparison argument with the homoclinic of the pendulum: for details, see Appendix 2 of [BCV].

We now show that this orbit, that by the above discussion satisfies the Euler-Lagrange equations in $[0,T]$, exhibits a drift of order one in the actions. Indeed:

- Considering small variations in the ϕ coordinates, it is easy to prove that $\phi(t)$ is continuous at $t=0$.
- Since, on $(-\infty,0)$, $\Big(\phi(t),q(t)\Big)$ realizes $h_1^\infty(z_1,x_1)$, Mather theory implies that the Legendre transform of $\Big(\phi',\phi,q',q\Big)\Big|_{t=0^-}$ lies on the stable manifold of the first KAM torus $\mathcal{T}_1$.

From these two facts, it follows that the initial action of the orbit above is close to the action of $\mathcal{T}_1$. With a similar argument, the final action is close to the action of $\mathcal{T}_N$, proving that the orbit considered has a drift of order one in its action variables.

Let us briefly discuss now the existence of such local minimum for G. It is reasonable to expect that, for big t_i, G and F are close to each other, and that the existence of a nondegenerate minimum of F implies the existence of a minimum of G. This is exactly what happens in our case: the proof of this, contained in Lemma 3 of [BCV], uses the dynamics on the whiskers and involves some surgery argument on minimal orbits, in order to compare suitable trajectories. This will give reciprocal bounds between h_i^t and h_i^∞, allowing a control of G with respect to F and, thus, the existence of the desired minimum.

We finally remark that one of the main reasons why we obtain a polynomial bound is that in one of the above mentioned surgeries and comparisons, the ergodization of a KAM torus under its Diophantine linear flow is involved: at this step, we use a well known result (see, for instance, [BGW]) asserting that the ergodization time of a Diophantine linear flow on a torus is polynomial.

References

[A] V.I. Arnold, *Instability of Dynamical Systems with Several Degrees of Freedom*, Soviet Mathematics, **5-1** (1964), 581.

[BB] M. Berti and P. Bolle, *Arnold's diffusion in nearly integrable isochronous systems*, preprint, 2000.

[Be] U. Bessi, *Arnold's Example with Three Rotators*, Nonlinearity, **10** (1997), 763, and *A λ-lemma for Transition Tori*, preprint, 1999.

[BCV] U. Bessi, L. Chierchia and E. Valdinoci, *Upper Bounds on Arnold Diffusion Time via Mather Theory*, Journal des Math. Pures et Appl., to appear.

[B] L. Biasco, tesi di laurea, Universitá Roma Tre, 1999.

[Bo] S.V. Bolotin, communication, International Workshop, Aussois, France, 1998.

[BGW] J. Bourgain, F. Golse and B. Wennberg, *On the Distribution of Free Path Lengths for the Periodic Lorentz Gas*, Comm. Math. Phys., **190** (1998), 491.

[CG] L. Chierchia and G. Gallavotti, *Drift and Diffusion in Hamiltonian Systems*, Annales I. H. P., Physique Theorique, **60** (1994), 1.

[CV] L. Chierchia and E. Valdinoci, *A note on the construction of Hamiltonian trajectories along heteroclinic chains*, Forum Math., **12** (2000), 247.

[Cr] J. Cresson *Symbolic Dynamics and Arnold Diffusion*, preprint, 1999.

[DGJS] A. Delshams, V. Gelfreich, À. Jorba and T. M. Seara, *Exponentially small splitting of separatrices under fast quasiperiodic forcing*, Comm. Math. Phys., **189** (1997), 35.

[DG] A. Delshams, P. Gutièrrez, *Splitting potential and the Poincarè-Melnikov method for whiskered tori in Hamiltonian systems*, J. Nonlinear Sci., **10** (2000), 433.

[G] G. Gallavotti, *Arnold's Diffusion in Isochronous Systems*, Mathematical Physics, Analysis and Geometry, **1** (1999), 295.

[GGM] G. Gallavotti, G. Gentile and V. Mastropietro, *A field theory approach to Lindstedt series for hyperbolic tori in three time scales problems*, Journal of Mathematical Physics, **40** (1999), 6430, and *Hamilton-Jacobi Equation, Heteroclinic Chains and Arnol'd Diffusion in three time scales systems* Nonlinearity, **13** (2000), 32.

[Gr] S.M. Graff, *On the conservation of Hyperbolic Invariant Tori for Hamiltonian Systems*, Journal of Diff. Equations, **15** (1974), 1.

[Ll] R. de la Llave, *A tutorial on KAM theory*, preprint, 1999.

[Ma] J. Mather *Action Minimizing Invariant Measures for Positive Definite Lagrangian Systems*, Math. Z., **207** (1991), 169, and *Variational Construction of Connecting Orbits*, Annales de l'Institut Fourier, **43** (1993), 169.

[M] J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton Univ. Press, 1973.

[N] N.N. Nekhoroshev, *An exponential estimate of the time of stability of nearly integrable Hamiltonian systems*, Russian Math. Surveys, **32** (1977), 1.

[T] D.V. Treshchev, *The mechanism of destruction of resonance tori of Hamiltonian Systems*, Math. USSR Sbornik, **68** (1991), 181.

[V] E. Valdinoci, *Families of whiskered tori for a-priori stable/unstable Hamiltonian systems and construction of unstable orbits*, Math. Phys. Electronic Journal, **6** 2000.

The University of Texas at Austin, TX 78712-1082
E-mail address: enrico@math.utexas.edu