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SPANS OF CERTAIN SIMPLE CLOSED CURVES CROSS ARCS

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Abstract. For each continuum X, where X is from the class of concave upward y symmetric simple closed curves, we calculate all of the spans of $X \times J$ where J is an interval. We calculate the various spans of $B \times J$ where $B = R^2 - U$ and U is the unbounded component of $R^2 - X$. Also, we calculate all the spans of Y where Y is the boundary of $B \times J$ in R^3.

1. Introduction

The concept of the span of a metric space was introduced by Lelek in 1964 [L1]. Later, variations of the span were introduced (cf [L2]and [L3]). Much work has been done on both the topological and geometric aspects of the various spans. In general it is difficult to determine the spans of even simple geometric objects. Also, the relationships of the various spans for a particular space are usually not easy to determine.

For a continuum X from the class of continua that we refer to as the concave upward y symmetric simple closed curves, we calculate the span, semispan, surjective span and surjective semispan of $X \times J$ where J is an interval. We calculate all of the spans of $B \times J$ where $B = R^2 - U$ and U is the unbounded component of $R^2 - X$. Also, we calculate the spans of $(X \times J) \cup X_0 \cup X_h$ where $X_j = B \times \{j\}$ and $J = [0, h]$.

2. Preliminaries

If X is a non-empty metric space, we define the *span of* X, $\sigma(X)$, to be the least upper bound of the set of real numbers α which

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satisfy the following condition: there exists a connected space C and continuous mappings $g,\ f : C \to X$ such that

$$(\sigma) \qquad g(C) = f(C)$$

and $\alpha \leq$dist$[g\,(c)\,, f\,(c)]$ for $c \in C$.

The definition does not require X to be connected, but to simplify our discussion we will now consider X to be connected. The surjective span $\sigma^*\,(X)$, the semispan $\sigma_0\,(X)$, and the surjective semispan $\sigma_0^*\,(X)$ are defined as above, except we change conditions (σ) to the following:

$$(\sigma^*) \qquad g(C) \;=\; f(C) \;=\; X,$$

$$(\sigma_0) \qquad g(C) \;\subseteq\; f(C),$$

$$(\sigma_0^*) \qquad g(C) \;\subseteq\; f(C) \;=\; X,$$

Equivalently (see [L1], p. 209), the span $\sigma\,(X)$ is the least upper bound of numbers α for which there exist connected subsets C_α of the product $X \times X$ such that

$$(\sigma)' \qquad p_1(C_\alpha) \;=\; p_2(C_\alpha)$$

and $\alpha\ \leq$dist(x, y) for $(x, y) \in C_\alpha$, where p_1 and p_2 denote the projections of $X \times X$ onto X, i.e., $p_1\,(x, y) = x$ and $p_2\,(x, y) = y$ for $x,\ y \in X$. Again, we will now consider X to be connected. The surjective span $\sigma^*\,(X)$, the semispan $\sigma_0\,(X)$, and the surjective semispan $\sigma_0^*\,(X)$ are defined as above, except we change conditions $(\sigma)'$ to the following (see [L3]):

$$(\sigma^*)' \qquad p_1(C_\alpha) \;=\; p_2(C_\alpha) \;=\; X,$$

$$(\sigma_0)' \qquad p_1(C_\alpha) \;\subseteq\; p_2(C_\alpha),$$

$$(\sigma_0^*)' \qquad p_1(C_\alpha) \;\subseteq\; p_2(C_\alpha) \;=\; X.$$

By a continuum we mean a nondegenerate, compact and connected metric space. We note that for a compact space X, C in the first set of definitions and C_α in the second set can be considered to be closed.

The following inequalities follow immediately from the definitions.

$$\begin{array}{ccccccccc} 0 & \le & \sigma^*(X) & \le & \sigma(X) & \le & \sigma_0(X) & \le & \operatorname{diam} X, \\ 0 & \le & \sigma^*(X) & \le & \sigma_0^*(X) & \le & \sigma_0(X) & \le & \operatorname{diam} X. \end{array}$$

It can easily be shown that, if J is an arc then $\sigma(J) = \sigma_0(J) = \sigma^*(J) = \sigma_0^*(J) = 0$. A simple consequence of this is that when X is a simple closed curve, $\sigma(X) = \sigma^*(X)$ and $\sigma_0(X) = \sigma_0^*(X)$.

To simplify our exposition we define the following sets, notation, and definitions. We define a metric ρ on $X \times J$ as follows

$$\rho((x,i),\ (y,j)) = \sqrt{(d(x,y))^2 + (i-j)^2},$$

where d is the metric on X.

We define q_1, $q_2 : (X \times J) \times (X \times J) \to X \times J$ by

$$q_1((x,l),\ (y,k)) = (x,l) \text{ and } q_2((x,l),\ (y,k)) = (y,k).$$

We define $r_1 : B \times J \to B$ and $r_2 : B \times J \to J$ by

$$r_1(x,t) = x \text{ and } r_2(x,t) = t,$$

where $B = R^2 - U$ and U is the unbounded component of $R^2 - X$. We let O denote the origin in R^2 or in R^3.

Let W, V and U be points on a concave upward y symmetric simple closed curve. By $\widehat{WV}$, we denote the shorter subarc on X determined by these two points. By $\widehat{WVU}$, we denote the subarc of X with endpoints W and U which contains V.

In [W3], we proved the following three theorems. In the following theorem we let $J = [0,h]$.

Theorem A. *Let X be a continuum. Suppose there exists $C \subseteq X \times X$ such that C is connected, for each $(x,y) \in C$, $d(x,y) \ge \sigma(X)$, $p_1(C) = p_2(C) = Y \subseteq X$, there exists $(x',y') \in C$ such that $d(x',y') = diamX$, and $(y',x') \in C$. Then $\sigma(X \times J) = min\{\sqrt{(\sigma(X))^2 + h^2},\ diamX\}$.*

Theorem B. *Let X be a continuum. Suppose there exists $C \subseteq X \times X$ such that C is connected, for any $(x,y) \in C$, $d(x,y) \ge \sigma_0(X)$, $p_1(C) \subseteq p_2(C) \subseteq X$, there exists $(x',y') \in C$ such that $d(x',y') = diamX$, and there exists $(z',x') \in C$ such that $d(z',x') = diamX$. Then $\sigma_0(X \times J) = min\{\sqrt{(\sigma_0(X))^2 + h^2}, diamX\}$.*

Let $a, b \in R^n \times \{0\}$ where $d(a,b) = d$. Let R be composition of a translation and a rotation in R^{n+1} where
$R : R^{n+1} \to R^{n+1}$ and $R(a) = (-\frac{d}{2}, 0, \cdots, 0)$, $R(b) = (\frac{d}{2}, 0, \cdots, 0)$.

Let

$R_T(Y) = \{y \in Y \mid R(y) = (x_1, \cdots, x_{n+1}) \text{ where } x_1 \geq 0, \frac{h}{2} \leq x_{n+1} \leq h\}$

$R_B(Y) = \{y \in Y \mid R(y) = (x_1, \cdots, x_{n+1}) \text{ where } x_1 \geq 0, 0 \leq x_{n+1} \leq \frac{h}{2}\}$

$L_T(Y) = \{y \in Y \mid R(y) = (x_1, \cdots, x_{n+1}) \text{ where } x_1 \leq 0, \frac{h}{2} \leq x_{n+1} \leq h\}$

$L_B(Y) = \{y \in Y \mid R(y) = (x_1, \cdots, x_{n+1}) \text{ where } x_1 \leq 0, 0 \leq x_{n+1} \leq \frac{h}{2}\}$

For a continuum X, let U be the unbounded component of $R^n - X$. Let $B = R^n - U$.

Theorem C. *Suppose X is a continuum contained in R^n which satisfies the hypothesis of Theorem A. Suppose that*

(i) *for all subcontinua $Z \subseteq B$, $\sigma_0(Z) \leq \sigma(X)$*
(ii) *Y is a continuum such that $X \times J \subset Y$ and $Y \subseteq B \times J$*
(iii) *there exists $y \in Y$ such that $R(y) = (0, \cdots, 0, \frac{h}{2})$ and*
(iv) *the sets $R_T(Y)$, $R_B(Y)$, $L_T(Y)$ and $L_B(Y)$ are each connected, where R is based on x' and y'.*
(v) *$Y \subset$ closure $B\left(y, \sqrt{(\frac{diamX}{2})^2 + (\frac{h}{2})^2}\right)$ where $R(y) = (0, \cdots, 0, \frac{h}{2})$.*

Then $\sigma^*(Y) = \sigma_0^*(Y) = min\{\sqrt{(\frac{diamX}{2})^2 + (\frac{h}{2})^2}, \sigma(X \times J)\}$.

Let f be a concave upward function where $f : [0,p] \to [0,q]$, $f(0) = q$ and $f(p) = 0$. Let $P = (p,0)$ and $-P = (-p,0)$. Let G_P denote the graph of f in R^2. Let G_{-P} denote the reflection of that graph through the *y-axis*. Let $X = G_P \cup G_{-P} \cup \overline{(-P)P}$. We refer to X as a concave upward y symmetric simple closed curve.

In [W2], we proved the following theorem.

Theorem W. *Let X be a concave upward y symmetric simple closed curve where $X = G_P \cup G_{-P} \cup \overline{(-P)P}$. Then* $\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = \min\{q, d(-P, G_P)\}$.

In the proof of this theorem we showed that the set

$$C = (\{-P\} \times G_P) \cup \left(\overline{(-P)P} \times \{Q\}\right) \cup$$
$$(\{P\} \times G_{-P}) \cup (G_P \times \{-P\}) \cup$$
$$(\{Q\} \times \overline{(-P)P}) \cup (G_{-P} \times \{P\})$$

satisfies these conditions:

i) $p_1(C) = p_2(C) = X$,
ii) C is connected,
iii) for each $(x,y) \in C$, $d(x,y) \geq \min\{q, d(-P, G_P)\}$.
iv) there exists a point $(x,y) \in C$, such that $d(x,y) = \min \{q, d(-P, G_P)\}$.

Also, it is clear that

v) $\operatorname{diam}(X) = \max\{2p, \sqrt{p^2+q^2}\}$ and $(-P,P)$, $(P,-P)$, $(-P,Q)$, $(Q,-P) \in C$.

In [W2], we also proved the following theorems.

Theorem W′. *Let Y be a continuum such that $Y \subseteq B$ where X is a concave upward y symmetric simple closed curve given by $X = G_P \cup G_{-P} \cup \overline{(-P)P}$ and B is the closure of the bounded component of $R^2 - X$. Then $\alpha(Y) \leq \alpha(X)$ where $\alpha = \sigma, \sigma_0, \sigma^*, \sigma_0^*$.*

Theorem W″. *If X is a concave upward y symmetric simple closed curve where $X = G_P \cup G_{-P} \cup \overline{(-P)P}$,* then $\sigma(B) = \sigma_0(B) = \min\{q, d(-P, G_P)\}$ where *B is the closure of the bounded component of $R^2 - X$.*

3. Main results

Theorem 1 *Let X be a concave upward y symmetric simple closed curve given by $X = G_P \cup G_{-P} \cup \overline{(-P)P}$ and let $J = [0, h]$. Then*

$$\sigma(X \times J) = \min\{\sqrt{(\sigma(X))^2 + h^2}, \operatorname{diam} X\}.$$

Proof: Note that the set C used in the proof of Theorem W and given above, satisfies the conditions of Theorem A. So, $\sigma(X \times J) = \min\{\sqrt{(\sigma(X))^2 + h^2}, \operatorname{diam} X\}$. Note that $\operatorname{diam} X = \max\{2p, \sqrt{p^2+q^2}\}$ and that by Theorem W, $\sigma(X) = \{\min\{q, d(-P, G_P)\}$. □

Theorem 2 *Let X be a concave upward y symmetric simple closed curve given by $X = G_P \cup G_{-P} \cup \overline{(-P)P}$ and let $J = [0, h]$.* Then

$$\sigma_0 (X \times J) = \min\{\sqrt{(\sigma_0 (X))^2 + h^2}, \mathrm{diam} X\}.$$

Proof: Similar to the proof of Theorem 1. Use Theorem W and Theorem B. □

Theorem 3. *Let X be a concave upward y symmetric simple closed curve given by $X = G_P \cup G_{-P} \cup \overline{(-P)P}$* with *$B$ the closure of the bounded component of $R^2 - X$ and let $J = [0, h]$.* Then

$$\sigma (B \times J) = \min\{\sqrt{(\sigma (B))^2 + h^2}, \mathrm{diam} B\}.$$

Proof: By Theorem W'', $\sigma(B) = \sigma(X)$. Note that the set C, given above and used in the proof of Theorem W, satisfies the conditions of Theorem B. □

Theorem 4. *Let X be a concave upward y symmetric simple closed curve given by $X = G_P \cup G_{-P} \cup \overline{(-P)P}$* with *$B$ the closure of the bounded component of $R^2 - X$. Then*

$$\sigma_0 (B \times J) = \min\{\sqrt{(\sigma_0 (B))^2 + h^2}, \mathrm{diam} B\}.$$

Proof: By Theorem W'', $\sigma_0(B) = \sigma(X)$. Note that the set C, given above and used in the proof of Theorem W, satisfies the conditions of Theorem B. □

Theorem 5. *Let X be a concave upward y symmetric simple closed curve given by $X = G_P \cup G_{-P} \cup \overline{(-P)P}$* with *$B$ the closure of the bounded component of $R^2 - X$.*
If $p \geq q$, then

$$\sigma^* (B \times J) = \sigma_0^* (B \times J) = \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sigma (B \times J)\}$$

If $p < q$, then

$$\sigma^* (B \times J) = \sigma_0^* (B \times J) = \min\{\sqrt{(\frac{p^2 + q^2}{2q})^2 + (\frac{h}{2})^2}, \sigma (B \times J)\}$$

Proof:

Case 1. $p \geq q$,

As noted before, X satisfies the conditions of Theorem A (see proof of Theorem 1). By Theorem W' and Theorem W, i of Theorem C is satisfied. Clearly conditions ii, iii,and iv,and v of Theorem C are satisfied. Hence,

$$\sigma^* (B \times J) = \sigma_0^* (B \times J) = \min\{\sqrt{\left(\frac{\text{diam}X}{2}\right)^2 + (\frac{h}{2})^2}, \sigma (B \times J)\}.$$

Case 2. $p < q \leq \sqrt{3}p$,

In this case, $q > p$ and $0 < \frac{q^2-p^2}{2q} < q$. Also, $d((-p, 0), (0, \frac{q^2-p^2}{2q})) = d((p, 0), (0, \frac{q^2-p^2}{2q})) = d((0, q), (0, \frac{q^2-p^2}{2q})) = \frac{q^2+p^2}{2q}$.

Let T_{-p} be the closure of the subset of B bound by $\overline{(0, \frac{q^2-p^2}{2q})\ (0, q)}$, $\overline{(0, \frac{q^2-p^2}{2q})\ (p, 0)}$ and G_P. Let T_p be the closure of the subset of B bound by $\overline{(0, \frac{q^2-p^2}{2q})\ (0, q)}$, $\overline{(0, \frac{q^2-p^2}{2q})\ (-p, 0)}$ and G_{-P}. Let T_q be the closure of the subset of B bound by $\overline{(-p, 0)\ (0, \frac{q^2-p^2}{2q})}$, $\overline{(0, \frac{q^2-p^2}{2q})\ (p, 0)}$ and $\overline{(-p, 0)\ (p, 0)}$.

Define subsets of $(B \times J) \times (B \times J)$ as follows:

$$\begin{aligned} C = \ & \{(-p, 0, 0)\} \times (T_{-p} \times [\frac{h}{2}, h]) \cup \{(-p, 0, h)\} \times (T_{-p} \times [0, \frac{h}{2}]) \cup \\ & \{(p, 0, 0)\} \times (T_p \times [\frac{h}{2}, h]) \cup \{(p, 0, h)\} \times (T_p \times [0, \frac{h}{2}]) \cup \\ & \{(0, q, 0)\} \times (T_q \times [\frac{h}{2}, h]) \cup \{(0, q, h)\} \times (T_q \times [0, \frac{h}{2}]) \\ D = \ & \{(-p, 0, 0)\} \times (G_P \times \{h\}) \cup \overline{(-p, 0, 0)\ (p, 0, 0)} \times \{(0, q, h)\} \cup \\ & \{(p, 0, 0)\} \times (G_{-P} \times \{h\}) \cup (G_P \times \{0\}) \times \{(-p, 0, h)\} \cup \\ & \{(0, q, 0)\} \times \overline{(-p, 0, h)\ (p, 0, h)} \cup (G_{-P} \times \{0\}) \times \{(p, 0, h)\} \cup \\ & \{(-p, 0, h)\} \times (G_P \times \{0\}) \cup \overline{(-p, 0, h)\ (p, 0, h)} \times \{(0, q, 0)\} \cup \\ & \{(p, 0, h)\} \times (G_{-P} \times \{0\}) \cup (G_P \times \{h\}) \times \{(-p, 0, 0)\} \cup \\ & \{(0, q, h)\} \times \overline{(-p, 0, 0)\ (p, 0, 0)} \cup (G_{-P} \times \{h\}) \times \{(p, 0, 0)\} \end{aligned}$$

$$E = \{((-p, 0, t)(p, 0, h - t)) \mid t \in [0, h]\}$$

Let $C^* = C \cup C^{-1} \cup D \cup D^{-1} \cup E \cup E^{-1}$. We can see that C^* is connected and $q_1(C^*) = q_2(C^*) = B \times J$. For each point $(x, y) \in C^*$,

$$\rho(x,y) \geq \min\{\sqrt{(\frac{q^2+p^2}{2q})^2 + (\frac{h}{2})^2}, \sqrt{(\min\{q, d(-P, G_P)\})^2 + h^2}, \text{diam} B\}.$$

Hence,

$$\sigma^*(B \times J) \geq \min\{\sqrt{(\frac{q^2+p^2}{2q})^2 + (\frac{h}{2})^2}, \sqrt{(\min\{q, d(-P, G_P)\})^2 + h^2}, \text{diam} B\}.$$

We need to show that

$$\sigma_0^*(B \times J) \leq \min\{\sqrt{(\frac{q^2+p^2}{2q})^2 + (\frac{h}{2})^2}, \sqrt{(\min\{q, d(-P, G_P)\})^2 + h^2}, \text{diam} B\}.$$

For each $(x, y, z) \in B \times J$, $\rho((x, y, z), (0, \frac{q^2-p^2}{2q}, \frac{h}{2}) \leq \sqrt{(\frac{q^2+p^2}{2q})^2 + (\frac{h}{2})^2}$. So $\sigma_0^*(B \times J) \leq \sqrt{(\frac{q^2+p^2}{2q})^2 + (\frac{h}{2})^2}$.

Let $D^* \subseteq (B \times J) \times (B \times J)$ be a connected set such that $q_1(D^*) \subseteq q_2(D^*) = B \times J$. Consider the functions $r_1 \circ q_1$, $r_1 \circ q_2 : D^* \to X$. The set D^* is connected, $r_1 \circ q_1$, $r_1 \circ q_2$ are continuous, and $r_1 \circ q_1(D^*) \subseteq r_1 \circ q_2(D^*) = B$. Hence, there exists a point $d^* \in D^*$ such that $d(r_1 \circ q_1(d^*), r_1 \circ q_2(d^*)) \leq \sigma_0^*(B)$ and $\rho(q_1(d^*), q_2(d^*)) \leq \sqrt{(\sigma(B))^2 + h^2} = \sqrt{(\min\{q, d(-P, G_P)\})^2 + h^2}$.

Now consider the functions $r_2 \circ q_1$, $r_2 \circ q_2 : D^* \to J$. The set D^* is connected, $r_2 \circ q_1$ and $r_2 \circ q_2$ are continuous, and $r_2 \circ q_1(D^*) \subseteq r_2 \circ q_2(D^*) = J$. Since J is an arc, there exists a point $d' \in D^*$ such that $d(r_2 \circ q_1(d'), r_2 \circ q_2(d')) = 0$. Hence $\rho(q_1(d'), q_2(d')) \leq \text{diam} X$.

Consequently

$$\sigma_0^*(B \times J) \leq \min\{\sqrt{(\frac{q^2+p^2}{2q})^2 + (\frac{h}{2})^2}, \sqrt{(\min\{q, d(-P, G_p)\})^2 + h^2}, \text{diam} X\},$$

and

$$\sigma^*(B\times J)=\sigma_0^*(B\times J)=\min\{\sqrt{(\frac{q^2+p^2}{2q})^2+(\frac{h}{2})^2},\sigma(B\times J).$$

Case 3. $\sqrt{3}p<q$

In this case, let C and D be the sets as defined in case 2. Let

$$E=\{((-p,0,t),\ (0,q,h-t))\mid t\in[0,h]\}$$

Let $C^*=C\cup C^{-1}\cup D\cup D^{-1}\cup E\cup E^{-1}$. The rest of the proof in this case is similar to case 2. In this case our conclusion is that

$$\sigma_0(B\times J)=\sigma_0^*(B\times J)=\min\{\sqrt{(\frac{q^2+p^2}{2q})^2+(\frac{h}{2})^2},$$
$$\sqrt{(\min\{q,d(-P,G_p)\})^2+h^2},\mathrm{diam}B\}$$
$$=\{\sqrt{(\frac{q^2+p^2}{2q})^2+(\frac{h}{2})^2},\sigma(B\times J)\}.\ \square$$

Let L be the line which is the perpendicular bisector of $\overline{(-P)\,Q}$. Let $(b,0)$ be the point where L intersects the *x-axis*.

Note the following:

If $p\geq q$ then $b\leq 0$.

If $p<q\leq\sqrt{3}p$ then $0<b\leq p$.

If $\sqrt{3}p<q$ then $p<b$.

Clearly, when $\sqrt{3}p<q$ the line L intersects $G_P-\{P,Q\}$. Also $L\cap G_P$ must contain exactly one point, say S. This is true since the line segment $\overline{SP}$ must be above the corresponding arc $\widehat{SP}$ on X. It is clear that the only point of intersection of $\overline{S\ (b,0)}$ and $\overline{SP}$ is S.

Let U_{-P} and U_Q be the two components of R^2-L where $-P\in U_{-P}$ and $Q\in U_Q$. Suppose that $p<q\leq\sqrt{3}p$ and $U_{-P}\cap G_P$ is not empty. Based on the construction of X, we see that there must be points R and T of G_P such that $\widehat{QR}\subset U_Q\cup L$, $\widehat{RT}\subset U_{-P}\cup L$ and $\widehat{TP}\subset U_Q\cup L$.

Let the points S, R and T be as given above. Let $S^{'}=(-x,y)$ where $S=(x,y)\,,R^{'}=(-x,y)$ where $R=(x,y)$ and $T^{'}=(-x,y)$ where $T=(x,y)$

Theorem 6. *Let X be a concave upward y symmetric simple closed curve.*
If $p \geq q$, then

$$\sigma^*(X \times J) = \sigma_0^*(X \times J) = \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2 + h^2}, \mathrm{diam}X\}.$$

If $p < q \leq \sqrt{3}p$ and $U_{-P} \cap G_P = \phi$, then

$$\sigma^*(X \times J) = \sigma_0^*(X \times J) = \min\{\sqrt{(\sigma^*(X))^2 + (\frac{h}{2})^2}, \sqrt{q^2 + (\frac{h}{2})^2}, \mathrm{diam}X\}.$$

If $p < q \leq \sqrt{3}p$ and $U_{-P} \cap G_P \neq \phi$, then

$$\sigma^*(X \times J) = \sigma_0^*(X \times J) = \min\{\sqrt{(d(-P, \widehat{RQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P, \widehat{TP}))^2 + (\frac{h}{2})^2}, \sqrt{(\sigma^*(X))^2 + h^2}, \sqrt{(d(Q, R))^2 + (\frac{h}{2})^2}, \sqrt{q^2 + (\frac{h}{2})^2}, \mathrm{diam}X\}.$$

If $\sqrt{3}p < q$, then

$$\sigma^*(X \times J) = \sigma_0^*(X \times J) = \min\{\sqrt{(d(-P, \widehat{SQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P, \widehat{SP}))^2 + h^2}, \sqrt{d(Q, S))^2 + (\frac{h}{2})^2}, \sqrt{q^2 + (\frac{h}{2})^2}, \mathrm{diam}X\}.$$

Proof. We consider the four cases.

Case 1: $p \geq q$

Let

$$
\begin{aligned}
C = & \{(-p,0,0)\} \times (\{(x,y) \in X \mid x \geq 0\} \times [\frac{h}{2}, h]) \cup \\
& (\{(x,y) \in X \mid x \leq 0\} \times [0, \frac{h}{2}]) \times \{(p,0,h)\} \cup \\
& \{((-p,0,t),(p,0,h-t)) \mid t \in [0,h]\} \cup \\
& \{(-p,0,h)\} \times (\{(x,y) \in X \mid x \geq 0\} \times [0, \frac{h}{2}]) \cup \\
& (\{(x,y) \in X \mid x \leq 0\} \times [\frac{h}{2}, h]) \times \{(p,0,0)\} \cup \\
& \overline{(-p,0,0)\ (p,0,0)} \times \{(0,q,h)\} \cup \\
& \{(p,0,0)\} \times (\{(x,y) \in X \mid x \leq 0\} \times [\frac{h}{2}, h]) \cup \\
& (\{(x,y) \in X \mid x \geq 0\} \times [0, \frac{h}{2}]) \times \{(-p,0,h)\} \cup \\
& \{((p,0,t),(-p,0,h-t)) \mid t \in [0,h]\} \cup \\
& \{(p,0,h)\} \times (\{(x,y) \in X \mid x \leq 0\} \times [0, \frac{h}{2}]) \cup \\
& (\{(x,y) \in X \mid x \geq 0\} \times [\frac{h}{2}, h]) \times \{(-p,0,0)\}.
\end{aligned}
$$

The set C is connected, $q_1(C) = q_2(C) = X \times J$, and for each $(x,y) \in C$, $\rho(x,y) \geq \min\{\sqrt{p^2 + (\frac{h}{2})^2},\ \sqrt{q^2 + h^2},\ 2p\}$. Hence, $\sigma^*(X \times J) \geq \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2 + h^2}, 2p\}$.

We need to show that $\sigma_0^*(X \times J) \leq \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2 + h^2}, 2p\}$. Clearly,

$d((x,y,z),\ (0,0,\frac{h}{2})) \leq d((-p,0,0),\ (0,0,\frac{h}{2})) = \sqrt{p^2 + (\frac{h}{2})^2}$ for all $(x,y,z) \in X \times J$. Hence, $\sigma_0^*(X \times J) \leq \sqrt{p^2 + (\frac{h}{2})^2}$.

Suppose $D^* \subseteq (X \times J) \times (X \times J)$ such that $q_1(D^*) \subseteq q_2(D^*) = X \times J$. Consider $p_1 \circ r_1 \circ q_1, p_1 \circ r_1 \circ q_2 : D^* \to [-p,p]$. These functions are continuous functions from a connected set into an arc. Hence, there exists a $d^* \in D^*$ such that $p_1 \circ r_1 \circ q_1(d^*) = p_1 \circ r_1 \circ q_2(d^*))$ and $\rho(q_1(d^*), q_2(d^*)) \leq \sqrt{q^2 + h^2}$. Consider the functions $r_2 \circ q_1, r_2 \circ q_2 : D^* \to J$ These are continuous functions from the connected set D^* into an arc such that $r_2 \circ q_1(D^*) \subseteq r_2 \circ q_2(D^*)$. Hence, there exists a $d^{'} \in D^{*}$ such that $d\left(r_2 \circ q_1(d^{'}), r_2 \circ q_2(d^{'})\right) = 0$. So $\rho(q_1(d^{'}), q_2(d^{'})) = \rho((x,t),(y,t)) \leq 2p =$diam$X$ where $t = r_2 \circ q_1(d^{'}) = r_2 \circ q_2(d^{'})$. Consequently,

$$\sigma_0^{*}(X \times J) \leq \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2+h^2}, 2p\} \text{ and}$$

$$\sigma^{*}(X \times J) = \sigma_0^{*}(X \times J) = \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2+h^2}, \text{diam}X\}.$$

Case 2. $p < q \leq \sqrt{3}p$ and $G_P \cap U_{-P} = \emptyset$

Let

$$\begin{aligned} C =& \{(-p,0,0)\} \times (G_P \times [\frac{h}{2}, h]) \cup (G_{-P} \times [0, \frac{h}{2}]) \times \{(p,0,h)\} \cup \\ & \{((-p,0,t),(p,0,h-t)) \mid t \in J\} \cup \\ & \{(-p,0,h)\} \times (G_P \times [0, \frac{h}{2}]) \cup (G_{-P} \times [\frac{h}{2}, h]) \times \{(p,0,0)\} \cup \\ & (\overline{(-p,0,0)\ (p,0,0)} \times [0, \frac{h}{2}]) \times \{(0,q,h)\} \cup \\ & (\overline{(-p,0,0)\ (p,0,0)} \times [\frac{h}{2}, h]) \times \{(0,q,0)\} \cup \\ & \{(p,0,0)\} \times (G_{-P} \times [\frac{h}{2}, h]) \cup (G_P \times [0, \frac{h}{2}]) \times \{(-p,0,h)\} \cup \\ & \{((p,0,t),(-p,0,h-t)) \mid t \in [0,h]\} \cup \\ & \{(p,0,h)\} \times (G_{-P} \times [0, \frac{h}{2}]) \cup (G_P \times [\frac{h}{2}, h]) \times \{(-p,0,0)\} \cup \\ & \{0,q,h)\} \times (\overline{(-p,0,0)\ (p,0,0)} \times [0, \frac{h}{2}]) \cup \\ & \{(0,q,0)\} \times (\overline{(-p,0,0)\ (p,0,0)} \times [\frac{h}{2}, h]). \end{aligned}$$

The set C is connected and $q_1(C) = q_2(C) = X \times J$. Also, for each $(x,y) \in C$,

$$\rho(x,y) \geq \min\{\sqrt{(d(-P,G_P))^2 + (\frac{h}{2})^2}, \sqrt{q^2 + (\frac{h}{2})^2}, 2p\}.$$

Hence,

$$\sigma^{*}(X \times J) \geq \min\{\sqrt{(d(-P,G_P))^2 + (\frac{h}{2})^2}, \sqrt{q^2 + (\frac{h}{2})^2}, 2p\}.$$

Now we need to show that

$$\sigma_0^*(X \times J) \leq \min\{\sqrt{(d(-P, G_P))^2 + (\frac{h}{2})^2}, \sqrt{q^2 + (\frac{h}{2})^2}, 2p\}.$$

For the point $(0, 0, \frac{h}{2})$, $\rho((0, 0, \frac{h}{2}), (x, y, z)) \leq \sqrt{q^2 + (\frac{h}{2})^2}$ for all points $(x, y, z) \in X \times J$. Hence $\sigma_0^*(X \times J) \leq \sqrt{q^2 + (\frac{h}{2})^2}$.

Since $G_P \cap U_{-P} = \emptyset$, for each $(x, y) \in G_P$,

$$d((x, y), (-p, 0)) \geq d((x, y), (0, q)).$$

By the construction of X we can see that for each $(x', y') \in X$,

$$d((x', y'), (x, y)) \leq d((x, y), (-p, 0)).$$

Consequently,

$$d((x, y, \frac{h}{2}), (x', y', 0)) \leq \sqrt{(d((-p, 0), (x, y)))^2 + (\frac{h}{2})^2}.$$

Hence

$$\sigma_0^*(X \times J) \leq \sqrt{(d(-P, G_P))^2 + (\frac{h}{2})^2}.$$

Let $D^* \subseteq (X \times J) \times (X \times J)$ such that D^* is connected and $q_1(D^*) \subseteq q_2(D^*) = X \times J$. Consider the functions $r_2 \circ q_1$, $r_2 \circ q_2 : D^* \to J$. These functions are continuous . Since D^* is connected and J is an arc, there exists a $d^* \in D^*$ such that $d(r_2 \circ q_1(d^*), r_1 \circ q_2(d^*)) = 0$ Hence, $\rho(q_1(d^*), q_2(d^*) \leq \text{diam} X$. Hence $\sigma_0^*(X \times J) \leq \text{diam} X$. Consequently,

$$\sigma_0^*(X \times J) \leq \min\{\sqrt{q^2 + (\frac{h}{2})^2}, \sqrt{(d(-P, G_P))^2 + (\frac{h}{2})^2}, 2p\}.$$

So,

$$\sigma^*(X \times J) = \sigma_0^*(X \times J) = \min\{\sqrt{q^2 + (\frac{h}{2})^2}, \sqrt{(\sigma^*(X))^2 + (\frac{h}{2})^2}, \text{diam} X\}.$$

Case 3. $p < q \leq \sqrt{3}p$ and $U_{-P} \cap G_P \neq \emptyset$

Let

$$\begin{aligned} C =&\{(-p,0,0)\} \times [(\widehat{TP} \times [\frac{h}{2},h]) \cup (\widehat{RT} \times \{h\}) \cup (\widehat{RQ} \times [\frac{h}{2},h])]\cup \\ &[(\widehat{-PT'} \times [0,\frac{h}{2}]) \cup (\widehat{T'R'} \times \{0\}) \cup (\widehat{R'Q} \times [0,\frac{h}{2}])] \times \{(p,0,h)\}\cup \\ &((\widehat{R'(-P)} \cup \overline{-PP} \cup \widehat{PR}) \times [0,\frac{h}{2}]) \times \{(0,q,h)\}\cup \\ &\{(0,q,0)\} \times [(\widehat{RP} \cup \overline{-PP} \cup \widehat{-PR'}) \times [\frac{h}{2},h]] \\ &\cup \{((-p,0,t),(p,0,h-t)) \mid t \in [0,h]\}\cup \\ &\{(-p,0,h)\} \times [(\widehat{TP} \times [0,\frac{h}{2}]) \ \cup (\widehat{RT} \times \{0\}) \cup (\widehat{RQ} \times [0,\frac{h}{2}])]\cup \\ &[(\widehat{-PT'} \times [\frac{h}{2},h]) \cup (\widehat{T'R} \times \{h\}) \cup (\widehat{R'Q} \times [\frac{h}{2},h])] \times \{(p,0,0)\}\cup \\ &((\widehat{R'(-P)} \cup \overline{(-P)P} \cup \widehat{PR}) \times [\frac{h}{2},h]) \times \{(0,q,0)\}\cup \\ &\{(0,q,h)\} \times ((\widehat{RP} \cup \overline{(-P)P} \cup \widehat{(-P)R'}) \times [0,\frac{h}{2}]). \end{aligned}$$

Let $D = C \cup C^{-1}$. We can see that D is connected and that $q_1(D) = q_2(D) = X \times J$. Also for each $((x,y,j),(x',y',j')) \in D$,

$$\begin{aligned} \rho((x,y,j),(x',y',j')) \geq \min\{&\sqrt{(d(-P,\widehat{RQ}))^2 + (\frac{h}{2})^2}, \\ &\sqrt{(d(-P,\widehat{TP}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P,G_P))^2 + h^2}, \\ &\sqrt{(d(Q,(\widehat{RP} \cup \overline{OP})))^2 + (\frac{h}{2})^2}, 2p\}. \end{aligned}$$

So,

$$\begin{aligned} \sigma^*(X \times J) \geq \min\{&\sqrt{(d(-P,\widehat{RQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P,\widehat{TP}))^2 + (\frac{h}{2})^2}, \\ &\sqrt{(d(-P,G_P))^2 + h^2}, \sqrt{(d(Q,(\widehat{RP} \cup \overline{OP})))^2 + (\frac{h}{2})^2}, 2p\}. \end{aligned}$$

We need to show that

$$\sigma_0^*(X \times J) \leq \min\{\sqrt{(d(-P, \widehat{RQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P, \widehat{TP}))^2 + (\frac{h}{2})^2},$$
$$\sqrt{(d(-P, G_P))^2 + h^2}, \sqrt{(d(Q, (\widehat{RP} \cup \overline{OP})))^2 + (\frac{h}{2})^2}, 2p\}.$$

Consider the vertical line through R and the horizontal line through R. Note that the subarc $\widehat{RP}$ of G_P is contained in the lower right quadrant determined by these two lines. Consequently, $d(Q, \widehat{RP}) = d(Q, R)$. Note that $d(Q, \overline{(-P)P}) = d(Q, O)$. Hence, $d(Q, \widehat{RP} \cup \overline{PO}) = \min\{d(Q, R), d(Q, O)\}$.

Consider the following observations:

Let $(x, y) \in \widehat{QR}$, then $d(-P, (x, y)) \geq d(Q, (x, y))$ and $d(-P, (x, y)) \geq d((x', y'), (x, y))$ for all $(x', y') \in X$. Hence

$$(*) \qquad \sigma_0^*(X \times J) \leq \sqrt{(d(-P, \widehat{RQ}))^2 + (\frac{h}{2})^2}.$$

Let $(x, y) \in \widehat{RT}$, then $d(Q, (x, y)) \geq d(-P, (x, y))$ and $d(Q, (x, y)) \geq d((x', y'), (x, y))$ for all $(x', y') \in X$. Also, $d(Q, \widehat{RP}) = d(Q, \widehat{RT}) = d(Q, R)$. So

$$(**) \qquad \sigma_0^*(X \times J) \leq \sqrt{(d(Q, R))^2 + (\frac{h}{2})^2}.$$

For any point $(0, 0, \frac{h}{2})$ and for any point $(x', y') \in X$,

$$\rho((0, 0, \frac{h}{2}), (x', y', t)) \leq \rho((0, 0, \frac{h}{2}), (0, q, 0))$$
$$= \sqrt{(d(Q, \overline{(-P)P}))^2 + (\frac{h}{2})^2} = \sqrt{q^2 + (\frac{h}{2})^2}.$$

Also $d(Q, \widehat{RP} \cup \overline{PO}) = \min\{d(Q, R), q\}$. Hence

$$(***) \qquad \sigma_0^*(X \times J) \leq \sqrt{(d(Q, \widehat{RP} \cup \overline{PO}))^2 + (\frac{h}{2})^2} = \sqrt{(\min\{d(Q, R), q\})^2 + (\frac{h}{2})^2}.$$

For each $(x,y) \in \widehat{TP}$, $d(-P,(x,y)) \geq d(Q,(x,y))$ and $d(-P,(x,y)) \geq d((x',y'),(x,y))$ for all $(x',y') \in X$. Hence,

$$(****) \qquad \sigma_0^*(X \times J) \leq \sqrt{(d(-P,\widehat{TP}))^2 + (\frac{h}{2})^2}.$$

Suppose $D^* \subseteq (X \times J) \times (X \times J)$ such that $q_1(D^*) \subseteq q_2(D^*)$ and for all $d^* \in D^*$, $\rho(q_1(d^*), q_2(d^*)) \geq \sigma_0^*(X \times J)$. Now consider the continuous functions $r_1 \circ q_1, r_1 \circ q_2 : D^* \to X$. Since D^* is connected and $r_1 \circ q_1(D^*) \subseteq r_1 \circ q_2(D^*) = X$, there is a $d^* \in D^*$ such that $d(r_1 \circ q_1(d^*), r_1 \circ q_2(d^*)) \leq \sigma_0^*(X) = \min\{q, d(-P, G_P)\}$. Hence,
(*****)

$$d(q_1(d^*), q_2(d^*)) \leq \sqrt{(\sigma_0^*(X))^2 + h^2} \leq \sqrt{(d(-P,G_P))^2 + h^2}.$$

Now consider the continuous functions $r_2 \circ q_1$, $r_2 \circ q_2 : D^* \to J$. Since D^* is connected and $r_2 \circ q_1(D^*) \subseteq r_2 \circ q_2(D^*) = J$, there is a $d^* \in D^*$ such that $d(r_2 \circ q_1(d^*), r_2 \circ q_2(d^*)) = 0$. Hence,

$$(******) \qquad d(q_1(d^*), q_2(d^*)) \leq 2p = \text{diam} X.$$

By $(*)$ through $(******)$, we see that

$$\sigma_0^*(X \times J) \leq \min\{\sqrt{(d(-P,\widehat{RQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P,\widehat{TP}))^2 + (\frac{h}{2})^2},$$
$$\sqrt{(d(-P,G_P))^2 + h^2}, \sqrt{(d(Q,(\widehat{RP} \cup \overline{OP})))^2 + (\frac{h}{2})^2}, 2p\}.$$

So

$$\sigma^*(X \times J) = \sigma_0^*(X \times J) = \min\{\sqrt{(d(-P,\widehat{RQ}))^2 + (\frac{h}{2})^2},$$
$$\sqrt{(d(-P,\widehat{TP}))^2 + (\frac{h}{2})^2}, \sqrt{(\sigma^*(X))^2 + h^2},$$
$$\sqrt{(d(Q,(\widehat{RP} \cup \overline{OP})))^2 + (\frac{h}{2})^2}, \text{diam} X\}.$$

Case 4. $\sqrt{3}p < q$

Let

$$\begin{aligned} C = \quad & \{(-p,0,0)\} \times [(\widehat{SP} \times \{h\}) \cup (\widehat{SQ} \times [\tfrac{h}{2}, h])] \cup \\ & [(\widehat{-PS'} \times \{0\}) \cup (\widehat{S'Q} \times [0, \tfrac{h}{2}])] \times \{(p,0,h)\} \cup \\ & \{(0,q,0)\} \times ((\widehat{SP} \cup \overline{(-P)P}) \times [\tfrac{h}{2}, h]) \cup \\ & ((\widehat{S'(-P)}) \cup \overline{(-P)P}) \times [0, \tfrac{h}{2}]) \times \{(0,q,h)\} \\ & \cup \{((-p,0,t),(0,q,h-t)) | \mathrm{t} \in [0,\mathrm{h}]\} \cup \\ & \{(-p,0,h)\} \times [(\widehat{SP} \times \{0\}) \ \cup (\widehat{SQ} \times [0, \tfrac{h}{2}])] \cup \\ & [(\widehat{-PS'} \times \{h\}) \cup (\widehat{S'Q} \times [\tfrac{h}{2}, h])] \times \{(p,0,0)\} \\ & \cup \{(0,q,h)\} \times ((\widehat{SP} \cup \overline{(-P)P}) \times [0, \tfrac{h}{2}]) \cup \\ & [(\widehat{S'(-P)}) \cup \overline{(-P)P}) \times [\tfrac{h}{2}, h]] \times \{(0,q,0)\}. \end{aligned}$$

Let $D = C \cup C^{-1}$. Clearly, D is connected and $q_1(D) = q_2(D) = X \times J$. Also for each $((x,y,j),(x',y',j')) \in D$,

$$\begin{aligned} \rho((x,y,j),(x',y',j')) \le \min\{ & \sqrt{(d(-P,\widehat{SQ}))^2 + (\frac{h}{2})^2}, \\ & \sqrt{(d(-P,\widehat{SP}))^2 + h^2}, \sqrt{(d(Q,(\widehat{SP} \cup \overline{(-P)P})))^2 + (\frac{h}{2})^2}, \\ & \sqrt{q^2+p^2}\}. \end{aligned}$$

So,

$$\begin{aligned} \sigma^*(X \times J) \ge \min\{ & \sqrt{(d(-P,\widehat{SQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P,\widehat{SP}))^2 + h^2}, \\ & \sqrt{(d(Q,(\widehat{SP} \cup \overline{(-P)P})))^2 + (\frac{h}{2})^2}, \sqrt{q^2+p^2}\}. \end{aligned}$$

We need to show that

$$\sigma_0^*(X \times J) \leq \min\{\sqrt{(d(-P,\widehat{SQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P,\widehat{SP}))^2 + h^2}, \sqrt{(d(Q,(\widehat{SP} \cup \overline{(-P)P})))^2 + (\frac{h}{2})^2}, \sqrt{q^2+p^2}\}.$$

For $(x, y) \in \widehat{QS}$, $d(-P,(x,y)) \geq d(Q,(x,y))$ and $d(-P,(x,y)) \geq d((x',y'),(x,y))$ for all $(x',y') \in X$. Hence,

(*) $$\sigma_0^*(X \times J) \leq \sqrt{(d(-P,\widehat{QS}))^2 + (\frac{h}{2})^2}.$$

For $(x, y) \in \widehat{SP}$, $d(Q,(x,y)) \geq d(-P,(x,y))$ and $d(Q,(x,y)) \geq d((x',y'),(x,y))$ for all $(x',y') \in X$. So,

(**) $$\sigma_0^*(X \times J) \leq \sqrt{(d(Q,\widehat{SP}))^2 + (\frac{h}{2})^2}.$$

For any point $(0,0,\frac{h}{2})$

$$\rho((0,0,\frac{h}{2}),(0,q,0)) = \sqrt{q^2 + (\frac{h}{2})^2},$$

and for all point $(x',y',t') \in X \times J$,

(***) $$\rho((x',y',t'),(0,0,\frac{h}{2})) \leq \sqrt{q^2 + (\frac{h}{2})^2}.$$

Consider the vertical line and the horizontal line through S. The arc $\widehat{SP}$ is in the lower right quadrant formed by these two lines. Hence, $d(Q,\widehat{SP}) = d(Q,S)$ and

(****) $$d(Q,\widehat{SP} \cup \overline{(-P)P}) = \min\{q, d(Q,S)\}.$$

Suppose $D^* \subseteq (X \times J) \times (X \times J)$ such that $q_1(D^*) \subseteq q_2(D^*)$ and for all $d^* \in D^*$, $\rho(q_1(d^*), q_2(d^*)) \geq \sigma_0^*(X \times J)$. Now consider the continuous functions $r_1 \circ q_1, r_1 \circ q_2 : D^* \to X$. Since D^* is connected and $r_1 \circ q_1(D^*) \subseteq r_1 \circ q_2(D^*) = X$, there is a $d^* \in D^*$ such that $d(r_1 \circ q_1(d^*), r_1 \circ q_2(d^*)) \leq \sigma_0^*(X) = \min\{q, d(-P, G_p)\}$. Hence,
(*****)
$$d(q_1(d^*), q_2(d^*)) \leq \sqrt{(\sigma_0^*(X))^2 + h^2} \leq \sqrt{(d(-P, G_P))^2 + h^2}.$$

Now consider the continuous functions $r_2 \circ q_1$, $r_2 \circ q_2 : D^* \to J$. Since D^* is connected and $r_2 \circ q_1(D^*) \subseteq r_2 \circ q_2(D^*) = J$, there is a $d^* \in D^*$ such that $d(r_2 \circ q_1(d^*), r_2 \circ q_2(d^*)) = 0$. Hence,

$$(******) \qquad d(q_1(d^*), q_2(d^*)) \le \sqrt{q^2 + p^2} = \text{diam} X.$$

By $(*)$ through $(******)$ given immediately above, we see that

$$\sigma_0^*(X \times J) \le \min\{\sqrt{(d(-P, \widehat{SQ}))^2 + (\frac{h}{2})^2}, \sqrt{(d(-P, \widehat{SP}))^2 + h^2},$$
$$\sqrt{(d(Q, (\widehat{SP} \cup \overline{(-P)P})))^2 + (\frac{h}{2})^2}, \sqrt{q^2 + p^2}\}.$$

So

$$\sigma^*(X \times J) = \sigma_0^*(X \times J) = \min\{\sqrt{(d(-P, \widehat{SQ}))^2 + (\frac{h}{2})^2},$$
$$\sqrt{(d(-P, \widehat{SP}))^2 + h^2}, \sqrt{(d(Q, (\widehat{SP} \cup \overline{(-P)P})))^2 + (\frac{h}{2})^2}, \text{diam} X\}. \ \square$$

In Theorems 7 and 8, let X be a concave upward y symmetric simple closed curve given by $X = G_P \cup G_{-P} \cup \overline{(-P)P}$.

Theorem 7. *Let* $Y = (X \times J) \cup X_0 \cup X_h$, *then* $\sigma(Y) = \sigma_0(Y) = \sigma(X \times J) =$
$\min\{\sqrt{(\sigma(X))^2 + h^2}, \text{diam } X\}$.

Proof: Since $Y \supset (X \times J)$, $\sigma(Y) \ge \sigma(X \times J)$. Since $Y \subset B \times J$, $\sigma(Y) \le \sigma(B \times J)$. Hence $\sigma(Y) = \sigma(B \times J) = \sigma(X \times J) = \min\{\sqrt{(\sigma(X))^2 + h^2}, \text{diam } X\}$. Similarly, $\sigma_0(Y) = \sigma_0(X \times J)$. $\square$

Theorem 8. *Let* $Y = (X \times J) \cup X_0 \cup X_h$,
If $p \ge q$, *then* $\sigma^*(Y) = \sigma_0^*(Y) = \sigma^*(X \times J)$.
If $p < q$, *then* $\sigma^*(Y) = \sigma_0^*(Y) = \min\{\sigma^*(X \times J), \sqrt{(\frac{q^2+p^2}{2q})^2 + h^2}\}$.

Proof: Case 1 $p \ge q$.
Let $C^* = C \cup D \cup D^{-1}$ where C is as in Theorem 6 case 1, and

$$\begin{aligned} D = \ & \{(-p,0,0)\} \times \{(x,y,h) \in X_h \mid x \ge 0\} \cup \\ & \{(x,y,0) \in X_0 \mid x \le 0\} \times \{(p,0,h)\} \cup \\ & \{(-p,0,h)\} \times \{(x,y,0) \in X_0 \mid x \ge 0\} \cup \\ & \{(x,y,h) \in X_h \mid x \le 0\} \times \{(p,0,0)\}. \end{aligned}$$

The set C^* is connected, $q_1(C^*) = q_2(C^*) = Y$, and for all $c^* \in C^*$,

$$\rho(q_1(c^*), q_2(c^*)) \geq \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2 + h^2}, 2p\}.$$

So, $\sigma^*(Y) \geq \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2 + h^2}, 2p\}$. Clearly, $\sigma_0^*(Y) \leq \sqrt{p^2 + (\frac{h}{2})^2}$. Since $(0, 0, \frac{h}{2}) \in Y$ and for all $(x', y', t') \in Y$,

$$\rho((0, 0, \frac{h}{2}), (x', y', t')) \leq \sqrt{p^2 + (\frac{h}{2})^2}.$$

By proofs similar to the ones in Theorem 6 case 1, showing that $\sigma_0^*(X \times J) \leq \sqrt{q^2 + h^2}$. and that $\sigma_0^*(X \times J) \leq 2p$, it can be shown that $\sigma_0^*(Y) \leq \sqrt{q^2 + h^2}$ and $\sigma_0^*(Y) \leq 2p$. Hence,

$$\sigma^*(Y) = \sigma_0^*(Y) = \min\{\sqrt{p^2 + (\frac{h}{2})^2}, \sqrt{q^2 + h^2}, 2p\} = \sigma^*(X \times J).$$

Case 2 $p < q \leq \sqrt{3}p$ and $G_P \cap U_{-P} = \emptyset$

Let

$$\begin{aligned} E = \quad & \{(-p, 0, 0)\} \times (T_{-p} \times \{h\}) \cup \{(p, 0, 0)\} \times (T_p \times \{h\}) \cup \\ & \{(0, q, 0)\} \times (T_q \times \{h\}) \cup \{(-p, 0, h)\} \times (T_{-p} \times \{0\}) \cup \\ & \{(p, 0, h)\} \times (T_p \times \{0\}) \cup \{(0, q, h)\} \times (T_q \times \{0\}). \end{aligned}$$

Let C be as in case 2 Theorem 6. Let $F = C \cup E \cup E^{-1}$. We observe that for all $(x, y, z) \in Y, \rho((x, y, z), (0, \frac{q^2 - p^2}{2q}, h)) \leq \sqrt{(\frac{q^2+p^2}{2q})^2 + h^2}$. This observation together with an argument similarly to the one in case 2 of Theorem 6 leads us to conclude that $\sigma_0^*(Y) = \sigma^*(Y) = \min\{\sigma^*(X \times J), \sqrt{(\frac{q^2+p^2}{2q})^2 + h^2}\}$.

Case 3 $p < q \leq \sqrt{3}p$ and $G_P \cap U_{-P} \neq \emptyset$.

Let D be as in case 3 in Theorem 6. Let $F = D \cup E \cup E^{-1}$. As in case 2, we conclude that $\sigma_0^*(Y) = \sigma^*(Y) = \min\{\sigma^*(X \times J), \sqrt{(\frac{q^2+p^2}{2q})^2 + h^2}\}$

Case 4 $\sqrt{3}p < q$.

Let D be as in case 4 in Theorem 6. Let $F = D \cup E \cup E^{-1}$. As in case 2 and case 3, we conclude that $\sigma_0^*(Y) = \sigma^*(Y) = \min\{\sigma^*(X \times J), \sqrt{(\frac{q^2+p^2}{2q})^2 + h^2}\}$. $\square$

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