Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Spring 2000
Pages 343-349

ON THE SURJECTIVE SPANS OF CONVEX SETS IN R^2

THELMA WEST

Abstract. Let B be a closed convex disc in the plane. A lower bound for the surjective span of B is determined. For any t such that $\frac{1}{2} \leq t \leq 1$, a closed convex disc in the plane is given such that the surjective span of B divided by the span of B is equal to t.

1. Introduction

The concept of the span of a metric space was introduced in [L1]. Various modified versions of the span have been defined since then (cf. [L2] and [L3]). In general it is difficult to calculate the spans of even simple geometric objects.

Let B be a closed convex set in the plane. It has been determined that the surjective span of B, $\sigma^*(B)$, is larger than or equal to one half of the span of B, $\sigma(B)$ (see [W2]). It has been shown that the span of B is equal to the breadth of B, $b(B)$ (see [T1]). We show that the surjective span of B is larger than or equal to the minimum of one half of the diameter and the breadth of B. Also, we give an example of closed disc B_t for each $t \in [\frac{1}{2}, 1]$ such that the surjective span of B_t divided by the span of B_t is equal to t.

2. Preliminaries

If X is a non-empty metric space, we define the span of X, $\sigma(X)$, to be the least upper bound of the set of real numbers α for which there exist connected subsets C_α of the product $X \times X$ such that

$$(\sigma) \qquad p_1(C_\alpha) = p_2(C_\alpha)$$

and $\alpha \leq \text{dist}(x, y)$ for $(x, y) \in C_\alpha$, where p_1 and p_2 denote the projections of $X \times X$ onto X, i.e., $p_1(x, y) = x$ and $p_2(x, y) = y$

for $x, y \in X$. We will now consider X to be connected. The surjective span $\sigma^*(X)$, the semispan $\sigma_0(X)$, and the surjective semispan $\sigma_0^*(X)$ are defined as above, except we change conditions (σ) to the following (see [L3])

$$(\sigma^*) \qquad p_1(C_\alpha) = p_2(C_\alpha) = X,$$

$$(\sigma_0) \qquad p_1(C_\alpha) \subseteq p_2(C_\alpha),$$

$$(\sigma_0^*) \qquad p_1(C_\alpha) \subseteq p_2(C_\alpha) = X.$$

Later Davis in [D] defined the symmetric span of X, $s(X)$, and the surjective symmetric span of X, $s^*(X)$. The definitions of $s(X)$ and $s^*(X)$ are the same as $\sigma(X)$ and $\sigma^*(X)$, respectively, except the requirement that $C_\alpha = C_\alpha^{-1}$ was added.

We note that for a compact space X, C_α can be considered to be closed. The following inequalities follow immediately from the definitions.

$$0 \leq \sigma^*(X) \leq \sigma(X) \leq \sigma_0(X) \leq \operatorname{diam} X,$$

$$(*) \qquad 0 \leq \sigma^*(X) \leq \sigma_0^*(X) \leq \sigma_0(X) \leq \operatorname{diam} X.$$

$$0 \leq s(X) \leq \sigma(X) \leq \operatorname{diam} X$$

$$0 \leq s^*(X) \leq \sigma^*(X) \leq \operatorname{diam} X.$$

Let X be a planar continuum. Let L_α denote the line passing through the origin such that the angle between the positive x-axis and L_α, measured counterclockwise, is α, where $\alpha \in [0, \pi)$. The directional diameter $d_\alpha(X)$ of X, in the direction α, is the length of the longest line segment (or segments) with endpoints on X, that is parallel to L_α.

The breadth of a continuum X is defined by

$$\inf\{d_\alpha(X) : \alpha \in [0, \pi)\}$$

and is denoted by $b(X)$. These definitions for X, a simple closed curve, were originally given in [T1, T2]. The notation for breadth was changed from $d(X)$ to $b(X)$ in [W1].

The following two results are used in this paper.

Theorem T. *Let B be a convex space in the plane. Let D be the simple closed curve which is the boundary of B. Then*

$$b(D) = \sigma(D) = \sigma_0(D) = \sigma^*(D) = \sigma_0^*(D) \text{ [T1]}.$$

Theorem W. *Let B be a convex space in the plane. Let D be the simple closed curve which is the boundary of B. For any continuum $Y \subseteq B$, $\tau(Y) \leq \tau(D)$ where $\tau = \sigma$, σ_0, σ^* or σ_0^** [W1].

Clearly, $b(D) = b(B) = \sigma(D) = \sigma(B)$.

3. Main results

Lemma. *Let B be a closed convex disc in the plane. Let $a, b \in \partial B = D$ such that $d(a,b) =$ diam D. Then there is a connected set $C \subset D \times D$ such that $p_1(C) = p_2(C) = D$, $d(p_1(c), p_2(c)) \geq b(D)$ for all $c \in C$, $C = C^{-1}$ and $(a,b) \in C$.*

Proof: Pick $a, b \in D$ such that $d(a,b) =$diam D. Let $h : S^1 \to D$ be a homeomorphism such that $h(e^{io}) = a$. We put an ordering on D in the following manner: let x, $y \in D$, then $x \leq y$ if and only if $h(e^{i\theta_1}) = x$, $h(e^{i\theta_2}) = y$, and $\theta_1 \bmod 2\pi \leq \theta_2 \bmod 2\pi$. Let $0 < \epsilon < b(D) \leq$diam$D$. We can pick a sequence of points x_1, x_2, $\cdots x_m$ on D where $x_1 = a < x_2 < \cdots < x_l = b < \cdots < x_m$, such that diam$(A_{i\,i+1}) \leq \frac{\epsilon}{2}$ and diam$(A_{m_1}) \leq \frac{\epsilon}{2}$, where $A_{i\,i+1}$ is the subarc on D with endpoints x_i and x_{i+1} with the smaller diameter. Also, A_{m_1} is the subarc on D with endpoints x_m and x_1 with the smaller diameter. Let $P = \bigcup_{i=1}^{m-1} \overline{x_i x_{i+1}} \cup \overline{x_m x_1}$, clearly P is a convex polygon and $H(D,P) \leq \frac{\epsilon}{2}$. It is clear from the definition of breadth that $b(D) \geq b(P)$. It has been shown that $b(D) - \epsilon \leq b(P)$ (see Theorem 1 in [W3]).

For any points $x_i, x_{i+1}, \cdots x_{i+j}$, where $j > 1$ such that $A_{i\,i+j} = \overline{x_i x_{i+j}}$, we eliminate the points x_{i+1} to x_{i+j-1}. Similarly, if $A_{i1} = \overline{x_i x_1}$ and $i < m$, we eliminate the points x_{i+1} to x_m.

We relabel the remaining points in the following manner:

$$a = P_1 < P_2 < \cdots < P_k = b < \cdots < P_n.$$

Clearly, for each A, where $A = A_{i\,i+1}$ or $A = A_{n\,1}$, either diam$(A) \leq \frac{\epsilon}{2}$ or A is a straight line segment. Clearly,

$$P = \bigcup_{i=1}^{m-1} \overline{x_i x_{i+1}} \cup \overline{x_m x}_1 = \bigcup_{i=1}^{n-1} \overline{P_i P_{i+1}} \cup \overline{P_n P_1}.$$

We can rotate and translate our whole space so that b is moved to the origin and a is moved to the point (0,diam P). Note that the

x-axis intersects P only at b and that the line,L,through a which is parallel to the x-axis intersects P only at a. This is true since a and b are of distance the diameter of P apart.

We can assume without loss of generality that the ordering of the vertices of P is clockwise. Let a' be the next vertex of P in clockwise ordering after a. Let $b^{'}$ be the next vertex of P in clockwise ordering after b. Not both a' and $b^{'}$ can be on the y-axis, otherwise P would not be a convex polygon. If one of these points is on the y-axis, we can assume without loss of generality that it is $b^{'}$. In which case, $b^{'} = a = P_1$ and $k = n$. Consider the angle α formed by the line segment $\overline{aa^{'}} = \overline{P_1 P_2}$ and the line L where $0 < \alpha < 90°$. Also, consider the angle β formed by the line segment $\overline{bb'}$ (where either $\overline{bb'} = \overline{P_k P_{k+1}}$ if $k < n$ or $\overline{bb'} = \overline{P_k P_1} = \overline{P_n P_1}$ if $k = n$) and the x-axis where $0 < \beta \leq 90°$. Again without loss of generality we can assume that $0 < \alpha \leq \beta < 90°$, since we could accomplish this relationship of the angles merely by changing the labels for the points a and b.

In [W1] we described the movements of two points F and G around a convex polygon such as P. We will define a set in $D \times D$ based on the movements of these two points. The two points move around P in the following way. The point F starts at $P_1 = a$ and G starts at $P_k = b$. On the first step G remains at P_k while F moves from P_1 to P_2. On each succeeding step, based on the given algorithm, one of the points remains at a vertex of P while the other point covers the next succeeding side of P. Such steps continue until F reaches P_k and G reaches P_1. We can repeat this pattern and let F move from P_k to P_1 (as G moved previously) and let G move from P_1 to P_k (as F moved previously).

We showed that the distance between F and G is always at least $b(P)$. Also, since $\sigma(P) = b(P)$ (see Theorem T), we know that at some time the distance between F and G is exactly $b(P)$.

In this section we use P_{n+1} as a second labeling for the vertex P_1. We showed in [W1] that the movements of F and G determine two increasing functions f and g. The function f is defined as follows

$$f : \{1, 2, \cdots, k\} \to \{k, \cdots, n, n+1\}$$

given by $f(1) = l_1 = k$ and $f(j) = l_j$ where P_{l_j} is a vertex of P such that F is at the vertex P_j and G is at the vertex P_{l_j} and l_j is the largest index for which this is true. By a previous observation, we see that $f(1) = k$ and $f(k) = l_k = n+1$. The function g is defined as follows

$$g : \{ k\,,\, k+1, \cdots , n,\, n+1\} \to \{1,\, 2, \cdots , k\,\}$$

given by $g(j) = l_j$ where P_{l_j} is a vertex of P such that G is at the vertex P_j and F is at the vertex P_{l_j} and l_j is the largest such index. By previous observation we see that $g(k) = l_k \geq 2$ and $g(n+1) = l_{n+1} = k$.

Let $m = \min\{l \mid f(l) = n+1\}$ and $j = \min\{ i \mid g(i) = m\}$. Clearly, $k \leq j < n+1$.

Case 1: $m < k$

We define C_ϵ as follows:

$$\begin{aligned} C_\epsilon = {} & P_k \times A_{1\ g(k)} \cup A_{k\ f\circ g(k)} \times P_{g(k)} \cup P_{f\circ g(k)} \times A_{g(k)\ g\circ f\circ g(k)} \\ & \cup \cdots \cup A_{j\ n+1} \times P_m \cup P_{n+1} \times A_{m\,k} \end{aligned}$$

Case 2: $m = k$

Let $l = \min\{ i \mid f(i) = j\}$. We define C_ϵ as follows:

$$\begin{aligned} C_\epsilon = {} & P_k \times A_{1\,g(k)} \cup A_{k\ f\circ g(k)} \times P_{g(k)} \cup P_{f\,\circ\, g(k)} \times A_{g(k)\ g\circ f\circ g(k)} \\ & \cup \cdots \cup P_j \times A_{l\,k} \cup P_{j\ n+1} \times P_k \end{aligned}$$

Since the point F covers the sides of P from $\overline{P_1 P_2}$ to $\overline{P_{k-1} P_k}$ and G covers the sides of P from $\overline{P_k P_{k+1}}$ to $\overline{P_n P_1}$, we see that

$$p_1\left(C_\epsilon \cup C_\epsilon^{-1}\right) = D,$$

$$p_2\left(C_\epsilon \cup C_\epsilon^{-1}\right) = D.$$

We have shown that when F is at a vertex P_i and G covers a side $\overline{P_l P_{l+1}}$ of P, that $d(P_i, \overline{P_l P_{l+1}}) \geq b(P) \geq b(D) - \epsilon$, clearly, $d(P_i,\, A_{l\ l+1}) \geq d\,(\,P_i\,, \overline{P_l P_{l+1}}\,)$. Hence, for all points $(x,y) \in C_\epsilon \cup C_\epsilon^{-1}$, $d(x,y) \geq b(D) - \epsilon$. Hence, there exists a connected set C in $D \times D$ such that $C = C^{-1}$, for each $(x,y) \in C$, $d(x,y) \geq b(D)$, and $p_1(C) = p_2(C) = D$. It is clear that $(a,b), (b,a) \in C$. $\square$

A simple consequence of this lemma is the following theorem which was previously proven in [DF].

Theorem 1. *Let B be a closed convex disc in the plane. Let D be the simple closed curve which is the boundary of B. Then $s(D) = b(D)$.*

Proof: It has been shown that $\sigma(D) = b(D)$ (see [T1]). From the lemma we see that $b(D) \leq s(D)$. By (*) we see that $s(D) \leq \sigma(D)$, so $s(D) = \sigma(D)$. $\square$

Theorem 2. *Let B be a closed convex set in the plane. Then*

$$s^*(B) \geq \min\{\frac{diamB}{2}, b(B)\}.$$

Proof: We know that $\sigma(B) = \sigma_0(B) = s(D) = s^*(D) = \sigma(D) = \sigma_0(D) = b(B) = b(D)$ where $D = \partial B$. Let $a, b \in B$ such that $d(a,b)$ =diam(B) =diam(D). Let C be the subset of $B \times B$ as established in the lemma. Let T be the composition of a rotation and a translation such that $T(b) = (\frac{d}{2}, 0)$ and $T(a) = (-\frac{d}{2}, 0)$. Let $B_L = \{x \in B |\ p_1(T(x)) \leq 0\}$ and $B_R = \{x \in B |\ p_1(T(x)) \geq 0\}$. Let $C^* = C \cup (B_L \times \{b\}) \cup (\{a\} \times B_R) \cup (\{b\} \times B_L) \cup (B_R \times \{a\})$. Clearly C^* is connected, $p_1(C^*) = p_2(C^*) = B$, $C^* = (C^*)^{-1}$, and for all $(x,y) \in C^*$, $d(x,y) \geq \min\{\frac{\text{diam}B}{2}, b(B)\}$. Consequently, $s^*(B) \geq \min\{\frac{\text{diam}B}{2}, b(B)\}$. $\square$

Corollary 1. *Let $B \subset R^2$ be a closed convex set in the plane then*

$$\sigma^*(B) \geq \min\{\frac{diamX}{2}, \sigma(B)\}.$$

So, $\frac{1}{2} \leq \frac{\sigma^*(B)}{\sigma(B)} \leq 1$.

Proof: Clearly, $\frac{\sigma^*(B)}{\sigma(B)} \leq \frac{\sigma(B)}{\sigma(B)} = 1$. If $\sigma^*(B) = \frac{\text{diam}B}{2}$ then $\frac{\sigma^*(B)}{\sigma(B)} = \frac{1}{2}(\frac{\text{diam}B}{\sigma(B)}) \geq \frac{1}{2}$. $\square$

Example: There exists a set B_t in the plane such that $\sigma^*(B_t)/\sigma(B_t) = t$, for $\frac{1}{2} \leq t \leq 1$.

Proof: Let E_t be an ellipse in the plane where the length of the minor axis is $2a$ and the length of the major axis is $2b = 4ta$. Let D_t be the bounded component of $R^2 - E_t$ and let $B_t = E_t \cup D_t$. Then $\sigma(B_t) = \sigma(E_t) = 2a$, $\sigma^*(B_t) = \min\{\frac{\text{diam}B_t}{2}, 2a\} = 2ta$ and $\sigma(B_t)/\sigma^*(B_t) = \frac{2ta}{2a} = t$. $\square$

References

[D] James F Davis, *The equivalence of zero span and zero semispan*, Proc. Amer. Math. Soc., **90, No.1** (1984), 133-138.

[DF] E. Duda and H.V.Fernandez, *Span and Plane Separating Continua*, to appear in the Houston Journal of Mathematics.

[L1] A. Lelek, *Disjoint mappings and the span of spaces*, Fund. Math., **55** (1964), 199-214.

[L2] A. Lelek, *An example of a simple triod with surjective span smaller than span*, Pacific J. Math., **64 No. 1** (1976), 207-215.

[L3] A. Lelek, *On the surjective span and semispan of connected metric spaces*, Colloq. Math., **37** (1977), 35-45.

[T1] Katarzyna Tkaczynska, *The span and semispan of some simple closed curves*, Proc. Amer. Math. Soc., **111, No. 1** (1991).

[T2] Katarzyna Tkaczynska, *On the span of simple closed curves*, Houston J. Math., **20, No. 3** (1994), 507-528.

[W1] Thelma West, *Concerning the spans of certain plane separating continua*, Houston J. Math., **25 No.4**, (1999).

[W2] Thelma West, *The Relationships of Spans of Convex Continua in* R^n, Proc. Amer. Math. Soc., **III, Number I** (1991).

[W3] Thelma West, *Spans of Spaces Contained in a Convex Disc*, accepted for publication in the Proceedings of the IV Joint Meeting of the American Mathematical Society and the Sociedad Mathematica Mexicana, Special Section in Continuum Theory, publisher Marcel Dekker.

University of Louisiana at Lafayette, Lafayette, Louisiana 70504-1010