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ON ANOTHER APPROACH FOR CONSTRUCTING COMPACTA WITH DIFFERENT DIMENSIONS dim AND ind

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Abstract

All known examples of compacta with noncoinciding dimensions dim and ind may be divided into two types: compacta constructed by the method of fan-products and compacta where the method of "two directions" is used. In the previous paper [6] two concrete methods of constructing compacta of the second type were proposed. Now another (more general) construction is described. It allows construction of compacta with noncoinciding dimensions dim and ind; to act in the class of the first-countable compacta; and to construct first-countable compacta which square and its product with some metrizable compactum has noncoinciding dimensions dim and ind. It is shown that nearly all known compacta of the second type are realizations of the described construction.

1. Introduction

In 1935 P.S.Alexandroff raised the question concerning the relationships between three main dimensions dim, ind and Ind in

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the case of compact spaces and later he proved the inequality $\dim X \le \mathrm{ind} X$ for any compactum X. First examples of compacta with noncoinciding dimensions dim and ind were constructed by A.Lunc and V.Lokucievskiĭ in 1949. Then more examples of compacta with additional besides dimensional properties appeared.

All these examples may be divided into two types: compacta constructed by the method of fan-products (V.V.Fedorchuk, V.V.Filippov, V.A.Chatyrko, B.A.Pasynkov and others) and compacta where the method of "two directions" is used, i.e., in the boundary of many open sets there is a "horisontal" subset or a "vertical" one of required dimension (see Key lemma). Examples constructed by A.Lunc, V.Lokucievskiĭ, P.Vopěnka, S.Mardešhić, B.A.Pasynkov, I.K.Lifanov, V.V.Filippov and some others are of the second type.

In our previous paper [6] we gave a list of properties of a compactum which allow to present two concrete methods of constructing compacta of the second type. Now another list of such a kind is given. It allows: to construct compacta with noncoincoding dimensions dim and ind acting, in particular, in the class of the first-countable compacta; to construct first-countable compacta with coinciding dimensions which squares and products with some metrizable compacta have noncoinciding dimensions dim and ind. It is shown that nearly all known compacta of the second type are realizations of the situation described in the paper.

2. Preliminaries

Below a space means a topological regular T_1-space. A compactum is a Hausdorff compact space. We use the following abbreviations: nbd(s) – neighbourhood(s); oddp O_1, O_2 in X – open dense and disjoint pair O_1, O_2 in a space X (i.e., O_1 and O_2 are open in X, $\mathrm{cl}(O_1 \cup O_2) = X$ and $O_1 \cap O_2 = \emptyset$); an oddp in X is called essential (respectively, maximal) if $\mathrm{cl}O_1 \cap \mathrm{cl}O_2 \neq \emptyset$ (respectively, $X \setminus (O_1 \cup O_2) = \mathrm{cl}O_1 \cap \mathrm{cl}O_2$); a map – a continuous mapping (between spaces).

A minimal partition ($\equiv$ a minimal 1-partition) in a space X is a subset F of X such that there exists a maximal oddp O_1, O_2 in X with $F = X \setminus (O_1 \cup O_2)$. For $n = 2, 3, \ldots$, a set $F \subset X$ is a minimal n-partition in X if F is a minimal partition in some minimal $(n-1)$-partition in X. The space X will be called a minimal 0-partition in X.

For a map $f : X \to Y$ and an oddp O_1, O_2 in Y, a point $x \in X$ is called (O_1, O_2, f)-**two-sided** if $fOx \cap O_i \neq \emptyset, i = 1, 2,$ for any nbd Ox of x.

For a family λ of subsets of a space X and $\Psi \subset X$, we put $\lambda \wedge \Psi = \{\Psi \cap L : L \in \lambda\}$.

3. Spreaded Tailings

Let we have spaces Φ, X, a map $r : X \to \Phi$, a cardinal number $\tau \geq (\max(\omega, \pi w(\Phi)))^+$ and two closed in X systems $\lambda = \{\Phi_\alpha : \alpha \in A\}$, $\mu = \{\Psi_\beta : \beta \in B\}$. Let $r_\alpha = r|_{\Phi_\alpha}, r_\beta = r|_{\Psi_\beta}, \alpha \in A, \beta \in B$.

Definition 3.1. The space X will be called a **spreaded τ-tailing** (τ-ST, for short) **of Φ with respect to r, λ and μ** ((τ, r, λ, μ)-ST of Φ, for short) if either $X = \emptyset$ for $\Phi = \emptyset$ or, for $\Phi \neq \emptyset$:

(1) r_α is a homeomorphism for any $\Phi_\alpha \in \lambda$;

(2) $|\mu| \geq \tau$;

(3) for any $\nu \subset \mu$ with $|\nu| \geq \tau$, there exists $\Phi_\alpha \in \lambda$ ($\Phi_\alpha = \Phi_\alpha(\nu)$) such that any nbd of Φ_α contains at least τ elements of ν.

Evidently, r is an onto map.

Key lemma. *Let X be a (τ, r, λ, μ)-ST of Φ and*

($*$) *for any oddp U_1, U_2 in X and any $\nu \subset \mu$ with $|\nu| \geq \tau$ either*

(a) *there exists $\Phi_\alpha \in \lambda$ and an open in Φ_α set $V \neq \emptyset$ such that $V \subset \mathrm{cl}U_1 \cap \mathrm{cl}U_2$ or*

(b) *there exist* $\Psi_\beta \in \nu$ *and an oddp* O_1, O_2 *in* Φ *such that* $r_\beta^{-1} O_i \subset \mathrm{cl} U_i, i = 1, 2$.

Proof. Take an oddp U_1, U_2 in X and $\nu \subset \mu$ with $|\nu| \geq \tau$.

Suppose (b) does not hold. Fix a π-base $\mathcal{B}$ in Φ of cardinality $< \tau$. Let, for $\Psi_\beta \in \nu$, $O_{i\beta}$ be the maximal open in Φ set such that $r^{-1} O_{i\beta} \subset \mathrm{cl} U_i, i = 1, 2$. Suppose, first, that $\Phi \setminus \mathrm{cl}(O_{1\beta} \cup O_{2\beta}) = \emptyset$. Then $O_{1\beta}, O'_{2\beta} = O_{2\beta} \setminus \mathrm{cl} O_{1\beta}$ is an oddp in Φ (may be unessential) such that $r_\beta^{-1} O_{1\beta} \subset \mathrm{cl} U_1, r_\beta^{-1} O'_{2\beta} \subset \mathrm{cl} U_2$. But this is impossible.

Now, suppose that $O_\beta = \Phi \setminus \mathrm{cl}(O_{1\beta} \cup O_{2\beta}) \neq \emptyset$. Then we can take $V_\beta \in \mathcal{B}$ such that $\emptyset \neq V_\beta \subset O_\beta$. Evidently, $r_\beta^{-1} W \cap \mathrm{cl} U_i \neq \emptyset$ for any non-empty open $W \subset V_\beta$ and $i = 1, 2$. Since $|\mathcal{B}| < \tau$, there exist $V \in \mathcal{B}$ and $\nu' \subset \nu$ with $|\nu'| \geq \tau$ such that $V = V_\beta \neq \emptyset$ for all $\Psi_\beta \in \nu'$. By (3), we can take $\alpha \in A$ such that any nbd of Φ_α contains at least τ elements of ν'. Identify Φ and Φ_α by means of r_α. Then r is a retraction of X onto $\Phi_\alpha \equiv \Phi$. Let $x \in V \equiv r_\alpha^{-1} V$ and Ox be a nbd of x in X. By Lemma 2.3 of [6], there exist a nbd W of x in Φ_α and a nbd O of Φ_α in X such that $O \cap r^{-1} W \subset Ox$. Evidently, we can suppose that $W \subset V$. If $\Psi_\beta \in \nu', \Psi_\beta \subset O$ then $\emptyset \neq r_\beta^{-1} W \cap \mathrm{cl} U_i \subset (r^{-1} W \cap O) \cap \mathrm{cl} U_i \subset Ox$ and so $Ox \cap \mathrm{cl} U_i \neq \emptyset$ for $i = 1, 2$. Hence $V \subset \mathrm{cl} U_1 \cap \mathrm{cl} U_2$. □

Remark. This lemma will play a very important role when we shall consider dimensional properties of spreaded tailings.

We need several assertions concerning spreaded tailings. The following one is evident.

Proposition 3.2. *Let* X *be a* (τ, r, λ, μ)*-ST of* Φ*. Then:*

(a) $T_{min}(X) = \bigcup \lambda \cup \bigcup \mu$ *is a* $(\tau, r|_{T_{min}(X)}, \lambda, \mu)$*-ST of* Φ*;*

(b) *if* X *is a subspace of* X' *and* $r = r'|_X$ *for a map* $r' : X' \to \Phi$ *then* X' *is a* (τ, r', λ, μ)*-ST of* Φ*.*

Below a (τ, r, λ, μ)-ST of Φ will be called **minimal** if $X = T_{min}(X)$, i.e., $X = \bigcup \lambda \cup \bigcup \mu$.

Definition 3.3. Let X be a (τ, r, λ, μ)-ST of Φ. It will be called **open** if all $r_\beta, \beta \in B$, are open.

The following is evident.

Proposition 3.4. *X is an open (τ, r, λ, μ)-ST of Φ iff $T_{min}(X)$ is an open $(\tau, r|_{T_{min}(X)}, \lambda, \mu)$-ST of Φ.*

Remark 3.5. It is evident that:
if X is an open (τ, r, λ, μ)-ST of Φ then $r_{min} = r|_{T_{min}(X)}$ is open and so if X is a minimal and open (τ, r, λ, μ)-ST of Φ then r is open.

Definition 3.6. Let X be a (τ, r, λ, μ)-ST of Φ. It will be called **(openly) c-hereditary** if, for any closed subset F of Φ, the inverse image $X_F = r^{-1}F$ is an (open) $(\tau, r_F = r|_{X_F}, \lambda_F = X_F \wedge \lambda, \mu_F = X_F \wedge \mu)$-ST of F.

Proposition 3.7. *Let X be a (τ, r, λ, μ)-ST of Φ. It is c-hereditary if Φ is normal.*

Proof. Conditions (1) and (2) of Definition 3.1 are evident for (we use the notations of Definition 3.6) any closed subset F of Φ, X_F, $\Phi_{F\alpha} = X_F \cap \Phi_\alpha, r_{F\alpha} = r_F|_{\Phi_{F\alpha}}$ and λ_F. Put $\Psi_{F\beta} = X_F \cap \Psi_\beta$ and $r_{F\beta} = r_F|_{\Psi_\beta}, \beta \in B$.

Take $\nu \subset \mu_F$ with $|\nu| \geq \tau$. Put $\nu' = \{\Psi_\beta \in \mu : \Psi_{F\beta} \in \nu\}$. Then there exists $\Phi_\alpha \in \lambda$ such that any nbd of Φ_α contains at least τ elements of ν'. Let W be a nbd of $\Phi_{F\alpha}$ in X_F. Then there exists a nbd U of $\Phi_{F\alpha}$ in X such that $W = U \cap X_F$. Identify Φ_α with Φ by means of r. Then $\Phi_{F\alpha}$ will be identified with F. By Lemma 2.3 [6], we can take nbds V of $\Phi_{F\alpha} \equiv F$ in $\Phi_\alpha \equiv \Phi$ and O of Φ_α in X such that $O \cap r^{-1}V \subset U$. It follows from this that if $\Psi_\beta \subset O$ then

$$\Psi_{F\beta} = X_F \cap \Psi_\beta \subset (r^{-1}V \cap X_F) \cap O \subset X_F \cap U = W. \qquad \square$$

Corollary 3.8. *If X is an open (τ, r, λ, μ)-ST of Φ and Φ is normal then X is an openly c-hereditary (τ, r, λ, μ)-ST of Φ.*

Definition 3.9. Let X be a (τ, r, λ, μ)-ST of Φ. It will be called **(openly) p-hereditary** if, for any minimal n-partition F in Φ, $n = 1, 2, \ldots$, and a minimal partition Φ' in F, there exists a maximal oddp O_1, O_2 in F such that $\Phi' = F \setminus (O_1 \cup O_2)$ and the space $X' = \mathrm{cl}(r^{-1}O_1) \cap \mathrm{cl}(r^{-1}O_2)$ is an (open) $(\tau, r' = r|_{X'}, \lambda' = X' \wedge \lambda, \mu' = X' \wedge \mu)$-ST of Φ'.

Proposition 3.10. *Let X be an open (τ, r, λ, μ)-ST of Φ. It is openly p-hereditary if Φ is normal. Moreover, for any closed subset F of Φ and an essential oddp O_1, O_2 in F (we use the notations of Definitions 3.9 and 3.6) X' is an open $(\tau, r', \lambda', \mu')$-ST of Φ' and (see Definition 3.6) $\lambda' = \lambda_{\Phi'}, \mu' = \mu_{\Phi'}$ (i.e., $\lambda' = r^{-1}\Phi' \wedge \lambda, \mu' = r^{-1}\Phi' \wedge \mu$).*

Proof. Since r_α and r_β are open, $r_\alpha^{-1}\Phi' = X' \cap \Phi_\alpha$ and $r_\beta^{-1}\Phi' = X' \cap \Psi_\beta$. Hence $\lambda' = \lambda_{\Phi'}, \mu' = \mu_{\Phi'}$ and so $\bigcup \lambda' \cup \bigcup \mu' = \bigcup \lambda_{\Phi'} \cup \bigcup \mu_{\Phi'} = T_{min}(X_{\Phi'})$. By Proposition 3.4 and Corollary 3.8, it follows that X' is an open $(\tau, r', \lambda', \mu')$-ST of Φ' (and $T_{min}(X') = T_{min}(X_{\Phi'})$). □

Remark 3.11. We have proved that:
$X' = X_{\Phi'} = r^{-1}\Phi'$ under the suppositions of Proposition 3.10 if $X = \bigcup \lambda \cup \bigcup \mu$, i.e., if X is a minimal (τ, r, λ, μ)-ST of Φ.

Remark 3.12. Let X be a minimal open (τ, r, λ, μ)-ST of Φ and Φ be normal. Then:

1) (by Definition 3.6 and Proposition 3.7) for any closed subset F of Φ (see Definition 3.6) X_F is a minimal $(\tau, r_F, \lambda_F, \mu_F)$-ST of F;

2) (by Proposition 3.10 and Remark 3.11) for any closed subset F of Φ and an essential oddp O_1, O_2 in F with $\Phi' = \mathrm{cl}O_1 \cap \mathrm{cl}O_2$, (see Definition 3.9) $X' \equiv X_{\Phi'}$ is a minimal $(\tau, r' \equiv r_{\Phi'}, \lambda' \equiv \lambda_{\Phi'}, \mu' \equiv \mu_{\Phi'})$-ST of Φ'.

Proposition 3.13. *Let X be a (minimal) open (τ, r, λ, μ)-ST of Φ and χ is a compactum with $\pi w(\chi) < \tau$. Then $\Pi = X \times \chi$ is a (minimal) open $(\tau, r_\Pi = r \times \mathrm{id}_\chi, \lambda_\Pi = \lambda \times \chi = \{\Phi_{\alpha\Pi} = \Phi_\alpha \times \chi : \alpha \in A\}, \mu_\Pi = \{\Psi_{\beta\Pi} = \Psi_\beta \times \chi : \beta \in B\})$-ST of $\Phi \times \chi$.*

Proof. Evidently, $\pi w(X \times \chi) < \tau$ and conditions (1), (2) for $r_{\alpha\Pi} = r_\Pi|_{\Phi_{\alpha\Pi}}$ and μ_Π hold.

Take $\nu \subset \mu_\Pi$ with $|\nu| \geq \tau$. Then $|\nu'| \geq \tau$ for $\nu' = \{\Psi_\beta \in \mu : \Psi_\beta \times \chi \in \nu\}$. By (3), there exists $\alpha \in A$ such that any nbd of Φ_α contains at least τ elements of ν'. Take a nbd O of $\Phi_{\alpha\Pi}$. Since χ is compact, the projection $\mathrm{pr} : X \times \chi \to X$ is closed and so there exists a nbd U of Φ_α such that $\mathrm{pr}^{-1} U \subset O$. It follows from this that if $\Psi_\beta \in \nu'$ and $\Psi_\beta \subset U$ then $\Psi_{\beta\Pi} \in \nu$ and $\Psi_{\beta\Pi} \subset O$. The openness of all $r_{\beta\Pi} = r_\beta \times \mathrm{id}_\chi$ is evident. If X is a minimal τ-ST of Φ then, evidently, Π is a minimal τ-ST of $\Phi \times \chi$. □

Proposition 3.14. *Let X be an open (τ, r, λ, μ)-ST of Φ and Φ be compact. Then for $\lambda_2 = \{\Phi_\alpha^2 : \alpha \in A\}, \mu_2 = \{\Psi_\beta^2 : \beta \in B\}$, the union $X_2 = \bigcup \lambda_2 \cup \bigcup \mu_2 \subset X^2$ is a minimal open $(\tau, r_2 = r^2|_{X_2}, \lambda_2, \mu_2)$-ST of Φ^2.*

Proof. Evidently, $\pi w(\Phi^2) < \tau$, all r_β^2 are open and conditions (1), (2) hold for r_α^2 and μ_2.

Take $\nu \subset \mu_2$ with $|\nu| \geq \tau$. Then $|\nu'| \geq \tau$ for $\nu' = \{\Psi_\beta \in \mu : \Psi_\beta^2 \in \nu\}$. By (3), there exists $\alpha \in A$ such that any nbd of Φ_α contains at least τ elements of ν'.

Take a nbd O of Φ_α^2 in X_2. There exists a nbd W of Φ_α^2 in X^2 such that $O = W \cap X_2$. Since Φ is compact, there exists a nbd U of Φ_α such that $U^2 \subset W$. Then $V = X_2 \cap U^2 \subset O$. If $\Psi_\beta \in \nu'$ and $\Psi_\beta \subset U$ then $\Psi_\beta^2 \in \nu$ and $\Psi_\beta^2 \subset X_2 \cap U^2 \subset O$.

The minimality of X_2 follows from the definition of minimality. □

4. Spreaded n-tailings

Let X be a (τ, r, λ, μ)-ST of Φ with $\lambda = \{\Phi_\alpha : \alpha \in A\}$, $\mu = \{\Psi_\beta : \beta \in B\}$, $r_\alpha = r|_{\Phi_\alpha}$ and $r_\beta = r|_{\Psi_\beta}$ for $\alpha \in A, \beta \in B$.

Definition 4.1. The space X will be called an n-(τ, r, λ, μ)-ST if, additionally,

(4) for any oddp O_1, O_2 in Φ and any $\Psi_\beta \in \mu$, the set of all (O_1, O_2, r_β)-two-sided points has dimension ind $\geq n$,

$n = 1, 2, \ldots$

Theorem 4.2. *Let X be an n-(τ, r, λ, μ)-ST of Φ and* $\mathrm{ind} V \geq m$ *for any open in Φ set $V \neq \emptyset$. Then*

$$\mathrm{ind} X \geq \max(m, n) \text{ for } m \neq n \text{ and } \mathrm{ind} X \geq n + 1 \text{ for } m = n.$$

Proof. The case when $m \neq n$ is evident. Let $m = n$ (and so, by Definition 4.1, $m = n > 0$).

Take (see (3)) $\alpha \in A$ such that any nbd of Φ_α contains at least τ elements of μ. Let $x \in \Phi_\alpha$ and a nbd Ox of x in Φ_α be such that ind $\mathrm{bd}_{\Phi_\alpha} \mathrm{cl} O \geq m - 1 \geq 0$ for all nbds O of x in Φ_α with $O \subset \mathrm{cl} Ox$. Choose an open in X set Ux so that $Ux \cap \Phi_\alpha = Ox$ and $Ux \subset r^{-1} r_\alpha Ox$. Take a nbd U_1 of x in X with $\mathrm{cl} U_1 \subset Ux$. Put $U_2 = X \setminus \mathrm{cl} U_1$.

If there exist $\alpha' \in A$ and a non-empty open in $\Phi_{\alpha'}$ set $V \subset \mathrm{cl} U_1 \cap \mathrm{cl} U_2$ then ind $\mathrm{bd} U_1 \geq \mathrm{ind} V \geq m = n$. Suppose that there are no such α' and V.

By Lemma 2.3 of [6] (if we identify Φ and Φ_α by means of r_α), there exist nbds O of x in Φ_α and W of Φ_α in X such that $W \cap r^{-1} r_\alpha O \subset U_1$. Evidently, $\mathrm{cl} O \subset Ox$. Let $\nu = \{\Psi_\beta \in \mu : \Psi_\beta \subset W\}$. Then, by Key lemma, there exist $\Psi_\beta \in \nu$ and an oddp O'_1, O'_2 in Φ such that $r_\beta^{-1} O'_i \subset \mathrm{cl} U_i, i = 1, 2$.

Put

$$O_1 = r_\alpha O \cup (O'_1 \cap r_\alpha Ox), \ O_2 = (O'_2 \setminus r_\alpha \mathrm{cl} O) \cup (\Phi \setminus r_\alpha \mathrm{cl} Ox).$$

Then

$$\begin{gathered} r_\beta^{-1} O_1 = r_\beta^{-1} r_\alpha O \cup r_\beta^{-1} (O'_1 \cap r_\alpha Ox) \subset (W \cap r^{-1} r_\alpha O) \cup \mathrm{cl} U_1 \subset \mathrm{cl} U_1, \\ r_\beta^{-1} O_2 \subset \mathrm{cl} U_2 \cup (X \setminus r^{-1} r_\alpha \mathrm{cl} Ox) \subset \mathrm{cl} U_2 \cup (X \setminus r^{-1} r_\alpha Ox) \\ \subset \mathrm{cl} U_2 \cup (X \setminus Ux) \subset \mathrm{cl} U_2 \cup (X \setminus \mathrm{cl} U_1) = \mathrm{cl} U_2. \end{gathered}$$

Since $\text{cl}O \subset Ox$ and O'_1, O'_2 are disjoint, the sets O_1, O_2 are also disjoint. Evidently, $\text{cl}(O_1 \cup O_2) = \text{cl}(O'_1 \cup O'_2) = \Phi$. Thus O_1, O_2 is an oddp in Φ.

Since $O \cup Ox \subset Ox$, we see that $O_1 \subset r_\alpha Ox$ and so (because r_α is a homeomorphism) $\text{bd}_\Phi \text{cl}O_1 \neq \emptyset$. Hence $\text{cl}O_1 \cap \text{cl}O_2 \neq \emptyset$. The set T of all (O_1, O_2, r_β)-two-sided points has, by (4), ind $T \geq n$ and $T \subset \text{cl}U_1 \cap \text{cl}U_2 \subset \text{bd}U_1$. Thus ind $\text{bd}U_1 \geq n$ and so ind $X \geq n+1$. □

Definition 4.3. The space X will be called:

- a **minimal** n-(τ, r, λ, μ)-ST of Φ if X is both a minimal (τ, r, λ, μ)-ST of Φ and an n-(τ, r, λ, μ)-ST of Φ;
- an **open** n-(τ, r, λ, μ)-ST of Φ if X is an open (τ, r, λ, μ)-ST of Φ and

  (5) ind $r_\beta^{-1} t \geq n$ for any $t \in \Phi$ and any $\Psi_\beta \in \mu$;
- a **weakly open** n-(τ, r, λ, μ)-ST of Φ if X is a (τ, r, λ, μ)-ST of Φ and

  (6) for any $t \in \Phi$ and any $\Psi_\beta \in \mu$, the set $\{x \in r_\beta^{-1} t : r_\beta$ is open at the point $x\}$ has dimension ind $\geq n$,

  $n = 1, 2, \ldots$.

Evidently:
X is an open n-(τ, r, λ, μ)-ST of $\Phi \Longrightarrow$
X is a weakly open n-(τ, r, λ, μ)-ST of $\Phi \Longrightarrow$
X is an n-(τ, r, λ, μ)-ST of Φ.

Definition 4.4. The space X is called an **(openly) c-hereditary** (respectively, **(openly) p-hereditary**) n-(τ, r, λ, μ)-ST of Φ if, for any closed subset F of Φ, (using the notations of Definition 3.6) X_F is an (open) n-$(\tau, r, \lambda_F, \mu_F)$-ST of F (respectively, for any closed subset F of Φ and any essential oddp O_1, O_2 in F (using the notations of Definition 3.9) X' is an (open) n-$(\tau, r', \lambda', \mu')$-ST of $\text{cl}O_1 \cap \text{cl}O_2$).

Proposition 3.7 implies the following:

Proposition 4.5. *If X is a weakly open n-(τ, r, λ, μ)-ST of Φ and Φ is normal then X is c-hereditary n-(τ, r, λ, μ)-ST of Φ.*

Corollary 3.8 and Proposition 3.10 imply the following:

Proposition 4.6. *If X is an open (and minimal) n-(τ, r, λ, μ)-ST of Φ and Φ is normal then X is an openly c-hereditary and openly p-hereditary n-(τ, r, λ, μ)-ST of Φ (and correspondent spreaded n-tailings are minimal).*

Propositions 3.13 and 3.14 give us the following:

Proposition 4.7. *Let X be an open (and minimal) n-(τ, r, λ, μ)-ST of Φ.*

1. *If χ is compact and $\pi w(\chi) < \tau$ then (using the notations of Proposition 3.13) Π is an open (and minimal)*

$$n\text{-}(\tau, r_\Pi, \lambda_\Pi, \mu_\Pi)\text{-}ST \text{ of } \Phi \times \chi.$$

2. *If Φ is compact then (using the notations of Proposition 3.14) X_2 is a minimal open m-$(\tau, r_2, \lambda_2, \mu_2)$-ST of Φ^2, where $m = \min\{\mathrm{ind}(r_\beta^2)^{-1}(t, t') = r_\beta^{-1}t \times r_\beta^{-1}t' : (t, t') \in \Phi^2\}$.*

5. Iterated Open Spreaded Tailings

Definition 5.1. A space X will be called:

1) an $(1, \tau, r, \lambda, \mu)$-ST of a space Φ if X is a minimal open 1-(τ, r, λ, μ)-ST of Φ;

m) an $(m, \tau, r, \lambda, \mu)$-ST of a space Φ if X is a minimal (τ, r, λ, μ)-ST of Φ and every $\Psi \in \mu$ is an $(m-1, \tau, r|_\Psi, \lambda_\Psi, \mu_\Psi)$-ST of $\Phi, m = 2, 3, \ldots$

Remark 5.2. The situation of Definition 5.1 may be generalizied. For example, “1-(τ, r, λ, μ)” in point 1) may be replaced by “n-(τ, r, λ, μ)” or the openness in 1) may be weakened using properties of hereditary type in 1) and m). The minimality in 1) and m) may also be omitted. But the choosen variant is simpler and it is sufficient for the results of section 7.

Proposition 5.3. *If a space X is an $(m, \tau, r, \lambda, \mu)$-ST of a space Φ then X is an open (τ, r, λ, μ)-ST of Φ and r is open, $m = 1, 2, \ldots$*

Proof. Let $m = 1$. Then X is an open (τ, r, λ, μ)-ST of Φ by Definition 5.1 and r is open by Remark 3.5.

Suppose that proposition is true for all $m < k$, $k > 1$, and let $m = k$. By the inductive hypothesis, (we use the notations of m) from Definition 5.1) $r|_\Psi$ is open for any $\Psi \in \mu$. Hence X is an open (τ, r, λ, μ)-ST of Φ. By Remark 3.5, r is open. □

Proposition 5.4. *Let X be an $(m, \tau, r, \lambda, \mu)$-ST of a normal space Φ and F is a closed subset of Φ. Then (see the notation of Definition 3.6) X_F is an $(m, \tau, r_F, \lambda_F, \mu_F)$-ST of F, $m = 1, 2, \ldots$*

Proof. It follows from Definitions 5.1 and 4.3 and Corollary 3.8 for $m = 1$. Suppose proposition is true for $m < k$ and let $m = k$, $k > 1$.

By Proposition 3.7 X_F is, evidently, a minimal $(\tau, r_F, \lambda_F, \mu_F)$-ST of F. Since every $\Psi_F \in \mu_F$ is equal to $\Psi \cap X_F = \Psi \cap r_\Psi^{-1} F$, where $r_\Psi = r|_\Psi$, for $\Psi \in \mu$, by the inductive hypothesis, Ψ_F is an $(m-1, \tau, r_F|_\Psi, \lambda_\Psi, \mu_\Psi)$-ST of F. Hence X_F is an $(m, \tau, r_F, \lambda_F, \mu_F)$-ST of F. □

Theorem 5.5. *Let X be an $(m, \tau, r, \lambda, \mu)$-ST of a compactum Φ with* $\dim \Phi \geq m$. *Then* $\operatorname{ind} X \geq m + 1$.

Proof. (It is similar to ones of Theorem 2.10 from [6] and Theorem 4.2.)

If $\dim\Phi > m$ then ind $\Phi \geq \dim\Phi \geq m+1$. Let $\dim\Phi = m$. Take an m-dimensional Cantor manofold $\Phi' \subset \Phi$ (see, for example, [1] or [3, Problem 3.2]). Then $\mathrm{ind}P \geq \dim P \geq m-1$ for every partitition P in Φ' and so $\mathrm{ind}V \geq m$ for any open in Φ' and non-empty set V. Since, by Proposition 5.4, $r^{-1}\Phi'$ is an $(m, \tau, r_{\Phi'}, \lambda_{\Phi'}, \mu_{\Phi'})$-ST of Φ', we can suppose, without loss of generality, that $\Phi' = \Phi$.

For $m = 1$, our assertion follows now from Theorem 4.2. Let it be true for all $m < k$, $k > 1$. Take $m = k$.

Let $\lambda = \{\Phi_\alpha : \alpha \in A\}$ and $\mu = \{\Psi_\beta : \beta \in B\}$. Choose $\alpha \in A$ such that any nbd of Φ_α contains at least τ elements of μ. Let $x \in \Phi_\alpha$ and a nbd Ox of x in Φ_α be such that $\mathrm{cl}Ox \neq \Phi_\alpha$. Take an open in X set U_1 so that $x \in U_1$ and $U_1 \subset r^{-1}r_\alpha Ox$ (where $r_\alpha = r|_{\Phi_\alpha}$). Put $U_2 = X \setminus \mathrm{cl}U_1$.

If there exist $\alpha' \in A$ and a non-empty open in $\Phi_{\alpha'}$ set $V \subset \mathrm{cl}U_1 \cap \mathrm{cl}U_2$ then ind $\mathrm{bd}U_1 \geq \mathrm{ind}V \geq k$. Suppose that there are no such α' and V.

By Lemma 2.3 of [6] (if we identify Φ and Φ_α by means of r_α), there exist nbds O of x in Φ_α and W of Φ_α in X such that $W \cap r^{-1}r_\alpha O \subset U_1$. Evidently, $O \subset Ox$. Let $\nu = \{\Psi_\beta \in \mu : \Psi_\beta \subset W\}$. Then by Key lemma there exist $\Psi_\beta \in \nu$ and an oddp O_1, O_2 in Φ such that (for $r_\beta = r|_{\Psi_\beta}$) $r_\beta^{-1}O_i \subset \mathrm{cl}U_i, i = 1, 2$. Since O_1, O_2 is an oddp in Φ, $r_\beta^{-1}r_\alpha O \subset W \cap r^{-1}r_\alpha O \subset U_1, O \neq \emptyset$ and r_β is onto, we see that $r_\alpha O \cap O_2 = \emptyset$ and $\emptyset \neq r_\alpha O \cap O_1 \subset O_1$. Since $O' = \Phi \setminus \mathrm{cl}\ r_\alpha Ox \neq \emptyset$ and $r_\beta^{-1}O' \subset \Psi_\beta \setminus r_\beta^{-1}\mathrm{cl}\ r_\alpha Ox \subset \Psi_\beta \setminus \mathrm{cl}\ r_\beta^{-1}r_\alpha Ox \subset \Psi_\beta \setminus \mathrm{cl}U_1 \subset U_2$, as above, $O' \cap O_1 = \emptyset$ and $\emptyset \neq O' \cap O_2 \subset O_2$. Hence $F = \mathrm{cl}O_1 \cap \mathrm{cl}O_2 \neq \emptyset$ is a partitition in Φ and so $\dim F \geq k-1$.

Since Ψ_β is an $((k-1), \tau, r_\beta, \lambda_\beta, \mu_\beta)$-ST of Φ, by Propositions 5.3, 3.10, Remark 3.11 and Proposition 5.4, $(X_{\beta F} = r_\beta^{-1}F)$= $\mathrm{cl}(r_\beta^{-1}O_1) \cap \mathrm{cl}(r_\beta^{-1}O_2)$ is a $(k-1, \tau, r_{\beta F} = r_\beta|_{X_{\beta F}}, \lambda_{\beta F}, \mu_{\beta F})$-ST of F. By the inductive hypothesis, $\mathrm{ind}X_{\beta F} \geq k$. Since $X_{\beta F} \subset \mathrm{cl}U_1 \cap \mathrm{cl}U_2$ and so $X_{\beta F} \subset \mathrm{bd}U_1$, we see that ind $\mathrm{bd}U_1 \geq k$. Thus $\mathrm{ind}X \geq k+1$. □

Proposition 5.6. *Let X be an $(m, \tau, r, \lambda, \mu)$-ST of a space Φ. Then:*

1. *If χ is compact and $\pi w(\chi) < \tau$ then (using the notations of Proposition 3.13) Π is an $(m, \tau, r_\Pi, \lambda_\Pi, \mu_\Pi)$-ST of $\Phi \times \chi$;*

2. *If Φ is compact then (using the notations of Proposition 3.14) X_2 is an $(m, \tau, r_2, \lambda_2, \mu_2)$-ST of Φ^2.*

Proof. Let us begin with point 1. It is true for $m = 1$ by Proposition 4.7. Let it be true for $m < k, k > 1$, and let $m = k$.

Then, by Propositions 5.3 and 3.13, Π is an open and minimal $(m, \tau, r_\Pi, \lambda_\Pi, \mu_\Pi)$-ST of $\Phi \times \chi$ and, by our inductive hypothesis, every $\Psi_\beta \times \chi$ (for $\mu = \{\Psi_\beta : \beta \in B\}$) is a $(k-1, \tau, r_{\beta\Pi}, \lambda_{\beta\Pi}, \mu_{\beta\Pi})$-ST of $\Phi \times \chi$ where $r_{\beta\Pi} = r_\Pi|_{\Psi_\beta \times \chi}$.

Point 2. is proved analogously by means of Propositions 4.7, 5.3 and 3.14. □

Addition to Section 5

In Lemma 2.6 of [6] the last line must look in the following (less strong) way:

$r^{-1}x \cap \bigcup\{R_\alpha : \alpha \in A_x, \Phi_\alpha \subset \cap\nu\} \subset \mathrm{bd}U \cap \mathrm{bd}(X \setminus \mathrm{cl}U)$ for any $x \in \mathrm{bd}_\Phi \mathrm{cl}O$.

In this weaker form, Lemma 2.6 is sufficient for obtaining Theorem 2.9 of [6], but it is not sufficient for obtaining Corollary 2.8 and Theorem 2.10 of [6]. The situation may be corrected in the following way.

The space X from Definition 2.1 of [6] will be called an open tailing (OT, for short) of Φ if the last two lines of point (2) of this definition are replaced by the following:

and, for any $\alpha \in A_x$ and $r_\alpha = r|_{\Phi_\alpha}$, we have ind $r_\alpha^{-1}x \geq 1$ and r_α is open.

Remark 2.2 and Corollary 2.4 of [6] will be true if tailings are replaced by OT in them.

If in Lemma 2.6 of [6] X is an OT of Φ then, instead of the last line of it, we can conclude that:

$$r^{-1}\mathrm{bd}_\Phi \mathrm{cl}O \cap \bigcup\{\Phi_\alpha : \alpha \in A, \Phi_\alpha \subset \cap\nu\} \subset \mathrm{bd}U \cap \mathrm{bd}(X \setminus \mathrm{cl}U)$$
$$\text{and } r_\alpha : r^{-1}\mathrm{bd}_\Phi \mathrm{cl}O \cap \Phi_\alpha \to \mathrm{bd}_\Phi \mathrm{cl}O \text{ is open for all } \alpha \in A.$$

It follows from this that Corollary 2.8 of [6] is true if tailings are replaced by open tailings (in particular, X is OT of Φ).

Theorem 2.10 of [6] and its proof are also true if tailings are replaced by open tailings. Thus Theorem 2.10 of [6] must be formulated in the following way:

Theorem 2.10 from [6]. *If X is an open tailing of a compactum Φ with* $\dim\Phi = n$, *then* $\mathrm{ind}X \geq n+1, n = 0, 1, 2, \ldots$.

Propositions 2.12 and 2.13 of [6] are also true for open tailings.

Since spaces $T(\Phi, K, c, \tau)$ (tailings of transfinite type) and $D(\Phi, K, c, \tau, \theta)$ (tailings of discrete type) from [6] are open tailings, all assertions of Section 3 concerning them are true.

All results of Section 4 of [6] (because they, in fact, concern open tailings) are also true (with their proofs).

6. One General Construction of Iterated Open Spreaded Tailings

Let $D = \{-1, 2\}, C$ be the standard Cantor set and c be some its irreducible monotone map onto the unit segment $I = [0, 1]$. Hence $|c^{-1}y| \leq 2$ for any $y \in I$.

Put $I^{m+1} = I_m \times \ldots \times I_1 \times I_0$, where $I_i = [0, 1],\ i = m, \ldots, 1, 0$, $m = 1, 2, \ldots$ (It is important for us that the first factor is I_m, the second factor is I_{m-1}, …, the last factor is I_0). Let π_i and π be the projections of the product (of sets) $L^m = I^{m+1} \times (D \cup C)$ onto the factor $I_i, i = m, \ldots, 0$, and $D \cup C$ respectively. We can take the lexicographic order and the topology induced by it on L^m. Thus L^m becomes a topological space. Evidently, it is a zero-dimensional first countable compactum. Its subspace $I^{m+1} \times D$ will be denoted by M^m.

Let Φ be a separable metrizable space. Fix an injection $in : \Phi \to I_0$. Let p and pr be the projections of $L^m \times \Phi$ onto L^m and Φ respectively. Then pr is open and perfect.

Define an equivalence relation E on $L^m \times \Phi$ supposing $(y,t)E(y',t')$ if $t = t'$ and either $y = y'$ or $\pi_i(y) = \pi_i(y')$ for $i = m, \ldots, 1, 0, in(t) = \pi_0(y)$ and $\pi(y), \pi(y') \in C$, $c(\pi(y)) = c(\pi(y'))$.

Put $L^m(\Phi) = L^m \times \Phi/E$. Let q be the natural quotient mapping of $L^m \times \Phi$ onto $L^m(\Phi)$. It is easy to verify that:
there exists a mapping $r_m : L^m(\Phi) \to \Phi$ such that
$\mathrm{pr} = r_m \circ q$ and r_m is continuous, open and perfect;
$L^m(\Phi)$ is Hausdorff and q is perfect and, since $L^m \times \Phi$ is Lindelöf (and so normal), $L^m(\Phi)$ is normal and Lindelöf;
$|q^{-1}x| \leq 2$ for any $x \in L^m(\Phi)$ and $L^m(\Phi)$ **is first countable**;
the restriction $r_{m,y,i}$ of r_m to $\Phi_{y,i} = q(\{y\} \times \{i\} \times \Phi)$ is a homeomorphism for $y \in I^{m+1}, i = -1, 2$;
the sets $\Psi_y = q(\{y\} \times C \times \Phi)$ are clopen in $L^m(\Phi)$ for $y \in I^{m+1}$, they are separable metrizable spaces, their subsets $q(\{y = (y_m, \ldots, y_1, y_0)\} \times C \times (\Phi \setminus (in)^{-1}y_0))$ are homeomorphic to $C \times (\Phi \setminus (in)^{-1}y_0)$ and $q(\{y = (y_m, \ldots, y_1, y_0)\} \times C \times (in)^{-1}y_0)$ are homeomorphic either to I for $(in)^{-1}y_0 \neq \emptyset$ or to $\emptyset$ for $(in)^{-1}y_0 = \emptyset$;
the restriction of q to $M^m \times \Phi$ is a homeomorphism and so we shall identify $q(M^m \times \Phi)$ and $M^m \times \Phi$ by q.

Since $M^m \times \Phi$ is Lindelöf and $\mathrm{ind} M^m = 0$, we have the following:

$\dim \Phi = \mathrm{ind}\, \Phi \leq \dim M^m \times \Phi \leq \mathrm{ind} M^m \times \Phi = \mathrm{ind} \Phi = \dim \Phi$.

As in Lemma 3.2 of [6], it is possible to prove that

$$\dim \Psi_y = \mathrm{ind} \Psi_y = \max(\dim \Phi, \dim I = 1).$$

It follows from this and Dowker's theorem [4] (see, for example, [1, Ch.4, §6, Theorem 15] that

$$\dim L^m(\Phi) = \max(\dim \Phi, 1).$$

For $y_m \in I_m = [0,1]$, put $\Psi_{y_m} = q(\{y_m\} \times I^m \times (D \cup C) \times \Phi)$, where $I^m = I_{m-1} \times \ldots \times I_0$. Evidently, Ψ_{y_m} may be identified with $L^{m-1}(\Phi)$ so that $r_{y_m} = r_m|_{\Psi_{y_m}}$ will be identified with r_{m-1}. It follows from this that r_{y_m} is open. Put $\mu_m = \{\Psi_{y_m} : y_m \in I_m\}$ and $\lambda_m = \{\Phi^-_{y_m} = \{y^-_m = (y_m, 0, \ldots, 0)\} \times \{-1\} \times \Phi, \Phi^+_{y_m} = \{y^+_m = (y_m, 1, \ldots, 1)\} \times \{2\} \times \Phi : y_m \in I_m\}$. Let $r^{-(+)}_{y_m} = r_m|_{\Phi^{-(+)}_{y_m}}$. We have already noted that $r^{-(+)}_{y_m}$ is a homeomorphism. Evidently, $|\mu_m| = \mathbf{c} \geq \tau = \omega^+ > w(\Phi)$. Let $\nu = \{\Psi_{y_m} : y_m \in A \subset I_m\}$ and $|A| \geq \tau$. There exists $z \in I_m$ such that any nbd of z in I_m contains at least τ points of A. Let O be a nbd of $\Phi_z = \Phi^-_z \cup \Phi^+_z$ in $L^m(\Phi)$. Then $O' = q^{-1}O$ is a nbd of $\Phi_z \equiv q^{-1}\Phi_z$ in $L^m \times \Phi$. **Now let Φ be compact**. (Let us note that **in this case $L^m(\Phi)$ is compact**.) Since the projection p is perfect in this case, there exists a nbd U of $(\{z^-\} \times \{-1\}) \cup (\{z^+\} \times \{2\})$ in L^m such that $p^{-1}U \subset O'$. For the sake of simplicity, we suppose that $0 < z < 1$. Then there exist $z_1 < z$ and $z_2 > z$ such that the half-intervals $(\{z^+_1\} \times \{2\}, \{z^-\} \times \{-1\}]$ and $[\{z^+\} \times \{2\}, \{z^-_2\} \times \{-1\})$ are contained in U. Hence $A' = (A \cap (z_1, z_2)) \setminus \{z\}$ has cardinality $\geq \tau$. Evidently, $\{y_m\} \times I^m \times (D \cup C) \times \Phi \subset p^{-1}U \subset O'$ for all $y_m \in A'$ and so $\Psi_{y_m} \subset O$ for all $y_m \in A'$. It follows from this that any nbd of either Φ^-_z or Φ^+_z contains at least τ elements of ν. We have proved that $L^m(\Phi)$ is an open and, evidently, minimal $(\omega^+, r_m, \lambda_m, \mu_m)$-ST of the compactum Φ.

Let $m = 1$ and $t \in \Phi$. We have already noted that, for $y_0 = in(t)$ and any $y_1 \in I_1$, the set $q(\{y_1, y_0)\} \times C \times \{t\})$ is homeomorphic to I and, evidently, it is contained in $r^{-1}_1 t$. Hence ind $r^{-1}_1 t \geq 1$. It follows from this that $L^1(\Phi)$ is an $(1, \omega^+, r_1, \lambda_1, \mu_1)$-ST of the compactum Φ.

Thus we have proved the following:

Proposition 6.1. *For any metrizable compactum Φ, the space $L^m(\Phi)$ is a $(m, \omega^+, r_m, \lambda_m, \mu_m)$-ST of Φ and* $\dim L^m(\Phi) = max(\dim \Phi, 1), m = 0, 1, 2, \ldots$

This proposition and Theorem 5.5 give us the following assertion.

Proposition 6.2. *If* Φ *is a metrizable compactum with* $\dim \Phi = m$ *then* $\dim L^m(\Phi) = m < m+1 \leq \mathrm{ind} L^m(\Phi), m = 1, 2, \dots$

We shall give more exact estimation of $\mathrm{ind}\, L^m(\Phi)$.

Proposition 6.3. *Let* Φ *be a metrizable compactum with* $\dim \Phi = n$. *Then* $\mathrm{ind}\ L^m(\Phi) = n+1$ *for* $m \geq n$ *and* $n \leq \mathrm{ind}\ L^m(\Phi) \leq n+1$ *for* $m < n, m = 1, 2, \dots, n = 0, 1, 2, \dots$

Proof. For $n = 0$ our assertion is evident. Let $n > 0$.

If $m = n$ then, by the previous proposition, $\mathrm{ind}\ L^m(\Phi) \geq n+1$. Suppose $m > n$. Then $L^m(\Phi)$ contains $L^n(\Phi)$ as a subspace and so $\mathrm{ind}\ L^m(\Phi) \geq n+1$. Let us prove that $\mathrm{ind} L^m(\Phi) \leq n+1$ (for $m \geq n$).

As it was noted, the sets $\Psi_y, y \in I^{m+1}$, are clopen in $L^m(\Phi)$ and they are metrizable compacta with $\mathrm{ind}\Psi_y = \dim \Phi = n$.

Hence $\mathrm{ind}_x L^m(\Phi) \leq n$ for any $x \in \cup\{\Psi_y : y \in I^{m+1}\}$. Now let $x \in \Phi_{y,i} = q(\{y\} \times \{i\} \times \Phi) \equiv \{y\} \times \{i\} \times \Phi$ for $y \in I^{m+1}, i = -1, 2$, and let U be a nbd of x in $L^m(\Phi)$. Since $\Phi_{y,i}$ is a retract of $L^m(\Phi)$, by Lemma 2.3 of [6], there exist nbds V of $t = r_{m,y,i}x$ in Φ and O of $\Phi_{y,i}$ in $L^m(\Phi)$ such that $r_m^{-1}V \cap O \subset U$. Take nbds V_1 and V_2 of t such that $\mathrm{cl}V_1 \subset V_2$, $\mathrm{cl}V_2 \subset V$. Let, for example, $i = 2$ and $y \neq (1, \dots, 1)$. Then, since p is closed, there exists $y' \in I^{n+1}$ such that $y < y'$ (in our lexicografical order) and $G = q([\{y\} \times \{2\}, \{y'\} \times \{-1\}) \times \Phi) \subset O$ (here $[\{y\} \times \{2\}, \{y'\} \times \{-1\})$ is a half-interval in the lexicografically ordered product $I^{m+1} \times (D \cup C)$). Then G is a nbd of x and $r_m^{-1}V \cap G \subset U$. Put $H = (r_m^{-1}V_1 \cap G) \cup \{r_m^{-1}V_2 \cap \Psi_z : y < z = (z_m, \dots, z_1, z_0) < y', z_0 \in in(\mathrm{bd}V_1)\}$. Then H is a nbd of x such that $H \subset U$ and $\mathrm{bd}H$ is homeomorphic to a subset of the product of the zero-dimensional compactum $[\{y\} \times \{2\}, \{y'\} \times \{-1\}]$ and $\mathrm{cl}V_2 \setminus V_1 \subset \Phi$. Hence $\mathrm{ind}\ \mathrm{bd}H \leq \mathrm{ind}\Phi = \dim \Phi = n$. We have proved that $\mathrm{ind}L^m(\Phi) \leq n+1$ and so $\mathrm{ind}L^m(\Phi) = n+1$.

Now, let $m < n$ (and so $n \geq 2$). Then $\mathrm{ind}L^m(\Phi) \geq \mathrm{ind}\Phi = \dim \Phi = n$. Since $L^m(\Phi)$ is a subset of $L^n(\Phi)$ and $\mathrm{ind}L^n(\Phi) = n+1$, we see that $\mathrm{ind}L^m(\Phi) \leq n+1$.

We have obtained:

Theorem 6.4. *Let Φ be a metrizable compactum with* $\dim \Phi = n > 0$. *Then $L^m(\Phi)$ is a first countable compactum with:*

$$\dim L^m(\Phi) = n \text{ for } m = 1, 2, \ldots;$$
$$\operatorname{ind} L^m(\Phi) = n + 1 \text{ for } m \geq n;$$
$$n \leq \operatorname{ind} L^m(\Phi) \leq n + 1 \text{ for } m < n.$$

7. Iterated Open Spreaded Tailings and Dimension of Products

Theorem 7.1. *Let χ_n be a metrizable compactum such that $\dim \chi_n = n, \dim \chi_n^2 = 2n - 1$ (see [2]) and let X_n be the discrete union of χ_n and $L^{2n-1}(Q^{n-1})$, where Q^{n-1} is Euclidian $(n-1)$-cube, $n = 2, 3, \ldots$. Then X_n is a first countable compactum and*

$$\dim X_n = \dim \chi_n = \operatorname{ind} X_n = \operatorname{ind} \chi_n = n,$$
$$\dim X_n \times \chi_n = \dim X_n^2 = 2n - 1 < 2n = \operatorname{ind} X_n \times \chi_n \leq \operatorname{ind} X_n^2.$$

Proof. We have $\dim L^{2n-1}(Q^{n-1}) = n - 1, \operatorname{ind} L^{2n-1}(Q^{n-1}) = n$ by Theorem 6.4. It follows from this that $\dim X_n = \operatorname{ind} X_n = n$.

Since $Q^{n-1} \times \chi_n \subset X_n \times \chi_n$, we see that

$$\begin{aligned} 2n-1 = \dim\ Q^{n-1} \times \chi_n &\leq \dim X_n \times \chi_n \\ &= \dim(L^{2n-1}(Q^{n-1}) \cup \chi_n) \times \chi_n \\ &= \max(\dim L^{2n-1}(Q^{n-1}) \times \chi_n, \dim \chi_n^2) \\ &\leq \max(\dim L^{2n-1}(Q^{n-1}) + \dim \chi_n, 2n-1) \\ &= 2n-1 \quad \text{(by Theorem 6.4).} \end{aligned}$$

Hence $\dim X_n \times \chi_n = 2n - 1$.

It follows from this that $2n - 1 = \dim X_n \times \chi_n \leq \dim X_n^2 = \max(\dim(L^{2n-1}(Q^{n-1}))^2, \dim L^{2n-1}(Q^{n-1}) \times \chi_n, \dim \chi_n^2) \leq \max(2n - 2, 2n - 1, 2n - 1) = 2n - 1$ (by Theorem 6.4).

Hence $\dim X_n^2 = 2n - 1$.

By Propositions 6.1 and 5.6, $\Pi = L^{2n-1}(Q^{n-1}) \times \chi_n$ is a $(2n-1, \omega^+, r_\Pi, \lambda_\Pi, \mu_\Pi)$-ST of $Q^{n-1} \times \chi_n$. Since $\dim Q^{n-1} \times \chi_n = 2n-1$, by Theorem 5.5, ind $\Pi \geq 2n$. Hence

$$2n \leq \mathrm{ind}\Pi \leq \mathrm{ind} X_n \times \chi_n \leq \mathrm{ind} X_n^2.$$

For any Tychonoff space Y and the segment Q^1, we have ind $Y \times Q^1 \leq$ ind $Y + 1$ (see [10, 7]). Hence ind $Y \times Q^n \leq$ ind $Y + n, n = 1, 2, \ldots$, and so $\mathrm{ind} L^{2n-1}(Q^{n-1}) \times Q^n \leq 2n$. By Hurewicz's theorem χ_n has a zero-dimensional map $f : \chi_n \to Q^n$. Then $g = \mathrm{id}_{L^{2n-1}(Q^{n-1})} \times f$ is also a zero-dimensional map between compacta and so

$$\mathrm{ind}\ L^{2n-1}(Q^{n-1}) \times \chi_n \leq \mathrm{ind}\ L^{2n-1}(Q^{n-1}) \times Q^n \leq 2n.$$

It follows from this that

$$\mathrm{ind}\ X_n \times \chi_n \leq \max(\mathrm{ind}\ L^{2n-1}(Q^{n-1}) \times \chi_n, \mathrm{ind}\chi_n^2) = 2n.$$

Thus ind $X_n \times \chi_n = 2n$.

8. Other Constructions of Spreaded Tailings

8.1. Open spreaded tailings

Let Φ be a metrizable compactum and $L^\omega(\Phi)$ be the disjoint union of Φ and $L^m(\Phi)$, $m = 1, 2, \ldots$ Define $r_\omega : L^\omega(\Phi) \to \Phi$ in the following way: $r_\omega|_\Phi = \mathrm{id}_\Phi, r_\omega|_{L^m(\Phi)} = r_m, m = 1, 2, \ldots$ Let the base of topology on $L^\omega(\Phi)$ consists of all open subsets of all $L^m(\Phi)$ and the sets $O(m, U) = r_\omega^{-1} U \cap \bigcup\{L^k(\Phi) : k \geq m\}$, where U is open in Φ. Evidently, r_ω is a retraction. It is open because all r_m and id_Φ are open.

Theorem 8.1. *Let* Φ *be a metrizable compactum with* $\dim \Phi = n > 0$. *Then*

$$\dim L^\omega(\Phi) = n < n + 1 = \mathrm{ind}\ L^\omega(\Phi).$$

Proof. Since, by Theorem 6.4, $\dim L^m(\Phi) = n$ for all m, Dowker's theorem mentioned above gives us the equality $\dim L^\omega(\Phi) = n$.

By the same Theorem 6.4 $\operatorname{ind} \cup \{L^m(\Phi) : m = 1, 2, \ldots\} = n + 1$. As in the proof of Proposition 6.3 for $x \in \Phi_{y,i}$ it is possible to prove that $\operatorname{ind}_x L^\omega(\Phi) \leq n + 1$ for all $x \in \Phi$.

Example 8.2. Let L^∞ be the product $I^\omega \times I_0 \times (D \cup C)$ with a topology induced by the lexicographic order and Φ be a metrizable compactum with $\dim \Phi > 0$. Evidently, L^∞ is a first countable compactum.

Similarly to the construction in Section 6, one can obtain a quotient space $L^\infty(\Phi)$ of the product $L^\infty \times \Phi$, a quotient map $q : L^\infty \times \Phi \to L^\infty(\Phi)$ and an open map $r : L^\infty(\Phi) \to \Phi$.

As in Section 6, put $\lambda = \{\Phi_x^- = \{(x, 0, \ldots)\} \times \{0\} \times \{-1\} \times \Phi, \Phi_x^+ = \{(x, 1, \ldots)\} \times \{1\} \times \{2\} \times \Phi : x \in I\}$ and $\mu = \{\Psi_x = q(([\{(x, 0, \ldots)\} \times \{0\} \times \{-1\}, \{(x, 1, \ldots)\} \times \{1\} \times \{2\}]) \times \Phi) : x \in I\}$. Then $L^\infty(\Phi)$ is an open minimal $(\omega^+, r, \lambda, \mu)$-ST of Φ. Moreover, each $\Psi_x, x \in I$, is homeomorphic to $L^\infty(\Phi)$. Hence, (using the same arguments as in the proof of Theorem 6.4) $\operatorname{ind} L^\infty(\Phi) = \dim L^\infty(\Phi) + 1 = \dim \Phi + 1$.

Remark 8.3. Let L be the set of countable sequences $\{\alpha_1, \alpha_2, \ldots\}$ of reals from $[0, 1]$ such that if there is 0 or 1 on some place of such a sequence then all its elements with greater indexes are 0 or 1 correspondently. L may be considered as a subset of I^ω and the topology on L is induced by the lexicographical order on I^ω. Evidently, L is a first countable compactum.

In the previous example one can use space $L \times I_0 \times (D \cup C)$ instead of the space L^∞.

Remark 8.4. Evidently, $L^\omega(\Phi)$ and $L^\infty(\Phi)$ may be considered as partial cases of ω-iterated open spreaded tailings.

8.2. Weakly open spreaded tailings

Example 8.5. Fix two irreducible, onto and monotone maps $c_i : C_i \to \Phi = [0,1]$ of copies of the standard Cantor set such that $|c_1^{-1}(t) \cup c_2^{-1}(t)| \le 3$ for any $t \in [0,1]$. Let $D = \{-1,2\}, I_1 = [0,\frac{1}{3}], I_2 = [\frac{2}{3},1]$ and $J = D \cup I_1 \cup I_2$. Take a lexicografically ordered product $I_0 \times J$ with the topology induced by this order, where $I_0 = [0,1]$. Evidently, $I_0 \times J$ is compact and $M = I_0 \times D \subset I_0 \times J$ is a 0-dimensional compactum (such that $M \setminus \{(0,-1),(1,2)\}$ is homeomorphic Alexandroff's "two arrows" space).

Let, for $x \in I_0$, Ψ_x be the discrete union of the topological products $I_i \times C_i, i = 1,2$; $p_x : \Psi_x \to I_1 \cup I_2$ be equal to the projection of $I_i \times C_i$ onto $I_i, i = 1,2$; r_x be equal to $c_i \circ q_i$ on $I_i \times C_i$, where q_i is the projection of $I_i \times C_i$ onto $C_i, i = 1,2$.

Put

$$\sigma(\Phi) = (M \times \Phi) \cup \bigcup\{\Psi_x : x \in I_0\};$$

and let r be equal: to the projection of $M \times \Phi$ onto Φ on $M \times \Phi$ and to r_x on $\Psi_x, x \in I_0$; p be equal to: the projection of $M \times \Phi$ onto M on $M \times \Phi$ and to p_x on $\Psi_x, x \in I_0$. Take the topology on $\sigma(\Phi)$ such that its base consists of all open subsets of all Ψ_x and all intersectons $r^{-1}O \cap p^{-1}[(x',2),(x'',-1)], r^{-1}O \cap p^{-1}(0,-1), r^{-1}O \cap p^{-1}(1,2)$, where O is open in Φ and $x', x'' \in I_0, x' < x''$. Evidently, $\sigma(\Phi)$ **is a first countable compactum** and r is a homeomorphism on all sets $\Phi_x^- = \{x\} \times \{-1\} \times \Phi, \Phi_x^+ = \{x\} \times \{2\} \times \Phi, x \in I_0$.

It is not difficult to prove that $\sigma(\Phi)$ is an $(\omega^+, r, \lambda = \{\Phi_x^-, \Phi_x^+ : x \in I_0\}, \mu = \{\Psi_x : x \in I_0\})$-ST of Φ. Fix $x \in I_0$ and $t \in \Phi$. Then there exists $i \in \{1,2\}$ such that $|c_i^{-1}t| = 1$. It follows from this that c_i is open in the unique point $z \in c_i^{-1}t$. Then r_x is open in every point of the segment $q_i^{-1}z = \{z\} \times I_i \subset r_x^{-1}t$. It follows from this that:

$\sigma(\Phi = I)$ **is a weakly open** 1-$(\omega^+, r, \lambda, \mu)$-ST of $\Phi = I$.

By Theorem 4.2, ind $\sigma(I) \geq 2$. Evidently, $\dim \Psi_x = \mathrm{ind}\Psi_x = 1$ for any $x \in I_0$ and so, for $\Psi = \cup\{\Psi_x : x \in I_0\}$, we have $\dim \Psi = \mathrm{ind}\Psi = 1$. Since $\dim M \times \Phi \leq \dim M + \dim \Phi = 1$, by Dowker's theorem mentioned above, $\dim \sigma(I) = 1$. As above, it is possible to prove that ind $\sigma(I) \leq 2$. Thus

$$\dim \sigma(I) = 1 < 2 = \mathrm{ind}\ \sigma(I).$$

Let $\sigma_i(I) = (M \times \Phi) \cup \bigcup\{I_i \times C_i \subset \Psi_x : x \in I_0\}, i = 1, 2$. Then, evidently, $\sigma_i(I)$ is compact and ind $\sigma_i(I) = 1$. Thus

$\sigma(I)$ **is the union of two subcompacta with** ind $= 1$.

Remark 8.6. The compactum $\sigma(I)$ is a simplification of **Filippov's compactum from** [8]. Moreover $\sigma(I)$ is a closed subset of Filippov's compactum.

It is easily seen that Filippov's compactum itself is a weakly open 1-$(\omega^+, r, \lambda, \mu)$-ST of I (for r, λ, μ defined in the similar to the case of $\sigma(I)$ way).

Example 8.7 (V. V Filippov, I. K. Lifanov [9])**.** A first countable space Z with $\dim Z = 1$ and ind$Z = 2$ (even ind$_z Z = 2$ at any point $z \in Z$).

Let L be the same set as in Remark 8.3 and B be the subset of all sequences with 0 or 1 as almost all elements. Evidently B is a dense subset of L.

"Putting a circle S^1 instead of a point x of the square I^2" means taking a set $(I^2 \setminus \{x\}) \cup S^1$ with basic nbds of points of $I^2 \setminus \{x\}$ the same as in I^2 and arditrary basic nbd of the point $(x, \varphi) \in S^1$ is the union of the arc $((x, \varphi_1), (x, \varphi_2))$ containing (x, φ) and the sector of the circle of some radius with center in x bounded by rays, starting in x under the angles φ_1, φ_2. Let denote this space I^2_x.

Put $Z' = L \times I$. The interval $(0, 1)$ can be decomposed in $\mathbf{c}$ countable dense and pairwise disjoint sets $P_\xi, \xi < \mathbf{c}$. We can also enumarate all points of the set $L \setminus B$ by all ordinals less than $\mathbf{c}$. Let l_ξ be the point enumarated by ξ. Put $D = \cup\{\{l_\xi\} \times P_\xi, \xi < \mathbf{c}\}$.

For any point $z \in Z'$ there is a monotone mapping $f'_z : Z' \to I^2$ such that $f_z'^{-1} f'_z(z) = z$ and the inverse image of a base at $f'_z(z)$ is a base at z. "Putting a circle instead of a point z of Z'" means taking a set $Z_z = (Z' \setminus \{z\}) \cup S^1(z)$ and the mapping $f_z : Z_z \to I^2_{f'_z(z)}$ such that $f_z|_{Z'\setminus\{z\}} = f'_z|_{Z'\setminus\{z\}}$ and $f_z|_{S^1(z)}$ is the identity mapping of $S^1(z)$. Bases at points $Z_z \setminus S^1(z)$ are the same as in Z' and, at points $s \in S^1(z)$, basic nbds are the inverse images of basic nbds of $f_z(s) \in I^2_{f'_z(z)}$. Let $Z = (Z' \setminus D) \cup \bigcup\{S^1(z) : z \in D\}$ and $g_z : Z \to Z_z$ be identical on $(Z' \setminus D) \cup S^1(z)$ and $g_z(S^1\{z'\}) = \{z'\}$ for $z' \in D \setminus \{z\}$. Take the topology on Z with a base consisting of inverse images of all open in Z_z sets for $z \in D$. Of course, the natural mapping $g : Z \to Z'$ (glueing every $S^1(z)$ into z) is continuous. Let $r : Z \to I$ be the composition of g and of the projection of the product Z' onto its factor I and pr is the mapping of Z onto L, defined in the same manner.

The space Z **is a minimal weakly open** 1-$(\omega_1, r, \lambda, \mu)$-ST of I, where $\lambda = \{\mathrm{pr}^{-1}\{l\} : l \in B \subset L\}$, $\mu = \{L_x = \mathrm{pr}^{-1}([\{x, 0, \ldots\}, \{x, 1, \ldots\}]) : 0 < x < 1\}$. It is not difficult to show that the restriction of r to any L_x is open at all points from $L_x \setminus D$, and for any $t \in I$, $|r^{-1}t \cap D| = 1$. Hence $\mathrm{ind}(r^{-1}t \setminus D) = 1$. By Theorem 4.2, $\mathrm{ind} Z \geq 2$.

The proof of equalities $\dim Z = 1$ and $\mathrm{ind}_z Z = 2$ at any point $z \in Z$ can be found in [9].

8.3. Two-sided spreaded tailings

Example 8.8. Let Φ be an n-dimensional metrizable compactum, $n = 1, 2, \ldots$, and $\Phi_3 = \{(O_1, O_2, t) : O_1, O_2$ is an oddp in Φ and $t \in \mathrm{cl} O_1 \cap \mathrm{cl} O_2\}$. Evidently, $|\Phi_3| \leq \mathbf{c}$. For any $K = (O_1, O_2, t) \in \Phi_3$ fix $t_i^j(K) \in O_j$ so that t is the limit of the sequence $t_i^j, i = 1, 2, \ldots, j = 1, 2$. Take a dense subset $P = \{p_i : i = 1, 2, \ldots\}$ of n-cube Q^n.

Let D and $I^2 = I_1 \times I_0$ be such as in Section 6. Fix an injection $in : \Phi_3 \to I_0$. Put $M^2 = I^2 \times D$ and take the lexicografic order and the topology induced by this order on M^2. Evidently, M^2 is a first countable and 0-dimensional compactum.

Let us "place $Q^n_y = Q^n$ between $\{y\} \times \{-1\}$ and $\{y\} \times \{2\}$" for $y = (y_1, y_0) \in I^2$. Take $T_n = M^2 \cup \bigcup\{Q^n_y : y \in I^2\}$ with the topology which subbase consists of all open subsets of all Q^n_y and all sets $O \cup \bigcup\{Q^n_y : \{y\} \times \{-1\} \cup \{y\} \times \{2\} \subset O\}$ for open in M^2 sets O. It is easy to see that the sets Q^n_y are clopen in T_n and that T_n is a first countable compactum.

Let π and pr be the projections of $T_n \times \Phi$ onto T_n and Φ respectively. They are perfect.

Put, for $y = (y_1, y_0) \in I^2, \Psi_y = \emptyset$ if $y_0 \notin in(\Phi_3)$ and $\Psi_y = Q^n_y \times \{t\} \cup \{(p_i, t^j_i(K)) : j = 1, 2, i = 1, 2, \dots\}$ if $(in)^{-1}y_0 = K = (O_1, O_2, t)$.

Now we can take

$$T_n(\Phi) = (M^2 \times \Phi) \cup \bigcup\{\Psi_y : y \in I^2\} \subset T_n \times \Phi$$

and $r = \mathrm{pr}|_{T_n(\Phi)}, r_y = r|_{\Psi_Y}, y \in I^2$.

Evidently, for $\Phi_{y,k} = \{y\} \times \{k\} \times \Phi, y \in I^2, k = -1, 2$, the restrictions $r_{yk} = r|_{\Phi_{yk}}$ are homeomorphisms.

In particular, for any $y_1 \in I_1$, the restrictions $r^-_{y_1} = r_{(y_1,0)(-1)}$ and $r^+_{y_1} = r_{(y_1,1)2}$ are homeomorphisms for $y_1 \in I_1$. Put $\Psi_{y_1} = (\{y_1\} \times I_0 \times D) \times \Phi \cup \bigcup\{\Psi_y : y = (y_1, y_0), y_0 \in I_0\}$ and $r_{y_1} = r|_{\Psi_{y_1}}, y_1 \in I_1$.

It is not difficult to prove that $T_n(\Phi)$ is an $(\omega^+, r, \lambda, \mu)$-ST of Φ for $\lambda = \{\Phi^-_{y_1} = \{(y_1, 0, -1)\} \times \Phi, \Phi^+_{y_1} = \{(y_1, 1, 2)\} \times \Phi : y_1 \in I_1\}$ and $\mu = \{\Psi_{y_1} : y_1 \in I_1\}$.

Take an oddp O_1, O_2 in Φ and $y_1 \in I_1$. Let $t \in \mathrm{cl}O_1 \cap \mathrm{cl}O_2$, $K = (O_1, O_2, t)$ and $y_0 = in(K)$. Then, for $y = (y_1, y_0)$, we see that $Q^n_{(y_1,y_0)} \subset \bigcap\{\mathrm{cl}\{(p_i, t^j_i(K)) : i = 1, 2, \dots\} : j = 1, 2\} \subset \mathrm{cl}\ r_y^{-1}O_1 \cap \mathrm{cl}\ r_y^{-1}O_2 \subset \mathrm{cl}\ r^{-1}O_1 \cap \mathrm{cl}\ r^{-1}O_2$. Hence

$$T_n(\Phi) \text{ is an } n\text{-}(\omega^+, r, \lambda, \mu)\text{-ST of } \Phi.$$

Theorem 8.9. *Let Φ be a metrizable compactum such that $\dim V = n$ for any open in Φ set $V \neq \emptyset$. Then $T_n(\Phi)$ is a first countable compactum with*

$$\dim T_n(\Phi) = n < n + 1 \leq \text{ind } T_n(\Phi).$$

Proof. The inequality $n+1 \leq \text{ind } T_n(\Phi)$ follows from Theorem 4.2. Since $\dim M^2 = 0$, we see that $\dim M^2 \times \Phi = n$. The inequalities $\dim \Psi_y \leq n$ for $y \in I^2$ and $\dim \Psi \leq n$ for $\Psi = \cup\{\Psi_y : y \in I^n\}$ are evident. Since Ψ is open in $T_n(\Phi)$, by Dowker's theorem mentioned above, $\dim T_n(\Phi) = n$.

Example 8.10. Let Φ be a metrizable compactum and Φ_C be the family of compact countable subsets of Φ. Evidently, $|\Phi_C| \leq \mathbf{c}$.

Let D and $I^{m+1} = I_m \times \ldots \times I_1 \times I_0,\ m \geq 0$, be such as in Section 6. Put $M^m = I^{m+1} \times D$ and take the topology induced by the lexicographic order on M^m. Fix an injection $in : \Phi_C \to I_0$.

Let us "place $I_y = [0,1]$ between $\{y\} \times \{-1\}$ and $\{y\} \times \{2\}$" for $y = (y_m, \ldots, y_0) \in I^{m+1}$ in the same way as it was described in the previous example. Let T^m denote the space obtained. It is a first countable ordered compactum.

Let π and pr be the projections of $T^m \times \Phi$ onto T^m and Φ respectively. They are open and perfect.

Put, for $y = (y_m, \ldots, y_0) \in I^{m+1}, \chi_y = \emptyset$ if $y_0 \notin in(\Phi_C)$ and $\chi_y = I_y \times K$ if $(in)^{-1}y_0 = K \in \Phi_C$.

Now we can take

$$T^m(\Phi) = (M^m \times \Phi) \cup \bigcup\{\chi_y : y \in I^{m+1}\} \subset T^m \times \Phi$$

and $r = \text{pr}|_{T^m(\Phi)}, r_y = r|_{\chi_y}, y \in I^{m+1}$.

It is not difficult to prove that, for $m > 0$, $T^m(\Phi)$ is an $(\omega^+, r, \lambda, \mu)$-ST of Φ for $\lambda = \{\Phi^-_{y_m} = \{y_m\} \times \{0, \ldots, 0\} \times \{-1\} \times \Phi, \Phi^+_{y_m} = \{y_m\} \times \{1, \ldots, 1\} \times \{2\} \times \Phi : y_m \in I_m\}$ and $\mu = \{\Psi_{y_m} = \pi^{-1}([\{y_m\} \times \{0, \ldots, 0\} \times \{-1\}, \{y_m\} \times \{1, \ldots, 1\} \times \{2\}]) : y_m \in I_m\}$. Thus Ψ_{y_m} is what is situated "between" $\Phi^-_{y_m}$ and $\Phi^+_{y_m}$ including these sets.

Below in this point $S^m(\Phi)$ will denote any compactum containing $T^m(\Phi)$ and having a map onto Φ such that its restriction to $T^m(\Phi)$ is equal to r.

Evidently,

$$(**) \qquad r^{-1}F = S^m(F) \text{ for any closed in } \Phi \text{ set } F.$$

As above,

$$\dim T^m(\Phi) = \max(\dim \Phi, 1).$$

Hence,

$$(***) \qquad \mathrm{ind}T^0(\Phi) \geq \dim T^0(\Phi) \geq 1 \text{ if } \dim \Phi \geq 0.$$

Let now $\dim \Phi = m > 0$.

Then, by $(**)$, we can suppose that Φ is an m-dimensional Cantor manifold and so $\mathrm{ind}V \geq m$ for any open subset $V \neq \emptyset$ of Φ.

Take a point $x \in \Phi' = \Phi_0^+$ and an open in Φ set O such that $r(x) \in O$ and $\mathrm{cl}O \neq \Phi$. It follows from this that $\dim \mathrm{bd}V \geq n-1$ for any open subset $V \neq \emptyset$ of O. Take a nbd U_1 of x in $T^m(\Phi)$ such that $\mathrm{cl}U_1 \subset r^{-1}O$. As above, there exist nbds V of $r(x)$ in Φ and W of Φ' in $T^n(\Phi)$ such that $\mathrm{cl}V \subset O$ and $r^{-1}V \cap W \subset U_1$. Evidently, $|\nu| = \mathbf{c}$ for the family ν of all Ψ_{y_m} containing in W. For oddp $U_1, U_2 = T^m(\Phi) \setminus \mathrm{cl}U_1$ in $T^m(\Phi)$, by Key lemma, either there exist non-empty and open in some $\Phi_{y_m}^-$ or $\Phi_{y_m}^+$ set G containing in $\mathrm{cl}U_1 \cap \mathrm{cl}U_2 \subset \mathrm{bd}U$ and hence $\mathrm{indbd}U \geq \mathrm{ind}G \geq m$ or there exist $y_m = t \in I_m$ and oddp O_1, O_2 in Φ such that $\Psi_t \subset W$ and, for $r_t = r|_{\Psi_t}$, we have $r^{-1}O_i \subset \mathrm{cl}U_i, i = 1, 2$. Since $r_t^{-1}V \subset U_1$ and $r_t^{-1}(\Phi \setminus \mathrm{cl}O) \subset T^n(\Phi) \setminus r^{-1}O \subset U_2$. Since r_t is surjective, $\emptyset \neq V \subset O_1$ and $\emptyset \neq \Phi \setminus \mathrm{cl}O \subset O_2$. It follows from this that $\dim \Phi' \geq m-1$ for $\Phi' = \mathrm{cl}O_1 \cap \mathrm{cl}O_2$.

Take a countable compactum $K \subset \Phi'$. There exists a countable compactum L in Φ such that $K = \mathrm{cl}(O_1 \cap L) \cap \mathrm{cl}(O_2 \cap L)$. Let in $L = y_0$. Then, for any $y = (t, y_{m-1}, \ldots, y_1, y_0)$, we have the equalities $\chi_y \cap r_t^{-1}\Phi' = (I_y \times L) \cap r_t^{-1} = I_y \times K = \mathrm{cl}U_1 \cap \mathrm{cl}U_2 \cap \chi_y$. It follows from this that $\chi_t \cap \mathrm{cl}U_1 \cap \mathrm{cl}U_2$ is $S^{m-1}(\Phi')$.

If $m = 1$ then, by $(***)$, $ind(\Psi_t \cap \mathrm{cl}U_1 \cap \mathrm{cl}U_2) \geq 1$ and so $\mathrm{indbd}U_1 \geq 1$ and $\mathrm{ind}T^1(\Phi) \geq 2$. Suppose for all $m < k, k > 1, \mathrm{ind}T^m(\Phi) \geq m$. Let $m = n$. Then $\mathrm{indbd}U_1 \geq \mathrm{ind}\Psi_t \cap \mathrm{cl}U_1 \cap \mathrm{cl}U_2 \geq \mathrm{ind}T^{k-1}(\Phi') \geq n-1$ and so $\mathrm{ind}T^n(\Phi) \geq n$. Hence

$$\dim T^m(\Phi) = \dim \Phi < \mathrm{ind}T^m(\Phi) \text{ for } \dim \Phi > 0.$$

It is possible to prove that $\mathrm{ind}T^m(\Phi) = \dim \Phi + 1$.

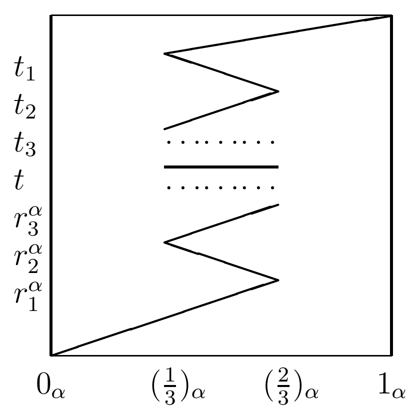
Remark 8.11. Using the same constructions as in point 8.1 one can obtain spaces $T^\omega(\Phi)$ and $T^\infty(\Phi)$ with the same dimensional properties.

These spaces may be regarded as cases of ω-iterated two-sided spreaded tailings.

Example 8.12. A chainable continuum L with $\mathrm{ind}L = 2$.

For each real number $t \in (0,1)$ choose a fixed decreasing sequence of reals $\{t_n\}$, converging to t. Let enumarate all increasing sequences of different rationals which limit belongs to the interval $(0,1)$ by ordinals $\alpha < \omega_{\mathbf{c}}$ and let $\{r_n^\alpha\}$ be the correspondent sequence. Let P denotes the segment of length $\omega_{\mathbf{c}}$ (the product of ordinals $< \omega_{\mathbf{c}}$ on half-interval $[0,1)$ with topology induced by the lexicographic order and the adjoint point $\omega_{\mathbf{c}}$ as a maximal element). We identify $\alpha < \omega_{\mathbf{c}}$ and $(\alpha, 0) \in P$. Points of $[\alpha, \alpha+1] \subset P$, are denoted by $x_\alpha, 0 \leq x \leq 1$.

Compactum L is a subset of $P \times I$. It consists of (see Fig.)



a) segments $\{\alpha\} \times I$, $0 < \alpha \leq \omega_{\mathbf{c}}$;

b) segments in the square $[\alpha, \alpha+1] \times I$, $\alpha < \omega_{\mathbf{c}}$ joining $(0_\alpha, 0)$ with $((\frac{2}{3})_\alpha, r_1^\alpha)$, $((\frac{2}{3})_\alpha, r_1^\alpha)$ with $((\frac{1}{3})_\alpha, r_2^\alpha)$,..., $((\frac{1}{3})_\alpha, t)$ with $((\frac{2}{3})_\alpha, t)$,..., $((\frac{2}{3})_\alpha, t_2)$ with $((\frac{1}{3})_\alpha, t_1)$, $((\frac{1}{3})_\alpha, t_1)$ with $(1_\alpha, 1)$, where t is the limit of $\{r_n^\alpha\}$ and $\{t_n\}$.

Put $\lambda = \{\{\omega_{\mathbf{c}}\} \times I\}$, $\mu = \{(\beta, \omega_{\mathbf{c}}] \times I \cap L : \beta < \omega_1\}$. If one takes any oddp O_1, O_2 in I then there exist $t \in \mathrm{cl}O_1 \cap \mathrm{cl}O_2$ and $t' > t$, $t' < 1$, such that the interval (t, t') belongs either O_1 or O_2. Without loss of generality let it be O_1. For any $\alpha < \omega_{\mathbf{c}}$, there exists $\beta > \alpha, \beta < \omega_{\mathbf{c}}$, such that $J_\beta = [\{(\frac{1}{3})_\beta, t\}, \{(\frac{2}{3})_\beta, t\}] \subset L$ and intervals $(r_{2m}^\alpha, r_{2m+1}^\alpha) \in O_2$ for all natural m. Then all points of J_β are (O_1, O_2, r)-two-sided. Hence, L **is a two-sided** 1-$(\mathbf{c}, r, \lambda, \mu)$-ST of I and thus $\mathrm{ind}L \geq 1$. Inequality $\mathrm{ind}L \leq 2$ is evident and the prove of chainability of L is the same as in [5].

Remark 8.13. The continuum L is simpler than S. Mardešić's continuum from [5] (L is its subcontinuum).

Remark 8.14. The concept of tailing in [6] may be generalized if in point (2) of Definition 2.1 one uses (O_1, O_2, r)-two-sided points instead of points of openness of r. Then the previous example would become a tailing of I in sence of [6].

8.4. Transfinite and discrete tailings as spreaded ones

We shall proof that transfinite and discrete tailings introduced in [6] are spreaded tailings.

First we shall use the notations of point 3.2 in [6].

It is not difficult to prove that:

any space $T(\Phi, K, c, \tau)$ is an open minimal $(\tau, r, \lambda = \{\Phi\}, \mu = \{\Psi_\alpha = (\{\beta : \alpha \leq \beta \leq \omega_\tau\} \times \Phi) \cup \bigcup\{q(C_\beta \times \Phi) : \alpha \leq \beta < \omega_\tau\} : \alpha < \omega_\tau\})$-ST of Φ and it is an 1-(τ, r, λ, μ)-ST of Φ if ind $K \geq 1$.

Similarly:

the compactum Y from Example 3.11 is a 1-$(\mathbf{c}, r, \lambda = \{I\}, \mu)$-ST of I; the compacta R, R and X from Examples 3.14, 3.16, 3.17 are 1-$(\omega^+, r, \lambda = \{I\}, \mu)$-ST of I.

Now we shall use the notations of point 3.3 in [6].

Let B be a set of pairwise disjoint families of indexes from A such that for any $x \in \Phi$ $|\beta \cap A_x| = 1$ for any $\beta \in B$. Since $|A_x| \geq \tau, x \in \Phi$, so $|B| \geq \tau$. We may also consider that $A = \bigcup\{\cup\beta : \beta \in B\}$. Evidently:
any space $D = D(\Phi, K, c, \tau, \aleph_0)$ is an open minimal $(\tau, r, \lambda = \{\Phi\}, \mu = \{\{\Phi \cup \Phi_\alpha : \alpha \in \beta\} : \beta \in B\})$-ST of Φ and it is an 1-(τ, r, λ, μ)-ST of Φ if ind $K \geq 1$.

Question 8.15. What are the relationships between tailings in sense of [6] and spreaded tailings?

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