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THE STRONG UNIVERSALITY OF CERTAIN STARLIKE SETS AND APPLICATIONS

Tadeusz Dobrowolski

Abstract

Following Banakh's idea [Ba], a version of strong universality in certain starlike sets is verified. Applications to an embedding problem for such starlike sets and sigma-compact absorbers follow.

This paper aims at revisiting Banakh's article [Ba]. In [Ba], extending the ideas from [DM1], [Do], [BD] [BC1], and from some other papers, T. Banakh made an important contribution to the topological classification of incomplete convex sets. Let us recall that the main tool for obtaining such a classification is the so-called absorbing set method (see e.g. [BP], [vM], and [BRZ]). This method requires to verify: (1) the so-called strong universality property of X, the space in question, for a class of spaces $\mathcal{C}$, (2) the Z_σ-property of X (both defined is Section 2), and (3) that X can be expressed as a countable union of elements of $\mathcal{C}$ that are closed in X. An intriguing, very concrete question is whether a σ-compact convex subset C of ℓ^2 is strongly universal for the class of compacta embeddable in C. An answer to this question is also unknown if, additionally, C has the Z_σ-property, cf. [DM2, Question 5.6]. In general, as shown in Section 2 there exists an example of a convex subset of ℓ^2 that has the Z_σ-property and is not strongly universal for the class of its closed subsets.

The central role in Banakh's approach [Ba] is played by certain special homeomorphisms, which, in our paper, are called

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almost cylindrical. Employing some properties of such homeomorphisms, Banakh verifies a certain version of the strong universality property for convex sets with almost internal points. Inspecting his reasoning, we re-examine the use of cylindrical homeomorphisms to extend his result to some sets X with $C \subseteq X \subseteq \bar{C}$ for some convex set C with an almost internal point. All applications that are listed in [Ba] can be carried over to such sets; we decided not to include them in this text (because this would be a formality only) with one exception related to embeddings of such sets into ℓ^2. We provide one more application concerning σ-compact absorbing sets that behave strangely with respect to the Cartesian product and do not admit any reasonable algebraic structure. These absorbing sets are versions of examples given in [BC2] (also cf. [Za]).

1. Mappings into Almost Convex Bodies

For $x = (x_i) \in \mathbb{R}^\infty$, we let $p_n(x) = (x_1, \dots, x_n) \in \mathbb{R}^n$ $p^n(x) = (x_{n+1}, x_{n+2}, \dots)$, and $\pi_i(x) = x_i$. We identify $p^n(x)$ with

$$(0, \dots, 0, x_{n+1}, x_{n+2}, \dots) \in \mathbb{R}^\infty$$

and, making an obvious identification of $\mathbb{R}^n$ with a subspace of $\mathbb{R}^\infty$, we view $p_n(x)$ as $(x_1, \dots, x_n, 0, \dots) \in \mathbb{R}^\infty$. The space $\mathbb{R}^\infty$ will be considered with the metric induced by the F-norm $\|x\| = \max\{2^{-i}\min\{|x_i|, 1\} \mid i \geq 1\}$. For two maps $f, g : X \to \mathbb{R}^\infty$, we let $\|f - g\| = \sup\{\|f(x) - g(x)\| \mid x \in X\}$.

We say that a homeomorphism $h \in H(\mathbb{R}^\infty)$ is n**-cylindrical** if there exists a homeomorphism $\bar{h} \in H(\mathbb{R}^n)$ such that $h(x) = (\bar{h}(p_n(x)), p^n(x))$, $x \in \mathbb{R}^\infty$. We say that h is **almost** n**-cylindrical** if, for some continuous function $\alpha : \mathbb{R}^n \to (0, \infty)$, $h(x) = (\bar{h}(p_n(x)), \alpha(p_n(x))p^n(x))$, $x \in \mathbb{R}^\infty$. The set of all n-cylindrical homeomorphisms and almost n-cylindrical homeomorphisms is denoted by $H_n(\mathbb{R}^\infty)$ and $H_n^a(\mathbb{R}^\infty)$, respectively. We let $H^a(\mathbb{R}^\infty) = \bigcup_{n=1}^\infty H_n^a(\mathbb{R}^\infty)$. Notice that $H_n(\mathbb{R}^\infty)$, $H_n^a(\mathbb{R}^\infty)$, $\bigcup_{n=1}^\infty H_n(\mathbb{R}^\infty)$, and $H^a(\mathbb{R}^\infty)$ are groups. Such n-cylindrical and almost n-cylindrical homeomorphisms have been frequently used in infinite-dimensional topology, especially in order to construct certain smooth maps on a topological vector space E. Then, $\mathbb{R}^\infty$ is replaced by E, and E is required to admit a splitting $E = E_n \oplus E^n$, where E_n is finite-dimensional.

Remark 1. Let E be a linear space with $\mathbb{R}^\infty_f \subseteq E \subseteq \mathbb{R}^\infty$, where $\mathbb{R}^\infty_f = \bigcup_{n=1}^\infty \mathbb{R}^n$. Since $p_n(x), p^n(x) = x - p_n(x) \in E$ for every $x \in E$, we have $h(E) = E$ for every homeomorphism $h \in H_n(\mathbb{R}^\infty) \cup H^a_n(\mathbb{R}^\infty)$. Actually, the above is true if E is replaced by $X \subseteq \mathbb{R}^\infty$ with $\mathbb{R}X + \mathbb{R}^\infty_f = X$.

We say that a convex set $C \subseteq \mathbb{R}^\infty$ is an **almost convex body** relative $\mathbb{R}^\infty_f \subset \mathbb{R}^\infty$ if,

(a) $0 \in \operatorname{int}_{\mathbb{R}^n}(C \cap \mathbb{R}^n)$ for every n, and

(b) $\mathbb{R}^\infty_f \cap C$ is dense in C.

Recall that a set $X \subseteq \mathbb{R}^\infty$ is **star-shaped** with respect to $x \in X$ if, for every $y \in X$, the segment $[y, x] = \{ty + (1-t)x \mid 0 \leq t \leq 1\}$ is contained in X. The **kernel**, $\ker(X)$, of X is the set of all $x \in X$ with respect to which X is star-shaped; $\ker(X)$ is always a convex set. Recall that by the **radial interior** of a set X that is star-shaped with respect to 0 we mean the set $\operatorname{rint}(X) = [0, 1)X$.

The following is a counterpart of [Ba, Lemma 2].

Lemma 1. *Let $C \subseteq \mathbb{R}^\infty$ be an almost convex body and K be a compactum with $K \subseteq \bar{C}$. For every $h \in H^a(\mathbb{R}^\infty)$, C absorbs $h(K \cap C)$. If X is such that* $\ker(X) = C$*, then X absorbs $h(K \cap X)$.*

Proof. If $h \in H_n(\mathbb{R}^\infty)$ then, using the fact that $0 \in \operatorname{int}_{\mathbb{R}^n}(C \cap \mathbb{R}^n)$ and the compactness of K, there exists $\beta \geq 1$ such that

$$p_n(h(K)) \cup p_n(K) \subset \pm\beta(C \cap \mathbb{R}^n).$$

Now, for $x \in K \cap X$, we have

$$\begin{aligned} h(x) &= p_n(h(x)) + (x - p_n(x)) \\ &\in \beta(C \cap \mathbb{R}^n) + X + \beta(C \cap \mathbb{R}^n) \\ &\subseteq (2\beta + 1)X. \end{aligned}$$

The last inclusion follows from the fact that X is starlike with respect to any point of C.
A similar argument can be used to show that X absorbs $g(K \cap X)$ for $g \in H^a(\mathbb{R}^\infty)$ of the form $g(x) = (p_n(x), \alpha(p_n(x))p^n(x))$. It is not difficult to show that the properties in question are preserved by the composition of homeomorphisms. Now, since every $h \in H^a(\mathbb{R}^\infty)$ is a composition of homeomorphisms considered above, the assertion of Lemma 1 follows. ☐

Lemma 2. *Let $A \subset \mathbb{R}^\infty$ and $B \subset \mathbb{R}^k$ be compacta, and $f : B \to \mathbb{R}^k$ be a map such that $[B \cup f(B)] \cap A = \emptyset$. Then, for $\varepsilon > 0$, there exist $n \geq k$ and $h \in H_n(\mathbb{R}^\infty)$ such that*

(i) $\|h - \mathrm{id}\| < \|f - \mathrm{id}_B\| + \varepsilon$,

(ii) $h|A = \mathrm{id}_A$,

(iii) $\|h|B - f\| < \varepsilon$.

Proof. Inspect [DM1, Lemma 1]. ☐

In the proof below we use the fact that the compacta are so-called Z-sets in $\mathbb{R}^\infty$. For information on **Z-sets** and **locally homotopy negligible sets** see [vM], [BRZ], and [To]. Here, we only recall that a closed subset A of an ANR space X is a Z-set if every map of a compactum into X can be arbitrarily closely approximated by maps whose ranges miss A; X is a Z_σ-space (or, has Z_σ-property) if it can be expressed as a countable union of Z-sets.

Lemma 3. *Let A and B be disjoint compacta in $\mathbb{R}^\infty$. Then, given a map $f : B \to \mathbb{R}^\infty$ and $\varepsilon > 0$ there exists an $h \in H^a(\mathbb{R}^\infty)$ such that* *(i)* $\|h - \mathrm{id}\,\| < \|f - \mathrm{id}_B\,\| + \varepsilon$,

(ii) $h|A = \mathrm{id}_A$,

(iii) $\|h|B - f\| < \varepsilon$.

Proof. Though the proof can easily be obtained by inspecting the proof of (a particular case of) [DM1, Proposition] we have decided to provide the details for it.

First, using the fact that A is a Z-set, we can approximate f by a map whose range is disjoint from A. So, we can assume that $f(B) \cap A = \emptyset$. Extend f to a map $\bar{f} : \mathbb{R}^\infty \to \mathbb{R}^\infty$. For $\varepsilon > 0$ there exists $\delta > 0$ such that

$$[b \in B \ \text{ and } \ \|b - b'\| < \delta] \Rightarrow \|\bar{f}(b) - \bar{f}(b')\| < \varepsilon/4.$$

Choose k so large that $2^{-k} < \delta$ and $p_k(B) \cap p_k(A) = \emptyset$. Let $B' = p_k(B)$ and approximate $\bar{f}|B'$ by $f' : B' \to \mathbb{R}^{k'}$ such that $k' \geq k$, $\|\bar{f}|B' - f'\| < \varepsilon/4$ and $f'(B') \cap A = \emptyset$. Increasing k', if necessary, we furthermore approximate f' by an embedding v (e.g., having the form $f' + w$ with $\|w\|$ sufficiently small) so that

$$\|v - \mathrm{id}_{B'}\,\| < \|f' - \mathrm{id}\,\| + \delta \ \text{ and } \ \|v - f'\| < \delta,$$

and $v(B) \cap A = \emptyset$. By Lemma 2, we can extend v to a homeomorphism $h' \in H_n(\mathbb{R}^\infty)$ that satisfies

$$\|h'|B' - f'\| < \varepsilon/4, \ \|h' - \mathrm{id}\,\| < \|f' - \mathrm{id}_{B'}\,\| + \varepsilon/4, \ \text{ and } \ h'|A = \mathrm{id}_A\,.$$

We need a "pseudohomotopy" $u = (u_t) : \mathbb{R}^\infty \times [0,1) \cup B \times \{1\} \to \mathbb{R}^\infty$ such that $u_1 = p_k|B$ and, for $t < 1$,

$$u_t \in H^a(\mathbb{R}^\infty), \ \ \|u_t - \mathrm{id}\,\| < \delta, \ \text{ and } \ u_t|A = \mathrm{id}_A\,.$$

Having such a u, it is easy to see that, for t suitably close to 1, $t < 1$, the map $h = h' \circ u_t$ may serve as a required homeomorphism that satisfies (i)–(iii).

To construct u, let first $\lambda : \mathbb{R}^k \to [0,1]$ be a continuous function with $\lambda|p_k(B) = 0$ and $\lambda|p_k(A) = 1$. Pick any homotopy $(\alpha_t)_{0\le t<1}$ of $[0,1]$ such that $0 < \alpha_t \le 1$ and $\alpha_t(1) = 1$ for all t, $\alpha_0 \equiv 1$, and $\lim_{t\to 1} \alpha_t(s) = 0$ for each $s < 1$. Finally, for $t < 1$, define

$$u_t(x) = (p_k(x), \alpha_t(\lambda(p_k(x)))p^k(x)). \qquad \square$$

For a set $A \subset \mathbb{R}^\infty$ and a map $f : A \to \mathbb{R}^\infty$, we write $f \in H^a(A)$ and say that f is **locally almost cylindrical** if, every point $x \in A$ admits a neighborhood U in A such that $f|U = h|U$ for some $h \in H^a(\mathbb{R}^\infty)$. If A is locally compact then $f \in H^a(A)$ if, for every compactum $K \subseteq A$, $f|K = h|K$ for some $h \in H^a(\mathbb{R}^\infty)$.

Lemma 4. *Given a map $f : K \to \mathbb{R}^\infty$, where $K \subset \mathbb{R}^\infty$ is a compactum, and a continuous function $\varepsilon : K \to [0,1]$, there exists a map $g : K \to \mathbb{R}^\infty$ such that*

(i) $\|g(x) - f(x)\| \le \varepsilon(x)$, $x \in K$;

(ii) $g|A \in H^a(A)$, *where* $A = K \setminus \varepsilon^{-1}(0)$.

Proof. We only need to slightly modify a procedure from the proof of [DM1, Theorem] (see [Ba]). Choose the tower $A_n = \{x \in K|\ \varepsilon(x) \ge 2^{-n}\}$; we have $\bigcup_{n=1}^\infty A_n = A$. Construct a sequence $\{g_n\} \subset H^a(\mathbb{R}^\infty)$ such that

$$\|g_n - g_{n-1}\| < 2^{-n},\ g_n|A_{n-1} = g_{n-1}|A_{n-1},$$

and

$$\|g_n(x) - f(x)\| < 2^{-n-2} \text{ for } x \in K \setminus \operatorname{int}(A_n).$$

The last condition (which in [DM1] was required only if $\varepsilon(x) = 0$) guarantees that, for $\bar f(x) = \lim g_n(x)$, $x \in K$, we have that

$$\|\bar f(x) - f(x)\| = \|g_n(x) - f(x)\| \le 2^{-(n-1)-2} = 2^{-n-1} < \varepsilon(x)$$

for $x \in A_n \setminus A_{n-1}$, and obviously that $\bar f(x) = f(x)$ when $\varepsilon(x) = 0$.

To construct the sequence $\{g_n\}$, let $g_1 \in H^a(\mathbb{R}^\infty)$ be such that $\|g_1|K - f\| < \frac{1}{8}$. Having found $g_1, \ldots, g_{n-1}$, apply Lemma 3 to the quadruple $[A = g_{n-1}(A_{n-1}),\ B = g_{n-1}(K \setminus \operatorname{int}(A_n)),\ f \circ g_{n-1}^{-1} : B \to \mathbb{R}^\infty,\ \varepsilon = 2^{-n-2}]$. Since, for $y = g_{n-1}(x) \in B$, we have $\|f \circ g_{n-1}^{-1}(y) - y\| \le \|f(x) - g_{n-1}(x)\| \le 2^{-n-1}$, by Lemma 3, there exists $h \in H^a(\mathbb{R}^\infty)$ such that $h|A = \mathrm{id}_A$, $\|h - \mathrm{id}\,\| < \|f \circ g_{n-1}^{-1}|B - \mathrm{id}_B\,\| + 2^{-n-2} < 2^{-n-1} + 2^{-n-2} < 2^{-n}$, and $\|h|B - f \circ g_{n-1}^{-1}|B\| < 2^{-n-2}$. We let $g_n = h \circ g_{n-1} \in H^a(\mathbb{R}^\infty)$. □

Lemma 5. *Let C be an almost convex body in $\mathbb{R}^\infty$. For a set $X \subseteq \bar{C}$ with $\ker(X) = C$, let $K \subseteq X$ be compact and B be a closed subset of K. Let $f : K \to X$ be a map such that $f|B$ is injective and $f(K \setminus B) \cap f(B) = \emptyset$, and let $\varepsilon > 0$. Then there exists an embedding $\bar{f} : K \to X$ such that*

(i) $\bar{f}|B = f|B$;

(ii) $\|\bar{f} - f\| < \varepsilon$;

(iii) $\bar{f}((K \cap C) \setminus B) \subseteq \operatorname{rint}(C)$;

(iv) $\bar{f}(K \setminus B) \subseteq \operatorname{rint}(X)$;

(v) $\bar{f}|K \setminus B \in H^a(K \setminus B)$; *in particular,* $\bar{f}^{-1}(E) \setminus B = (K \cap E) \setminus B$ *for every linear space E such that $\mathbb{R}^\infty_f \subseteq E \subseteq \mathbb{R}^\infty$.*

Proof. The proof follows the arguments of [Ba, Lemma 7 and 8]. Namely, a sequence of compacta

$$\emptyset = K_{-2} = K_{-1} = K_0 \subset K_1 \subset K_2 \subset \cdots,\ \ K_n \subset \operatorname{int}(K_{n+1}),\ \text{ and } \bigcup_{n=-2}^\infty K_n = K \setminus B,$$

a sequence of natural numbers $k_0 < k_1 < k_2 \cdots$, and a sequence of embeddings $\{g_n : K \to \mathbb{R}^\infty\}_{n=0}^\infty$ with $g_n|K \setminus B \in H^a(K \setminus B)$ and $g_0|B = f|B$ are constructed so that

(1_n) $g_n(K_n) \subset \mathrm{rint}(X)$ and $g_n(K_n \cap C) \subset \mathrm{rint}(C)$;

(2_n) $g_n(x) = g_{n-1}(x)$ for $x \in K_{n-2} \cup (K \setminus \mathrm{int}(K_{n+1}))$;

(3_n) $g_n(x) \in [p_{k_{n+1}}(g_{n-1}(x)), g_{n-1}(x)]$ that is $\pi_i(g_n(x)) = \pi_i(g_{n-1}(x))$ for all $i \leq k_{n+1}$ and $\pi_i(g_n(x)) \in [0,1]\pi_i(g_{n-1}(x))$ for $i > k_{n+1}$.

Having done this, let $\bar{f}(x) = \lim g_n(x)$, $x \in K$. Note that conditions (i)-(v), except (ii), follow (for the "particular" part of (v) apply the statement of Remark 1); (ii) will be checked below.

The inductive procedure of finding the sequence $\{g_n\}$ starts with the construction of g_0 that will determine sequences (K_n) and (k_n). We will additionally require that g_0 satisfies

(4) $g_0|B = f|B$;

(5) $\|g_0 - f\| < \varepsilon/2$;

(6_n) $p_{k_n} \circ g_0(K_n \setminus K_{n-3}) \subset \mathrm{int}_{\mathbb{R}^{k_n}}(C \cap \mathbb{R}^{k_n}) = \mathrm{rint}(C \cap \mathbb{R}^{k_n})$;

(7_n) $p_{k_n} \circ g_0(K_n) \cap p_{k_n} \circ g_0(K \setminus \mathrm{int}(K_{n+1})) = \emptyset$.

In order to fulfill (6_n) and (7_n), we first replace f by (a close to f) map f' that satisfies these conditions for some sequence (K_n), and next we find g_0 by an application of Lemma 4. We use the fact that the complement of $\mathrm{rint}(C) \cap (\mathbb{R}^\infty_f \setminus f(B))$ is a locally homotopy negligible set in $X \setminus f(B)$ to find a map f' such that $f'(K \setminus B) \subset \mathrm{rint}(C) \cap \mathbb{R}^\infty_f$, $f'|B = f|B$ and $f'(K \setminus B) \cap f'(B) = \emptyset$. Moreover, we can easily additionally achieve that, locally, $f'|K \setminus B$ takes values in a certain $\mathbb{R}^n \subset \mathbb{R}^\infty$, that is,

$$(*) \qquad \forall\ (x \in K \setminus B)\ \exists\ (n \text{ and a neighborhood } U_x \text{ of } x)\ [f'(U_x) \subset \mathrm{int}_{\mathbb{R}^n}(C \cap \mathbb{R}^n)].$$

Agree that the original f has the above properties of f'. Define inductively (K_n) and (k_n) as follows. Let $K_0 = \emptyset$. Choose a number $k_0 \in \mathbb{N}$ with $2^{-k_0} < \varepsilon/2$ (this choice yields (ii) by an application of (5) because $p_{k_2}(\bar{f}(x)) = p_{k_2}(g_0(x))$ implies

$\|\bar{f}(x) - g(x)\| \leq 2^{-k_2} < 2^{-k_0} < \varepsilon/2)$. Further, let $K_1 = \{x \in K \mid d(x, B) \geq 1\}$. By the compactness of K_1 and by $(*)$, conclude that $f(K_1) \subset \mathrm{int}_{\mathbb{R}^n}(C \cap \mathbb{R}^n)$ for some $n \geq k_0$. Using $f(K_1) \cap f(B) = \emptyset$, note that $p_{k_1}(f(B)) \cap p_{k_1}(f(K_1)) = \emptyset$ for some $k_1 > n$. Enlarge B to an open set $U \subset K$ to have $p_{k_1}(f(U)) \cap p_{k_1}(f(K_1)) = \emptyset$, and let $K_2 = (K \setminus U) \cup \{x \in K \mid d(x, B) \geq \frac{1}{2}\}$. It follows that, for such $k_1 > k_0$,

$$f(K_1) \subset \mathrm{int}_{\mathbb{R}^{k_1}}(C \cap \mathbb{R}^{k_1}) \text{ and } p_{k_1}(f(K_1)) \cap p_{k_1}(f(K \setminus \mathrm{int}(K_2))) = \emptyset.$$

Continue this process inductively and observe that conditions (6_n) and(7_n) hold with g_0 replaced by f.

Let $\varepsilon_n = \frac{1}{3}\min(d(p_{k_n}(f(K_n)), p_{k_n}(f(K \setminus \mathrm{int}(K_{n+1})))), d(f(K_{n+2}), \mathbb{R}^{k_{n+2}} \setminus C))$ and $\varepsilon_0 = \varepsilon/2$. Pick a continuous function $\varepsilon : K \to [0,1]$ with $\varepsilon^{-1}(0) = B$ and $\varepsilon(x) < \min(\varepsilon_0, \ldots, \varepsilon_{n+2})$ for $x \in K_n \setminus K_{n-1}$. Lemma 4 provides a map $g_0 : K \to \mathbb{R}^\infty$, $g_0|K \setminus B \in H^a(K \setminus B)$ such that $\|g_0(x) - f(x)\| \leq \varepsilon(x) < \varepsilon/2$, $x \in K$. It is evident that g_0 satisfies (4) and (5). For $x \in K_n \setminus K_{n-3}$, $f(x) \in \mathrm{int}_{\mathbb{R}^{n_k}}(C \cap \mathbb{R}^{n_k})$; hence $p_{n_k}(f(x)) = f(x)$ and

$$\begin{aligned} \|p_{n_k}(g_0(x)) - f(x)\| &= \|p_{k_n}(g_0(x)) - p_{k_n}(f(x))\| \\ &\leq \|g_0(x) - f(x)\| \\ &\leq \varepsilon(x) < \varepsilon_{n-2}. \end{aligned}$$

Since $\varepsilon_{n-2} \leq \frac{1}{3}d(f(K_n), \mathbb{R}^{k_n} \setminus C)$, we conclude that $p_{k_n}(g_0(x)) \in \mathrm{int}_{\mathbb{R}^{k_n}}(C \cap \mathbb{R}^{k_n})$, showing (6_n). To verify (7_n), fix $y \in K \setminus \mathrm{int}(K_{n+1})$ and let $x \in K_m \setminus K_{m-1} \subset K_n$ for some $1 \leq m \leq n$. Then, we have $3\varepsilon_m \leq \|p_{k_m}(f(x)) - p_{k_m}(f(y))\|$, and $\varepsilon(x), \varepsilon(y) < \varepsilon_m$. Therefore,

$$\|p_{k_n}(g_0(x)) - p_{k_n}(g_0(y)\| \geq \|p_{k_m}(g_0(x)) - p_{k_m}(g_0(y))\| \geq$$

$$\begin{aligned} \|p_{k_m}(f(x)) - p_{k_m}(f(y))\| &- \|p_{k_m}(f(x)) - p_{k_m}(g_0(x))\| \\ &- \|p_{k_m}(f(y)) - p_{k_m}(g_0(y))\| \geq \end{aligned}$$

$$3\varepsilon_m - \|f(x) - g_0(x)\| - \|f(y) - g_0(y)\| \geq 3\varepsilon_m - \varepsilon(x) - \varepsilon(y) \geq \varepsilon_m > 0.$$

By (6_n) and (7_n), $g_0(B) \cap g_0(K \setminus B) = \emptyset$; this together with the fact that, $g_0|K \setminus B \in H^a(K \setminus B)$, yields that g_0 is an embedding. This finishes the construction of g_0.

Suppose that embeddings $g_0, \ldots, g_{n-1}$ have been constructed so that (1_n)–(3_n), $n \geq 1$, are satisfied. Suppose additionally that

(8_{n-1}) $p_{k_{n+1}} \circ g_{n-1}(K_n \setminus K_{n-2}) \in \operatorname{int}_{\mathbb{R}^{k_{n+1}}}(C \cap \mathbb{R}^{k_{n+1}})$

(9_{n-1}) $p_{k_{n+1}} \circ g_{n-1}(K_n \setminus \operatorname{int}(K_{n-1})) \cap p_{k_{n+1}} \circ g_{n-1}(K_{n-2} \cup (K \setminus \operatorname{int}(K_{n+1}))) = \emptyset$;

are satisfied. (Notice that (8_0) follows from (6_0), for $K_1 \setminus K_{-1} \subset K_2 \setminus K_{-1}$, and that (9_0) is a direct consequence of (7_0), for if the sets in (9_0) have a point in common then the sets in (7_0) do as well).

We are ready to define a required embedding $g_n : K \to \mathbb{R}^\infty$. Since $g_{n-1}|K \setminus B \in H^a(K \setminus B)$, there exists $h \in H^a(\mathbb{R}^\infty)$ such that $g_{n-1}|K_n = h|K_n$; hence, by Lemma 1, for some $\beta \geq 1$,

$$g_{n-1}(K_n) \subseteq h(K_n) \subseteq \beta X \quad \text{and} \quad g_{n-1}(K_n \cap C) \subseteq \beta C.$$

By (8_{n-1}) there exists $0 < \delta < 1$ with $p_{k_{n+1}} \circ g_{n-1}(K_n \setminus \operatorname{int}(K_{n-1})) \subseteq (1 - \frac{\delta}{2})C$. Employing (9_{n-1}), pick a continuous function $\alpha : \mathbb{R}^{k_{n+1}} \to [\frac{\delta}{2\beta}, 1]$ such that $\alpha(x) = 1$ for $x \in p_{k_{n+1}} \circ g_{n-1}(K_{n-2} \cup (K \setminus \operatorname{int}(K_{n+1})))$ and $\alpha(x) = \frac{\delta}{2\beta}$ for $x \in p_{k_{n-1}} \circ g_{n-1}(K_n \setminus \operatorname{int}(K_{n-1}))$, set

$$\begin{aligned} h(x) &= p_{k_{n+1}}(x) + \alpha(p_{k_{n+1}}(x)) p^{k_{n+1}}(x) \\ &= \alpha(p_{k_{n+1}}(x))x + (1 - \alpha(p_{k_{n+1}}(x))) p_{k_{n+1}}(x), \end{aligned}$$

and let $g_n = h \circ g_{n-1}$. It is clear that g_n is an embedding such that $g_n|K \setminus B \in H^a(K \setminus B)$. Evidently, (2_n) and (3_n) are satisfied. To verify the first assertion of (1_n), let $x \in K_n$; if $x \in K_{n-2}$ then the assertion follows from (2_n) and (1_{n-1});

if $x \in K_n \setminus K_{n-1}$ then $g_{n-1}(x) \in \mathrm{rint}(X)$ by (1_{n-1}), $p_{k_{n+1}}(g_{n-1}(x)) \in \mathrm{int}_{\mathbb{R}^{k_{n+1}}}(C \cap \mathbb{R}^{k_{n+1}})$ by (8_{n-1}), and hence $g_n(x) = \frac{\delta}{2\beta} g_{n-1}(x) + (1 - \frac{\delta}{2\beta}) p_{k_{n+1}}(g_{n-1}(x)) \in \frac{\delta}{2} X + (1-\delta)C \in \mathrm{rint}(X)$. A similar argument works for the proof of the second part of the assertion.

To verify (8_n), we first note that $g_n(x) \in [p_{k_{n+1}}(g_{n-1}(x)), g_{n-1}(x)]$ and $g_{n-1}(x) \in [p_{k_n}(g_{n-2}(x)), g_{n-2}(x)]$ by (3_n) and (3_{n-1}), respectively. Hence, we have

$$p_{k_{n+2}}(g_n(x)) \in [p_{k_{n+1}}(g_{n-1}(x)), p_{k_{n+2}}(g_{n-1}(x))] = [p_{k_{n+1}}(g_0(x)), p_{k_{n+2}}(g_0(x))] \text{ and}$$
$$p_{k_{n+2}}(g_{n-1}(x)) \in [p_{k_n}(g_{n-2}(x)), p_{k_{n+2}}(g_{n-2}(x))] = [p_{k_n}(g_0(x)), p_{k_{n+2}}(g_0(x))]$$

(as well as $p_{k_{n+1}}(g_{n-1}(x)) \in [p_{k_n}(g_0(x)), p_{k_{n+1}}(g_0(x))]$) for $x \in K \setminus K_n$ and $x \in K \setminus K_{n-1}$, respectively, because, in these cases, $g_{n-1}(x) = g_0(x)$ and $g_{n-2}(x) = g_0(x)$ by (2_{n-1})–(2_1). It follows that $p_{k_{n+2}}(g_n(x)) \in [p_{k_{n+1}}(g_0(x)), p_{k_{n+2}}(g_0(x))]$ and $p_{k_{n+2}}(g_0(x)) \in \mathrm{conv}(p_{k_n}(g_0(x)), p_{k_{n+1}}(g_0(x)), p_{k_{n+2}}(g_0(x)))$ for $x \in K \setminus K_n$ and $x \in K \setminus K_{n-1}$, respectively. Now, pick $x \in K_{n+1} \setminus K_{n-1}$; if $x \in K_{n+1} \setminus K_n \subset K \setminus K_n$ [resp., if $x \in K_n \setminus K_{n-1} \subset K \setminus K_{n-1}$] then an application of (6_n) and (6_{n-1}) [resp., (6_i) for $i = n, n+1, n+2$] yields (8_n) (use the fact that $K_n \setminus K_{n-1}$ is contained in each of $K_n \setminus K_{n-3}$, $K_{n+1} \setminus K_{n-2}$, and $K_{n+2} \setminus K_{n-1}$).

To show (9_n), notice that for $x \in K_{n+1} \setminus \mathrm{int}(K_n)$ we have $p_{k_{n+1}}(g_n(x)) = p_{k_{n+1}}(g_0(x))$ by an application of (3_n) and (2_{n-1})–(2_1); also, if $y \in K_m \setminus K_{m-1} \subset K_n$, $1 \le m \le n-1$, we have $p_{k_m}(g_n(y)) = p_{k_m}(g_{m-1}(y)) = p_{k_m}(g_0(y))$, and if $y \in K \setminus \mathrm{int}(K_{n+2})$ then $p_{k_{n+1}}(g_n(y)) = p_{k_{n+1}}(g_0(y))$. Now, if (9_n) were not true, i.e., if $p_{k_{n+2}}(g_n(x)) = p_{k_{n+2}}(g_n(y))$ for some $x \in K_{n+1} \setminus \mathrm{int}(K_n)$ and $y \in K_n \cup (K \setminus \mathrm{int}(K_{n+2}))$, then either $p_{k_m}(g_0(y)) = p_{k_m}(g_0(x))$ (if $y \in K_m \setminus K_{m-1}$) or $p_{k_{n+1}}(g_0(x)) = p_{k_{n+1}}(g_0(x))$ (if $y \in K \setminus \mathrm{int}(K_{n+2})$), which contradicts either (7_m) or (7_{n+1}), respectively. □

2. Strong Universality for Sets with Almost Internal Points

Recall that, for a convex set C, $0 \in C$ is **almost internal** (see [BP, p.160]) if the set $A = \{a \in C | \ \exists(\varepsilon > 0)\ [-\varepsilon a \in C]\}$ is dense in C.

Lemma 6. *Let C be an infinite-dimensional relatively compact convex subset of a Fréchet space having $0 \in C$ as an almost internal point, and let X, $X \subseteq \bar{C}$, be such that* $\ker(X) = C$. *For every compact set $K \subseteq X$ with $K \cap \operatorname{span}(C) = K \cap C$, every $\bar{B} = B \subseteq X$, every map $f : K \to X$ with $f(K \setminus B) \cap f(B) = \emptyset$, $f|B$ is injective, and $f^{-1}(C) \cap B = B \cap C$, and every $\varepsilon > 0$ there exists an embedding $\bar{f} : K \to X$ satisfying*

(i) $\bar{f}|B = f|B$;

(ii) $d(\bar{f}, f) < \varepsilon$;

(iii) $\bar{f}(K \setminus B) \subset \operatorname{rint}(X)$;

(iv) $\bar{f}^{-1}(C) = K \cap C$.

Proof. By our assumption, $\bar{C}$ is compact, and there exists a countable set A with the property that, for every $a \in A$, $-\varepsilon a \in C$ for some $\varepsilon > 0$. In what follows we adopt an argument of [Ba, Lemma 9] that is a variation of a reasoning from [BP, p. 160]. There exists a biorthogonal sequence $\{(x_n, x_n^*)\}_{n=1}^{\infty}$ (i.e., $x_m^*(x_n) = \delta_n^m$) such that $\operatorname{span}\{x_n | \ n \geq 1\} = \operatorname{span}(A)$ and $\{x_n^*\}$ is total when restricted to the Fréchet space $E = \overline{\operatorname{span}}(A)$. It follows that $T(x) = (x_m^*(x))$, $x \in E$, is a continuous linear operator into $\mathbb{R}^\infty$ whose restriction to $\bar{C}$ yields an affine embedding onto $T(\bar{C})$. Moreover, $T(C)$ is an almost convex body relative $\mathbb{R}^\infty_f$ (this is because $T(A)$, a subset of $T(C \cap \operatorname{span}(A)) = T(C) \cap \mathbb{R}^\infty_f$, is dense in $T(C)$). Clearly, $\ker(T(X)) = T(C)$.

Hence, we can identify $T(X)$ and $T(C)$ with X and C, respectively, and apply Lemma 5 to obtain a required approximation $\bar{f}$. To show (iv) use Lemma 5(iv)-(v) (if $\bar{f}(x) \in C$ for some $x \in K \setminus B$ then $\bar{f}(x) \in \operatorname{span}(C)$, hence $x \in K \cap \operatorname{span}(C) = K \cap C$). □

Lemma 7. *Let C be an infinite-dimensional convex subset of a Fréchet space having $0 \in C$ as an almost internal point, and let X, $X \subseteq \bar{C}$ be such that* $\ker(X) = C$. *For every compact set $L \subseteq X$ there exists a relatively compact infinite-dimensional convex set $D \subset C$ such that*

(i) 0 *is an almost internal point of D;*

(ii) $D = \bar{D} \cap C$ *is dense in* $\bar{D} \cap X = X'$;

(iii) $L \subseteq X' \subseteq \ker(X') = D$;

(iv) $K \cap D = K \cap C$ *for every compact set $K \subseteq L$;*

(v) $\mathrm{span}(D) \subseteq \mathrm{span}(C)$ *and, consequently, for $K \subseteq L$ with* $K \cap \mathrm{span}(C) = K \cap C$, *we have* $K \cap \mathrm{span}(D) = K \cap D$.

Proof. Take a sequence $A = \{a_n\} \subset C$ so that $L \subseteq \bar{A}$, for every n there exists $\varepsilon_n > 0$, $\varepsilon_n < 2^{-n}$, such that $-\varepsilon_n a_n \in C$ and $\bar{A}$ is infinite-dimensional and compact. Then $D = \overline{\mathrm{conv}}(A \cup \{-\varepsilon_n a_n |\ n \geq 1\}) \cap C$ satisfies (i)–(v). □

Lemma 8. *Let C be an infinite-dimensional convex subset of a Fréchet space having $0 \in C$ as an almost internal point and let X be such that $C \subseteq X \subseteq \bar{C}$. Every compact set* $K \subseteq \mathrm{rint}(\bar{C}) \cap X$ *is a Z-set in X.*

Proof. By the fact that the complement of X is locally homotopy negligible in $\bar{C}$ and by properties of Z-sets, we can assume that $C = X = \bar{C}$ (we also can assume that C is separable).

If C is compact, then using the fact that 0 is an almost internal point of C, by [BP, **?**], we conclude that K is a Z-set. A similar argument works also in the case where C is locally compact. If C is not locally compact then every compact subset of C is a Z-set. □

Recall that a space X is **strongly K-universal** if for every closed set $B \subseteq K$ and every map $f : K \to X$ such that $f|B$ is a Z-embedding can be approximated by Z-embeddings $\bar{f}$ with $\bar{f}|B = f|B$. This approximation is uniform if K is compact, and uniform with respect to every bounded metric on X, in general. This notion has a relative counterpart, the **strong** (K, K')**-universality** of a pair (X, X'). Namely, assuming additionally that $f^{-1}(X') \cap B = B \cap K'$, we require that $\bar{f}^{-1}(X') = K'$. In the same manner, the **strong** (K, K', K'')**-universality** of (X, X', X'') is defined (when we write (A, A') or (A, A', A'') we always mean that $A'' \subseteq A' \subseteq A$).

Theorem. *Let X be a subset of a Fréchet space such that* $\ker(X) = C$ *is infinite-dimensional having* 0 *as an almost internal point and $X \subseteq \bar{C}$. Then, for every compactum $K \subseteq X$,*

(i) X is K-universal;

(ii) if $K \cap C = K \cap \operatorname{span}(C)$, then (X, C) is $(K, K \cap C)$-strongly universal.

Proof. Let us justify (ii), the proof of (i) is similar.

Pick a map $f : K \to X$ that restricts to some compactum $B \subseteq K$ to a Z-embedding with $f^{-1}(C) \cap B = B \cap C$, and let $\varepsilon > 0$. Employing the fact that C is infinite-dimensional, we can additionally assume that $f(K \setminus B) \cap f(B) = \emptyset$. Set $L = K \cup f(K)$. Apply Lemma 7 to find a relatively compact convex infinite-dimensional set D with properties (i)-(v). Let $X' = \bar{D} \cap X$. For the triple $(f,\ D,\ X')$ obtain the embedding $\bar{f} : K \to X'$ satisfying (i)-(iv) of Lemma 6.

By (iv) of Lemma 7, it easily follows that $\bar{f}(K \cap C) \subset C$. If for $x \in K$ we have $\bar{f}(x) \in C \cap X' \subseteq C \cap \bar{D}$, then $x \in D \subseteq C$.

Since $\bar{f}(K \setminus B) \subset \operatorname{rint}(X') \subset \operatorname{rint}(\bar{D}) \cap X \subset \operatorname{rint}(\bar{C}) \cap X$, by Lemma 8, we infer that every compact subset of $\bar{f}(K \setminus B)$ is a Z-set in X. This together with the fact that $\bar{f}(B)$ is a Z-set in X, gives that $\bar{f}(K)$ is a Z-set in X. □

Remark 2. The argument of Theorem actually shows that when X and C satisfy the above hypothesis then, for every compactum $K \subseteq \bar{C}$ such that $K \cap C = K \cap \operatorname{span}(C)$ and $K \cap X = K \cap \operatorname{span}(X)$, the triple $(\bar{C}, X, C)$ is strongly $(K, K \cap X, K \cap C)$-universal. Using the results [BRZ, 1.7.9 and 5.2.6] (see [BC2]), we obtain the strong $(K \cap X, K \cap C)$ universality of (X, C). In the same manner, we can obtain the strong $(K, K \cap X)$-universality of $(\bar{C}, X)$ and the strong $K \cap X$ universality of X for a compactum $K \subseteq \bar{C}$ such that $K \cap X = K \cap \operatorname{span}(X)$.

Below we provide an example showing the strong universality does not hold, in general, for the class of all closed subsets of a convex set. However, we do not know whether the hypothesis concerning the existence of the almost internal point of C is essential in Theorem and Remark 2.

Example 1. Let B be the unit open ball of ℓ^2, and let S be the unit sphere, the boundary of B. Let A be an incomplete subset of S. Then $C = B \cup A$ is a convex set that is not strongly universal for A (which is a closed subset of C). Simply, the constant map of A with value $0 \in B$ cannot be approximated by closed embeddings. (Letting, e.g., $A = \mathbb{Q}$ be the space of rationals, we see that C could even be obtained as a countable union of closed completely metrizable sets.) This contrasts with the results of [BRZ, 5.3.2 and 5.3.5], where A is assumed to be a **closed** subset of $\operatorname{span}(A)$. Notice also that the strong universality of X discussed in Remark 2 is relative to the class of sets that are closed in $\operatorname{span}(X)$; our example shows that the hypothesis 'of being closed' is essential. The space C does not have the Z_σ-property. To obtain an example with this extra property let $C' = (A' \times \{0\}) \cup (B \times B') \subset \ell^2 \times \ell^2$, where $A' \subset S \times \{0\}$, $B' = B \cap \ell^2_f$, and ℓ^2_f is the linear subspace of ℓ^2 consisting of finite sequences. It is clear that C' is a convex Z_σ-space. It is clear that if A is a space that cannot be represented as a countable union of closed completely metrizable sets (e.g., $A = \mathbb{Q}^\infty$), then the constant map of A with value $0 \in C'$ cannot be approximated by closed embeddings.

As indicated previously, the group of homeomorphisms $H^a(\mathbb{R}^\infty)$ (and its counterpart $H^a(E)$ for an arbitrary Fréchet space E) is useful in constructing homeomorphisms with some additional properties (such like diffeomorphisms). Any set X satisfying the assumptions of Theorem has the so-called compact homeomorphism property, i.e., every homeomorphism $h : K \to L$ between compacta K and L in X can be extended to a selfhomeomorphism $\bar{h}$ of X. For the reason stated above, it would be interesting to know whether $\bar{h}$ can be taken so that $\bar{h}|X \setminus K \in H^a(X \setminus K)$ and $\bar{h}^{-1}|X \setminus L \in H^a(X \setminus L)$. This is the case if X is a linear space, see [Do].

3. Applications to an Embedding Problem and Certain Sigma-compact Absorbers

Certain convex subsets C of Fréchet spaces are homeomorphic to convex subsets of ℓ_2. This is known when C is a linear space that is either σ-compact [BD] or, more generally, is contained in a σ-compact space [BC1]. The convex case, which we follow below, was treated in [Ba].

Proposition 1. *Assume that X is an F_σ subset of a linear space L, a subspace of a Fréchet space E, and assume that X is contained in a σ-compact subset of E. If* $\ker(X) = C$ *is infinite-dimensional having* 0 *as an almost internal point and $X \subset \bar{C}$, then X is homeomorphic to a subset X' of ℓ_2 with* $\ker(X')$ *dense in $\bar{X}'$. If, moreover, C is an F_σ subset of* $\operatorname{span}(C)$ *then the pairs* (X, C) *and* $(X', \ker(X'))$ *are homeomorphic.*

Proof. We we will only take on the absolute case. The case when $\bar{C}$, the closure of C in E, is locally compact is a consequence of the fact that, according to [BD], there exists an affine map of $\bar{C}$ into ℓ_2.

Assume that $\bar{C}$ is nonlocally compact. As in [BD], we will show that there exists an injective affine (not necessary continuous) transformation T of X (actually, a σ-compact linear subspace that contains X) into ℓ_2 such that X and $T(X)$ are so-called **$\mathcal{C}$-absorbers** for a certain class $\mathcal{C}$. By the Uniqueness Theorem on absorbers, X and $T(X) = X'$ are then homeomorphic. To check that X, X' are $\mathcal{C}$-absorbers we must identify the class $\mathcal{C}$ and verify the following conditions:

(1) X and X' are Z_σ-spaces;

(2) X and X' are countable unions of elements of $\mathcal{C}$;

(3) X and X' have the strong $\mathcal{C}$-universality property.

We may assume that $L = \operatorname{span}(X)$. Find a representation $X = \bigcup_{n=1}^{\infty} X_n$ such that X_n is closed in L and $X_n \subseteq A_n$ for some compactum $A_n \subseteq \bar{C}$; we may assume that both $\{X_n\}$ and $\{A_n\}$ are increasing sequences. Define $\mathcal{C}$ to be the class of homeomorphs of elements of $\{X_n\}$. Now, for X, condition (2) is a triviality, and (3) is obtained by an application of Remark 2 to $K = \bar{X}_n$. Condition (1) for X follows from the fact that compacta are Z-sets in nonlocally compact $\bar{C}$, and the complement of X is locally homotopy negligible in $\bar{C}$.

If we defined T to be continuous on each A_n then obviously $X' = T(X)$ would satisfy (2). In order to achieve (1) and (3), we must arrange that T is continuous on some compact enlargements of each A_n. Take a sequence $D = \{d_n\} \subset C$ such that for every n there exists $\varepsilon_n > 0$ with $-\varepsilon_n d_n \in C$. For each n, we can find a set $D_n \subseteq D$ such that $\tilde{A}_n$, the closure of D_n, is compact and $A_n \subseteq \tilde{A}_n$. If only T is continuous on each $\tilde{A}_n$ then $T(D)$ is dense in $T(X) = X'$; hence, 0 would be an almost internal point, and Remark 2 yields (3). The following construction of the enlargements is that of [Ba]. By the fact that X_n is a Z-set in X, for each n there exists a homotopy $\alpha_n : \operatorname{conv}\{d_1, \dots, d_n\} \times [0,1] \to C$ such that $\alpha_n(\cdot, 0) = \mathrm{id}$ and $\operatorname{im}(\alpha_n(\cdot, t)) \cap X_n = \emptyset$ for all $t > 0$. Write $B_n = \operatorname{im}(\alpha_n)$. Letting $\{x_n^*\}$ to be a sequence of continuous linear functionals that separate the points of $\bar{L}$ be such that $|x_n^*(x)| < 2^{-n}$ for all $x \in \tilde{A}_n \cup B_n$, we see that $T(x) = (x_n^*(x))$ ia a linear transformation that continuously maps each $\tilde{A}_n \cup B_n$ into ℓ_2. Since $T(\bar{X}_n)$ is compact and $T(X_n) = T(X) \cap T(\bar{X}_n)$, $T(X_n)$ is closed in X'. To show that $T(X_n)$ is a Z-set in X' for a given $\varepsilon > 0$, use the density of $T(D)$ in X', to ε-approximate a map $\varphi : Q \to X'$ by a map $\varphi' : Q \to \operatorname{conv}\{T(d_1), \dots, T(d_m)\}$ for some $m \geq n$.

Let $\varphi_t(q) = T(\alpha_m(T^{-1}(\varphi'(q), t))$. By the continuity of T on B_m, there exists $t_0 > 0$ such that φ_{t_0} is ε-close to φ' and obviously $\mathrm{im}(\varphi_{t_0})$ misses $T(X_m) \supset T(X_n)$. □

Remark 3. It is not difficult to make adjustments in the above proof to show that, under respective assumptions of Proposition 1, there exist σ-absorbers Y and Y' with the following inclusions $X \subseteq Y \subset \bar{C}$ and $X' \subseteq Y' \subseteq \bar{X}' \subset \ell_2$ such that the pairs (Y, X) and (Y', X') are homeomorphic [resp., the triples (Y, X, C) and $(Y', X', \ker(X'))$ are homeomorphic].

Consider the infinite-dimensional compact convex ellipsoid $M = \{(x_i) \in \ell_2 |\ \sum_1^\infty i^2 x_i^2 \leq 1\}$. Any separable metric space Z embeds onto a linearly independent subset of the pseudoboundary $S = \{(x_i) \in M |\ \sum_1^\infty i^2 x_i^2 = 1\}$ of M. One always can find such an embedding so that additionally there exists a countable set $D \subset S \setminus Z$ linearly independent and dense in S. Let

$$P = \mathrm{span}(D) \cap M,$$

which can obviously be expressed as a countable union of sets homeomorphic with finite-dimensional cubes. We will consider the following counterpart of the space $\Omega(Z)$ introduced in [BC2]

$$\omega(Z) = \{tz + (1-t)p |\ z \in Z,\ p \in P,\ 0 \leq t \leq 1\} \subset M.$$

Proposition 2. *For a sigma-compact space Z, the space $\omega(Z)$ is an absorber for $\mathcal{C}$, the class of all compacta embeddable in $Z \times P$; moreover, the complement of $\omega(Z)$ is locally homotopy negligible in M.*

Proof. Express $Z = \bigcup_{n=1}^\infty Z_n$ and $P = \bigcup_{n=1}^\infty P_n$ as countable unions of compacta Z_n and P_n. For every pair of compacta $K \subset Z$ and $L \subset P$, write $\omega(K, L) = \{tz + (1-t)p |\ z \in K,\ p \in L,\ 0 \leq t \leq 1\}$. Since the map $(z, p, t) \to tz + (1-t)p$ is injective, $\omega(K, L)$ is homeomorphic to $K \times L \times [0, 1]$; hence, $\omega(K, L)$ is a compactum embeddable in $Z \times P$. As $\omega(Z) = \bigcup_{n=1}^{\infty} \omega(Z_n, P_n)$, we have that $\omega(Z)$ is countable union of elements of $\mathcal{C}$.

Since the complement of $P \setminus S$ in $M \setminus S$ is locally homotopy negligible, $\omega(Z)$ is a Z_σ-space. Note also that the complement of $\omega(Z)$ is locally homotopy negligible in M. The strong universality of $\omega(Z)$ for the class $\mathcal{C}$ follows from Theorem because $P \subseteq \ker(\omega(Z)) \subseteq \bar{P} = M$, and 0 is an almost interior point of P. □

Corollary 1. *Let Z be that of Proposition 2. For every m, the product $\omega(Z)^m$ is an absorber for the class of compacta embeddable in $Z^m \times P$; moreover, the complement of $\omega(Z)^m$ is locally homotopy negligible in M^m. Hence, $\omega(Z)^m$ is homeomorphic to $\omega(Z^m)$.*

Proof. The only thing that needs justification is the strong universality property of $\omega(Z)^m$ for the class of all its compacta. This is, however, a consequence of Theorem because $P^m \subset \ker(\omega(Z)^m) \subset M^m$. □

For a set $A \subseteq Z^m$, we say that $\varphi : A \to Y$ is **fiberwise injective** if φ is injective on each set $A \cap F_i$, where $F_i = \pi_i^{-1}(p)$, $p \in Z^{m-1}$, $1 \leq i \leq m$, and $\pi_i : Z^m \to Z^{m-1}$ is the projection given by $\pi_i((z_i)) = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)$.

Lemma 9. *For every map $\Phi : \omega(Z)^m \to \omega(Z)^n$ there exist an open set $U \subseteq Z^m$ and an embedding $i : \Phi(j(U)) \to Z^n \times I^q$, where $j : Z^m \to \omega(Z)^m$ stands for canonical embedding. In particular, if there exists a fiberwise injective map (resp. an embedding) $\Phi : \omega(Z)^m \to \omega(Z)^n$, $n < m$, then there exists an open set $U \subseteq Z^m$ and a fiberwise map (resp. an embedding) $\varphi : U \to Z^n \times I^q$ for some positive integer q.*

Proof. Identify Z^m with a subspace of $\omega(Z)^m$. By Corollary 1, $\Phi(Z^m)$ is covered by countable many compact subsets of $Z^n \times P$. By the Baire Theorem, there exists an open subset U of Z^m so that $\Phi(U) \subset Z^n \times I^q$ for some q. □

In what follows, we closely mimic a reasoning of [BC2, Theorem 6.7]. Let C be the Cook's continuum [Co]. This is a hereditary indecomposable continuum such that every map of a subcontinuum A of C into C is either a constant map or the inclusion map. We let

$$K = \prod_{i=1}^{\infty} A_i,$$

where A_i are pairwise disjoint subcontinua of C. For the purpose of Corollary 3, we note that K is strongly infinite-dimensional, see [En] for terminology.

Proposition 3. *The above compactum K has the property that for no open set $U \subseteq K^m$ there exists a fiberwise injective map $\varphi : U \to K^n \times I^q$ for $n < m$ and some q.*

Proof. We check our assertion when $m = 3$ and $n = 2$ only; a similar argument works for arbitrary $n < m$. Take an open set $U \subseteq K \times K \times K$ and suppose we have a fiberwise injective map $\varphi : U \to K \times K \times I^q$ for some $q \in \mathbb{N}$, and write $\varphi(z) = ((\varphi_i^1(z)), (\varphi_i^2(z)), \varphi^3(z)) \in (\prod_{i=1}^{\infty} A_i) \times (\prod_{i=1}^{\infty} A_i) \times I^q = K \times K \times I^q$ for $z \in U$. There exists k and $x_1, x_2, x_3 \in \prod_{i=1}^{k-1} A_i$ such that $(\{x_1\} \times \prod_{i=k}^{\infty} A_i) \times (\{x_2\} \times \prod_{i=k}^{\infty} A_i) \times (\{x_3\} \times \prod_{i=k}^{\infty} A_i) \subset U$. For further considerations we can assume that $k = 1$ and that $U = (\prod_{i=1}^{\infty} A_i^1) \times (\prod_{i=1}^{\infty} A_i^2) \times (\prod_{i=1}^{\infty} A_i^3)$, where $A_i^j = A_i$. Let $\pi_i^j : U \to A_i^j$, $j = 1, 2, 3$, be an obvious projection. We claim that, for every i and $l = 1, 2$, φ_i^l is either a constant map or equals π_i^j for some $j = 1, 2, 3$. If this is true then T, the complement of the set $\{(i,j)| \ (\exists\ l)\ (\varphi_i^l = \pi_i^j)\}$, is infinite. Therefore, for some j_0, say $j_0 = 1$, the set $S = \{i| \ (i, j_0) \in T\}$ is infinite. Now, for fixed $(x_2, x_3) \in (\prod_{i=1}^{\infty} A_i^2) \times (\prod_{i=1}^{\infty} A_i^3)$, and $x \in \prod_{i \notin S} A_i^1$, we have that $((\varphi_i^1), (\varphi_i^2))$ restricted to the set $L = (\prod_{i \in S} A_i^1) \times \{(x, x_1, x_2)\}$, a subset of the fiber $K \times \{(x_1, x_2)\}$, is a constant map. By the property of φ, $\varphi^3|L : L \to I^q$ is an embedding, which is impossible because L is infinite-dimensional.

For the sake of completeness, let us repeat the argument of [BC2] showing that if φ_i^l is not a constant map it must be π_i^j for some j. If φ_i^l is not a constant map then there are $z = (z_{i,j}), z' = (z'_{i,j}) \in U$ such that $\varphi_i^l(z) \neq \varphi_i^l(z')$; moreover, we can assume that z and z' differ each from the other at one, say (i, j), coordinate only. Consider $U = A_i^j \times M$ and, for a fixed $m \in M$, define $\varphi_m : A_i^j = A_i \to A_i$ by $\varphi_m(a) = \varphi_i^l(a, m)$. By the above property of Cook's continuum and the connectedness of M we have that: either, for all $m \in M$, φ_m is a constant map, or, for all $m \in M$, $\varphi_m = \mathrm{id}_{A_i}$. However, since $z = (a, m)$ and $z' = (a', m)$ for some $a, a' \in A_i^j$ we have $\varphi_m(a) = \varphi_i^l(z) \neq \varphi_i^l(z') = \varphi_m(a')$, consequently, the first case of the above alternative does not occur. Therefore, $\varphi_i^l = \pi_i^j$ (and obviously $l = j$). □

Corollary 2. *For the compactum K of Proposition 3, there is no fiberwise injective map of $\omega(K)^m$ into $\omega(K)^n$ with $n < m$. In particular, we have that*

(i) there is no embedding of $\omega(K)^m$ into $\omega(K)^n$ for $n < m$, and hence the spaces $\omega(K)^m$ and $\omega(K)^n$ are nonhomeomorphic for $n \neq m$;

(ii) writing $\mathcal{C}_m$ for the class of compacta embeddable in $K^m \times P$, it follows that $\mathcal{C}_m$ are pairwise distinct for distinct m's; hence, $\mathcal{C}_m$ is not a multiplicative class, however, it has the property that $L \times [0, 1] \in \mathcal{C}_m$ for every $L \in \mathcal{C}$;

(iii) there is no group structure or convex structure on any $\omega(K)^m$.

Proof. To see (i) combine Lemma 9 and Proposition 3. For (ii), assume K^m is embeddable in $K^n \times P$ for some $n < m$. By the Baire Theorem, some open subset of K^m embeds into $K \times I^q$ for some q. This, however, contradicts the assertion of Proposition 3. To show (iii) notice that every space Z carrying either a group structure or a convex structure admits a fiberwise injective map $\varphi : Z \times Z \to Z$ (as noticed in [BC2], for the case of convex set, $\varphi(z, z') = \frac{1}{2}(z + z')$ will do). However, $\omega(K)^{2m}$ does not admit any fiberwise injective map into $\omega(K)^m$. □

Now, we describe yet another example of a compactum that may replace the compactum K in Corollary 2(i). For an ordinal $\alpha < \omega_1$ consider the Smirnov cube S_α (see, e.g., [En, p. 333]). A result of Reńska [Re] implicitly provides, for each α, a compactum R_α such that every open subset of R_α contains a copy of S_α and $\operatorname{trind} X_\alpha \leq \alpha + 1$, where trind (respectively, trInd) stands for the small (respectively, large) transfinite dimension; we always have $\operatorname{trind} S_\alpha \leq \alpha$ and $\operatorname{trInd} S_\alpha = \alpha$. We thank E. Pol for the following simpler example of such a compactum.

Example 2. Take a null-sequence $\{C_n\}$ of pairwise disjoint Cantor sets in the Cantor set C such that every nonempty open subset of C contains some C_n. For every n, pick a map f_n of C_n onto S_α. Let R_α be the adjoint space obtaining by attaching S_α to C via maps f_n. The compactum $R = R_\alpha$ is as required.

Proof. We will show that $\operatorname{trInd} R \leq \alpha + 1$. Write for q the quotient map. Let $\mathcal{F}_n$ be a finite clopen cover of C with diameters $< 1/n$ and such that $d(F, F') > 1/n$ for $F \neq F'$, $F, F' \in \mathcal{F}_n$. Let A and B be closed disjoint subsets of R. There exists $k \in \mathbb{N}$ such that $d(q^{-1}(A), q^{-1}(B)) > 1/k$. Let $U_A = \bigcup\{F \in \mathcal{F}_n | A \cap q^{-1}(F) \neq \emptyset\}$; in the same way define U_B. Then U_A and U_B are clopen subsets of C with $U_A \cap U_B = \emptyset$, $U_A \cup U_B = C$, $U_A \supset q^{-1}(A)$, and $U_B \supset q^{-1}(B)$. It follows that $D = q(U_A) \cap q(U_B)$ is a partition between A and B. Since $d(U_A, U_B) > 1/k$ and $\operatorname{diam} C_n \to 0$, there exists $m \in \mathbb{N}$ such that if C_n has a point in common with both U_A and U_B then $n \leq m$. We conclude that $D \subset \bigcup_{n=1}^{m} q(C_n)$; hence, A is contained in a disjoint finite union of copies of the Smirnov cube S_α. Consequently, $\operatorname{trInd} D \leq \alpha$, and therefore $\operatorname{trInd} R \leq \alpha + 1$ (and obviously $\operatorname{trind} R \leq \alpha + 1$). □

Proposition 4. *For the above compactum $R = R_{\alpha_0}$ with $\alpha_0 = \omega^\omega$, no open set of R^m admits an injective map into $R^n \times I^q$ for $n < m$ and some q.*

Proof. We only will show details for the case of $m = 2$ and $n = 1$.

Aiming at a contradiction, suppose that an open subset of $R \times R$ admits an injective map into $R \times I^q$. The property of R described above implies that $S_\alpha \times S_\alpha$ embeds into $R \times I^q$. Since $\operatorname{trind}(R \times I^q) \leq \alpha(+)1 + q$, we arrive to a contradiction if only we know that $\operatorname{trind}(S_\alpha \times S_\alpha) > \alpha(+)n$ for arbitrary n. This is, however, a consequence of Chatyrko's result [Ch] stating that $\operatorname{trind}(S_\alpha \times S_a) = \operatorname{trind}(S_{\alpha(+)\alpha})$, which yields that $\operatorname{trind}(S_{\alpha_0} \times S_{\alpha_0}) = \alpha_0(+)\alpha_0 = \omega^\omega(+)\omega^\omega$; here, we use the facts that (1) $\alpha_0(+)\alpha_0$ is a so-called invariant ordinal, that is, $\alpha_0(+)\alpha_0 = \omega^\omega \cdot \gamma$ for some γ, and (2) $\operatorname{trind} S_\beta = \beta$ for invariant ordinals [Lu]. □

Corollary 3. *The statements (i) and (ii) of Corollary 2 hold when K is replaced by the compactum R. In particular, there is no embedding of $\omega(R)^m$ and $\omega(R)^n$ are nonhomeomorphic for $n \neq m$.*

Moreover, for every n, $\omega(R)^n$ is countable-dimensional, and $\omega(K)^m$ and $\omega(R)^n$ are nonhomeomorphic for any m and n.

Proof. The first statement follows in the same way as (i) and (ii) of Corollary 2. The 'moreover' part is a consequence of the facts that R^n is countable-dimensional, and K is strongly infinite-dimenisonal, see [En]. □

We thank R. Pol and E. Pol for pointing out to us the space R and its properties described above.

There exists yet another example of a compactum that may replace K in Corollary 2(i) when $n = 1$ and $m = 2$. Such a compactum Q involving a hereditary infinite-dimensional continuum was discovered by J. Kulesza. It turns out that the spaces $\omega(K)$, $\omega(R)$, and $\omega(Q)$ are pairwise nonhomeomorphic. It is likely that there are uncountably many such examples.

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Department of Mathematics, Pittsburg State University, Pittsburg, KS 66762

E-mail address: tdobrowo@mail.pittstate.edu