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Department of Mathematics & Statistics  
Auburn University, Alabama 36849, USA  
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ON PROPERTIES OF RELATIVE  
METACOMPACTNESS AND  
PARACOMPACTNESS TYPE

Elise M. Grabner, Gary C. Grabner and Kazumi Miyazaki

**Abstract**

We study several natural relative properties of metacompactness and paracompactness types and the relationships among them. Connections to other relative topological properties are also investigated.

**Theorem** Suppose  $C$  and  $F$  are subspaces of the  $T_3$  space  $X$ . If  $C$  is strongly metacompact in  $X$  and  $F$  is strongly countably compact in  $X$  then  $C \cap F$  is compact in  $X$ .

**Theorem** A  $T_2$  space  $X$  is compact if and only if it is normal and strongly metacompact in every larger regular space.

**Example** A Tychonoff space  $X$  having a subset  $C$  which is 2– paracompact in  $X$  but not metacompact in  $X$  from outside.

**Theorem** Suppose  $f : X \rightarrow Y$  is a closed mapping onto  $Y$  and  $C \subseteq X$ . If  $C$  is cp-metacompact in  $X$  then  $f(C)$  is cp-metacompact in  $Y$ .

In [AG1] and [AG2] Arkhangel'skiĭ and Genedi introduce the thesis that for any topological property  $\mathcal{P}$  one can associate

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a relative property characterized in terms of a subspace  $Y$  of a space  $X$ , in such a way that it coincides with  $\mathcal{P}$  whenever  $Y = X$ . Naturally one would expect that for a given topological property there should be a variety of relative topological properties associated with it. We study several natural relative properties of metacompactness and paracompactness types and the relationships among them. Connections to other relative topological properties introduced in [AG1] and [A1] are also investigated.

Throughout this paper all spaces are  $T_1$ . Ordinals will have the order topology and subsets of topological spaces will have the subspace topology. For any collection  $\mathcal{A}$  of subsets of a set  $X$ , any  $C \subseteq X$  and any  $x \in X$ ,  $(\mathcal{A})_C = \{A \in \mathcal{A} : C \cap A \neq \emptyset\}$ ,  $(\mathcal{A})_x = \{A \in \mathcal{A} : x \in A\}$  and  $st(x, \mathcal{A}) = \cup(\mathcal{A})_x$ . If  $X$  is a set,  $\mathcal{H}$  a collection of subsets of  $X$  and  $C \subseteq X$  then  $\mathcal{H}$  is said to be (locally) point finite on  $C$  provided  $C \subseteq \cup \mathcal{H}$  and for every  $x \in C$  the collection  $(\mathcal{H})_x$  is finite (there is an open neighborhood  $V$  of  $x$  such that  $\{H \in \mathcal{H} : V \cap H \neq \emptyset\}$  is finite).

## 1. Metacompact (paracompact) in $X$

Suppose  $C$  is a subset of the space  $X$ . The following definitions of the most natural properties of relative paracompactness type are due to Gordienko, [Go]. The subspace  $C$  is said to be 1 – paracompact in  $X$  provided every open cover of  $X$  has an open refinement locally finite on  $C$ . The subspace  $C$  is 2 – paracompact in  $X$  provided every open cover of  $X$  has an open partial refinement covering  $C$  and locally finite on  $C$ . The subspace  $C$  is 3 – paracompact in  $X$  provided every open cover of  $X$  has a partial refinement consisting of sets open in  $C$  locally finite on  $C$ .

By replacing “locally finite” with “point finite” in Gordienko’s definitions we obtain relative metacompact analogs. We say that a subspace  $C$  of a space  $X$  is strongly metacompact in  $X$  provided every open cover of  $X$  has an open refinement point

finite on  $C$ . For a subspace  $C$  of a space  $X$  we say that  $C$  is metacompact in  $X$  provided every open cover of  $X$  has an open partial refinement point finite on  $C$ . Clearly for a space  $X$  strongly metacompactness in  $X$  is a natural relatively metacompact analog of 1- paracompactness in  $X$  and metacompactness in  $X$  is the corresponding relative metacompact analog of both 2- and 3- paracompactness in  $X$ .

**Theorem 1.** *If  $C$  is a metacompact (paracompact) subspace of a space  $X$  then  $C$  is metacompact (3- paracompact) in  $X$ .*

This implication cannot be reversed even for 1- paracompactness in a space  $X$ . For example, let  $X = \omega_1 + 1$ . Since  $X$  is compact every subset  $C$  of  $X$  is 1-paracompact in  $X$ . Thus  $\omega_1$  is 1- paracompact in  $X$  but is not even metacompact. Clearly, for any space  $X$  and  $C \subseteq X$ , if  $C$  is strongly metacompact (1- paracompact) [2-paracompact] in  $X$  then  $C$  is metacompact (2-paracompact) [3- paracompact] in  $X$ . None of these implications can in general be reversed.

A (partial) refinement  $\mathcal{V} = \{V_s : s \in \mathcal{S}\}$  of a cover  $\mathcal{U} = \{U_s : s \in \mathcal{S}\}$  of a set  $X$  is said to be precise if  $V_s \subseteq U_s$  for every  $s \in \mathcal{S}$ . Every open cover of a metacompact (paracompact) space has a precise point finite (locally finite) open refinement, [E]. The following is a natural relative metacompact (paracompact) analog of this useful property.

**Theorem 2.** *Suppose that  $\mathcal{U}$  is an open cover of a space  $X$ ,  $C \subseteq X$  and  $\mathcal{V}$  is an open (partial) refinement of  $\mathcal{U}$  that is point finite or locally finite on  $C$ . Then  $\mathcal{U}$  has a precise open (partial) refinement  $\mathcal{U}^*$  that is point finite or locally finite on  $C$ .*

**Corollary 3.** *Suppose  $C$  and  $D$  are subsets of the topological space  $X$ . If  $C$  and  $D$  are strongly metacompact (1- paracompact) in  $X$  then  $C \cup D$  is strongly metacompact (1- paracompact) in  $X$ .*

Suppose  $C$  and  $D$  are subsets of the topological space  $X$  and  $\mathcal{P}$  represents one of {strongly metacompact, metacompact, 1-, 2-, 3- paracompact}. It is obvious but never-the-less useful that if  $C$  is  $\mathcal{P}$  in  $X$  and  $D \subseteq C$  then  $D$  is  $\mathcal{P}$  in  $X$ . We frequently use this fact without comment.

**Corollary 4.** *Suppose that  $C$  is a subset of a space  $X$  such that both  $C$  and  $X \setminus C$  are strongly metacompact (1- paracompact) in  $X$ . Then  $X$  is metacompact (paracompact).*

Like many other covering properties metacompactness and paracompactness are closed hereditary. In fact,  $F_\sigma$  subspaces of metacompact (paracompact) spaces are metacompact (paracompact).

**Theorem 5.** *Suppose that  $C$  is a subset of  $X$  and  $C$  is  $\mathcal{P}$  in  $X$  where  $\mathcal{P}$  represents one of {strongly metacompact, metacompact, 1-, 2-, 3- paracompact}. If  $F$  is any  $F_\sigma$  subspace of  $X$  then  $F \cap C$  is  $\mathcal{P}$  in  $F$ .*

*Proof.* Suppose that the subspace  $C$  is strongly metacompact in the space  $X$  and that  $F = \cup\{F_i : i < \omega\}$  where, for each  $i < \omega$ ,  $F_i$  is a closed subset of  $X$ . Let  $\mathcal{U}$  be a collection of open subsets of  $X$  that covers  $F$ . For each  $i < \omega$ , let  $\mathcal{U}_i = \mathcal{U} \cup \{X \setminus F_i\}$  and  $\mathcal{V}_i$  an open refinement of  $\mathcal{U}_i$  point finite on  $C$ .

Let  $\mathcal{W}_0 = \{U \in \mathcal{V}_0 : U \cap F_0 \neq \phi\}$ . For  $0 < i < \omega$ , let  $\mathcal{W}_i = \{U \setminus (\cup\{F_j : j < i\}) : U \in \mathcal{V}_i \text{ and } U \cap F_i \neq \phi\}$ . Set  $\mathcal{W} = \cup\{\mathcal{W}_i : i < \omega\}$ . Notice that for each  $i < \omega$ , each element of  $C$  is in at most finitely many members of  $\mathcal{W}_i$ ,  $\mathcal{W}_i$  is an open partial refinement of  $\mathcal{U}$  and if  $x \in F$  and  $i = \min\{j < \omega : x \in F_j\}$  then  $x \in \cup\mathcal{W}_i$ . Also note that if  $x \in F_i$  then  $x \notin W$  for any  $W \in \mathcal{W}_j$  where  $i < j < \omega$ . Therefore for each  $x \in F$ ,  $(\mathcal{W})_x \subseteq \cup\{\mathcal{W}_k : k \leq i\}$  and so for each  $x \in C \cap F$  the collection  $(\mathcal{W})_x$  is nonempty and finite. Thus  $\mathcal{W}$  is an open partial refinement of  $\mathcal{U}$  covering  $F$  point finite on  $C \cap F$ . Hence  $F \cap C$  is strongly metacompact in  $F$ . (The proofs of the other cases are similar.)  $\square$

**Corollary 6.** *Suppose that  $C$  is a  $F_\sigma$  subset of a space  $X$ . Then  $C$  is metacompact (3-paracompact) in  $X$  if and only if  $C$  is metacompact (paracompact).*

We can now give a simple characterization of strongly metacompact in  $X$  for closed subsets of  $X$ .

**Theorem 7.** *Suppose  $C$  is a closed subset of a space  $X$ . The following are equivalent:*

1.  $C$  is strongly metacompact in  $X$ ,
2.  $C$  is metacompact in  $X$ ,
3.  $C$  is metacompact.

This characterization of strongly metacompact in a space  $X$  for closed subsets does not hold for  $F_\sigma$  subsets of  $X$ .

**Example 8.** (Example 82 [SS]) Let  $X$  be the upper half real plane with the tangent disk topology,  $P = \mathbb{R} \times (0, \infty)$  and  $L = X \setminus P = \mathbb{R} \times \{0\}$ . Since  $L$  is a closed discrete subset of  $X$ , by Theorem 7,  $L$  is strongly metacompact in the nonmetacompact space  $X$ . Hence by Corollary 4 the subspace  $P$  is not strongly metacompact in  $X$  even though  $P$  is an open metrizable  $F_\sigma$  subset of  $X$ .

Also notice that 1- paracompactness in a space  $X$ , for closed subsets, cannot be characterized in the same manner as strongly metacompactness in  $X$  as in Theorem 7. For example even though the set  $L$  of the above example is a closed discrete subset of  $X$  it is not even 2-paracompact in  $X$ . However it is easily seen that a closed subset  $C$  of a space  $X$  is 3- paracompact in  $X$  if and only if  $C$  is paracompact. Also, for a normal space  $X$ , a closed subspace  $C$  of  $X$  is 1- paracompact in  $X$  if and only if it is 2- paracompact in  $X$ .

A space  $X$  is called nearly metacompact (paracompact) provided that if  $\mathcal{U}$  is an open cover of  $X$  then there is a dense set  $D \subseteq X$  and an open refinement  $\mathcal{V}$  of  $\mathcal{U}$  (locally-finite) point

finite on  $C$ . The class of nearly metacompact spaces was introduced in [HL], studied further in [G] as  $d$ -metacompact spaces and in [GGV] among others. The class of nearly paracompact space was introduced in [G] as  $d$ -paracompact spaces.

Notice that if  $\mathcal{V}$  is an open cover of a space  $X$  and  $C$  is a subset of  $X$  such that  $\{V \in \mathcal{V} : V \cap C \neq \phi\}$  is finite (countable) then  $\{V \in \mathcal{V} : V \cap \overline{C} \neq \phi\}$  is also finite (countable). However an open cover  $\mathcal{V}$  of a space  $X$  that is point finite (locally finite) on a subset  $C$  of  $X$  need not be point finite on  $\overline{C}$ . For a space  $X$  and a dense subset  $C$  of  $X$  we say that  $X$  is nearly metacompact (paracompact) on  $C$  provided  $C$  is strongly metacompact (1-paracompact) in  $X$ . Gavrusenko made the useful observation that a nearly metacompact space in which the isolated points are dense is nearly paracompact. In fact, if  $X$  is a nearly metacompact space in which the set  $S$  of isolated points is dense then  $X$  is nearly paracompact on  $S$ . This seems like a very special situation but as suggested by the following question it is not yet clear just how special.

**Question 1.** *Suppose that  $X$  is a nearly metacompact (paracompact) space. Does there exist a dense subset  $C$  of  $X$  such that  $X$  is nearly metacompact (paracompact) on  $C$ ?*

The following useful theorem demonstrates the relationship between a subset  $C$  of a space  $X$  being strongly metacompact in  $X$  and nearly metacompactness.

**Theorem 9.** *For a space  $X$  and a subset  $C$  of  $X$ ,  $C$  is strongly metacompact in  $X$  if and only if  $\overline{C}$  is nearly metacompact on  $C$ .*

Since, as observed earlier, a closed paracompact subspace of a space  $X$  need not be 1-paracompact in  $X$ , the corresponding relationship between a subset  $C$  of a space  $X$  being 1-paracompact in  $X$  and the nearly paracompactness of  $\overline{C}$  does not hold. However the following are easily seen.

**Theorem 10.** *Suppose that  $X$  is a space and  $C$  is a subset of  $X$ . If  $C$  is 1- paracompact in  $X$  then  $\overline{C}$  is nearly paracompact on  $C$ . Also if  $\overline{C}$  is nearly paracompact on  $C$  then  $C$  is 3- paracompact in  $X$ .*

A collection  $\mathcal{S}$  of sets is said to be *monotone* provided for all  $S, S' \in \mathcal{S}$  either  $S \subseteq S'$  or  $S' \subseteq S$ . More generally, collection  $\mathcal{S}$  of sets is said to be *directed* provided for all  $S, S' \in \mathcal{S}$  there is a  $T \in \mathcal{S}$  such that  $S \cup S' \subseteq T$ . A space  $X$  is compact (paracompact) [metacompact] if and only if every monotone open cover of  $X$  has a finite subcover (locally–finite open refinement) [point–finite open refinement] [AU] ([M]) [[S]].

**Theorem 11.** *Suppose that  $C$  is a subset of the space  $X$ . The subspace  $C$  is strongly metacompact (1- paracompact) in  $X$  if and only if every directed open cover  $\mathcal{U}$  has an open refinement that is point finite (locally finite) on  $C$ .*

Clearly, metacompactness in  $X$ , and 2 – (3–) paracompactness in  $X$  can also be characterized in terms of directed open covers as above. However in Theorem 11, directed cannot be replaced with monotone. In the following example we give a regular space  $X$  and a dense subspace  $A$  of  $X$  such that every monotone open cover of  $X$  has an open refinement locally finite on  $A$  but  $A$  is not 1-paracompact (or even strongly metacompact) in  $X$ .

**Example 12.** Let  $Y = \prod_{i=1}^{\infty} (\omega_i + 1)$ . For each natural number  $k$ , let  $X_k = (\prod_{i=1}^k (\omega_i + 1)) \times (\prod_{i=k+1}^{\infty} (\omega_i))$  and let  $X = \cup_{k=1}^{\infty} X_k$  with the usual subspace topology, [Mis].

Every open cover of  $X$  has a subcover of cardinality less than or equal to  $\aleph_{\omega_0}$  and every open cover of  $X$  with cardinality less than  $\aleph_{\omega_0}$  has a countable subcover [Mis]. Let  $\mathcal{V}$  be a monotone open cover of  $X$  of cardinality  $\aleph_{\omega_0}$ . Then there is a cardinal  $\lambda \leq \aleph_{\omega_0}$  and a subcover  $\mathcal{V}' = \{V_\alpha : \alpha < \lambda\}$  of  $\mathcal{V}$  such that if



$\alpha < \beta < \lambda$  then  $V_\alpha \subseteq V_\beta$ . If  $\lambda < \aleph_{\omega_0}$  then  $\mathcal{V}'$  has a countable subcover. If  $\lambda = \aleph_{\omega_0}$  then  $\{V_{\omega_i} : i < \omega\}$  is a countable subcover of  $\mathcal{V}'$ . Thus as observed by Dennis Burke, every monotone open cover of the space  $X$  has a countable subcover. Hence  $X$  is  $\aleph_1$ -compact and so every point countable open cover of  $X$  has a countable subcover.

For each natural number  $k$ , let  $A_k = (\prod_{i=1}^k (\omega_i + 1)) \times (\prod_{i=k+1}^\infty \{0\})$  and note that  $A_k$  is compact. Let  $A = \cup_{k=1}^\infty A_k$  and notice that  $A$  is a dense subset of  $X$ . Suppose  $\mathcal{U} = \{U_i : i = 1, 2, \dots\}$  is an open cover of  $X$  such that  $U_1 \subseteq U_2 \subseteq \dots$ . Since for every natural number  $i$ ,  $A_i$  is a compact subset of  $X$ , w.o.l.g. assume that  $A_i \subseteq U_i$ . Since  $X$  is regular, for every natural number  $i$  let  $V_i$  be an open neighborhood of the compact set  $A_i$  such that  $\overline{V_i} \subseteq U_i$ . Let  $W_1 = U_1$  and for  $k = 2, 3, \dots$ , let  $W_k = U_k \setminus \cup \{\overline{V_i} : i = 1, 2, \dots, k-1\}$ . Let  $\mathcal{W} = \{W_k : k = 1, 2, \dots\}$ . Clearly  $\mathcal{W}$  is an open cover of  $X$ . For  $x \in A$  let  $k = \min\{i : x \in A_i\}$ . Then  $x \in A_k \subseteq V_k$ . Note that if  $j$  is a natural number and  $V_k \cap W_j \neq \emptyset$  then  $j \leq k$ . Therefore  $V_k$  is a neighborhood of  $x$  meeting only finitely many members of  $\mathcal{W}$ . Hence every monotone open cover of  $X$  has an open refinement locally finite on  $A$ .

For the natural number  $i$  and all  $\alpha < \omega_i$  let  $G(\alpha, i) = \{x \in X : x = \{x_k\}_{k=1}^\infty, x_i < \alpha\}$  and  $\mathcal{G}_i = \{G(\alpha, i) : \alpha < \omega_i\}$ . Then  $\mathcal{G} = \cup_{i=1}^\infty \mathcal{G}_i$  is an open cover of  $X$  with cardinality  $\aleph_{\omega_0}$  which does not have a subcover of with lower cardinality [Mis]. Therefore as noted above this open cover of  $X$  cannot have a point countable open refinement. Suppose  $\mathcal{U}$  is an open refinement of  $X$  and  $x = \{x_i\}_{i=1}^\infty \in X$  such that  $(\mathcal{U})_x$  is uncountable. Also suppose for all  $\alpha < \omega_1$ ,  $U_\alpha$  is a distinct member of  $(\mathcal{U})_x$ . For each  $\alpha < \omega_1$ , let  $S_\alpha$  be a finite set of natural numbers such that for each  $i \in S_\alpha$  there is an open neighborhood  $W(i, \alpha)$  of  $x_i$  in  $\omega_i + 1$  such that  $x \in X \cap (\cap \{\pi^{-1}(W(i, \alpha)) : i \in S_\alpha\}) \subseteq U_\alpha$ . Since there are only countably many such finite sets, there is a finite set  $S$  of natural numbers and an uncountable  $T \subseteq \omega_1$  such that  $S = S_\alpha$  for all  $\alpha \in T$ . For every natural number  $i$

define  $x_i^* = \begin{cases} x_i, & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}$ . Then  $x^* = \{x_i^*\}_{i=1}^\infty \in A \cap U_\alpha$  for all  $\alpha \in T$ . Hence  $\mathcal{U}$  is not point countable on  $A$ . Thus  $A$  is not strongly metacompact in  $X$ . (Note that the set  $A$  is 2-paracompact in  $X$ .)

We do not know if being metacompact or 2-,3- paracompact in a space can be characterized in terms of monotone open covers. However if we make the obvious modifications to Junnila's proof that metacompact (paracompact) spaces can be characterized in terms of monotone open covers [Ju], we get the following lemma.

**Lemma 13.** *For a space  $X$  and  $C \subseteq X$ , if every monotone open cover of  $X$  has an open refinement point finite (locally finite) on  $C$  then  $C$  is metacompact (2-paracompact) in  $X$ .*

For a closed subset of a spaces  $X$ , using Theorem 7 and the 3- paracompact analog, monotone open covers can be used to characterize strongly metacompactness and 3- paracompactness in  $X$ .

**Theorem 14.** *For a space  $X$  and a closed subset  $C$  of  $X$ , the subspace  $C$  is strongly metacompact (3- paracompact) in  $X$  if and only if every monotone open cover of  $X$  has an open refinement point finite on  $C$  (a partial refinement consisting of sets open in  $C$  locally finite on  $C$ ).*

**Theorem 15.** *Suppose that the space  $X$  has Lindelöf degree  $L(X) \leq \omega_1$  and  $C$  is an  $F_\sigma$  subset of  $X$ . Then  $C$  is strongly metacompact in  $X$  if and only if every monotone open cover of  $X$  has an open refinement point finite on  $C$ . If  $X$  is normal then  $C$  is 1- paracompact in  $X$  if and only if every monotone open cover of  $X$  has an open refinement locally finite on  $C$ .*

*Proof.* Suppose that  $C = \cup\{F_i : i < \omega\}$  where for every  $i < \omega$ ,  $F_i$  is a closed subset of  $X$  and that every monotone open cover of (the normal space)  $X$  has an open refinement point finite (locally

finite) on  $C$ . Let  $\mathcal{U}$  be an open cover of  $X$  and, since  $L(X) \leq \omega_1$ , suppose that  $\mathcal{U} = \{U(\alpha) : \alpha < \omega_1\}$ . For each  $\gamma < \omega_1$ , let  $U^*(\gamma) = \cup\{U(\alpha) : \alpha \leq \gamma\}$  and let  $\mathcal{U}^* = \{U^*(\gamma) : \gamma < \omega_1\}$ . Clearly  $\mathcal{U}^*$  is a monotone open cover of  $X$ . Since  $\mathcal{U}^*$  has an open refinement, point finite (locally finite) on  $C$ , by Theorem 2, it has a precise open refinement  $\mathcal{V} = \{V(\alpha) : \alpha < \omega_1\}$  point finite (locally finite) on  $C$ . Since every monotone open cover of  $X$  has an open refinement point finite (locally finite) on  $C$ , by Lemma 13,  $C$  is metacompact (2- paracompact) in  $X$ . Let  $\mathcal{W}$  be an open partial refinement of  $\mathcal{U}$  point finite (locally finite) on  $C$ . For each  $\gamma < \omega_1$  let  $\alpha_\gamma : \omega \rightarrow [0, \gamma]$  be onto.

[metacompact case] For each  $\gamma < \omega_1$  and  $i < \omega$ , let  $H(\gamma, i) = (V(\gamma) \cap U(\alpha_\gamma(i))) \setminus (\cup\{F_n : n \leq i\})$ . Since for all  $\gamma < \omega_1$   $V(\gamma) \subseteq U^*(\gamma) = \cup\{U(\alpha_\gamma(i)) : i < \omega\}$ ,  $V(\gamma) \setminus C \subseteq \cup\{H(\gamma, i) : i < \omega\}$ . Since  $\mathcal{V}$  is an open cover of  $X$ , for each  $\gamma < \omega_1$ , the collection  $\mathcal{H} = \{H(\gamma, i) : \gamma < \omega_1 \text{ and } i < \omega\}$  is an open partial refinement of  $\mathcal{U}$  covering  $X \setminus C$ . To see that  $\mathcal{H}$  is point finite on  $C$ , let  $x \in C$  and  $n < \omega$  such that  $x \in F_n$ . The open cover  $\mathcal{V}$  is point finite on  $C$ , so the set  $A = \{\gamma < \omega_1 : x \in V_\gamma\}$  is finite. Hence  $(\mathcal{H})_x \subseteq \{H(\gamma, i) : \gamma \in A \text{ and } i \leq n\}$  which is a finite set and thus  $x$  is in only finitely many members of  $\mathcal{H}$ . The collection  $\mathcal{H} \cup \mathcal{W}$  is an open refinement of  $\mathcal{U}$  point finite on  $C$ .

[paracompact case] Since  $X$  is normal, for each  $n < \omega$ , let  $G_n^*$  and  $G_n$  be disjoint open subsets of  $X$  such that  $X \setminus (\cup(\mathcal{W})_C) \subseteq G_n^*$  and  $\cup\{F_i : i \leq n\} \subseteq G_n$ . For each  $n < \omega$ , let  $M_n = \cap\{G_k^* : k \leq n\}$ . For each  $\gamma < \omega_1$  and  $i < \omega$ , let  $H(\gamma, i) = (V(\gamma) \cap U(\alpha_\gamma(i))) \cap M_i$ . Note that for all  $\gamma < \omega_1$ ,  $V(\gamma) \setminus (\cup(\mathcal{W})_C) \subseteq \cup\{H(\gamma, i) : i < \omega\}$  and so  $\mathcal{H} = \{H(\gamma, i) : \gamma < \omega_1 \text{ and } i < \omega\}$  is an open partial refinement of  $\mathcal{U}$  covering  $X \setminus (\cup(\mathcal{W})_C)$ . Let  $x \in C$  and  $n < \omega$  such that  $x \in F_n$ . Since  $\mathcal{V}$  is locally finite on  $C$ , let  $O$  be an open neighborhood of  $x$  such that  $\Gamma_x = \{\gamma < \omega_1 : O \cap V(\gamma) \neq \emptyset\}$  is finite. Since for all  $\gamma < \omega_1$  and  $i < \omega$ ,  $H(\gamma, i) \subseteq V(\gamma)$ ,  $\{\gamma < \omega_1 : O \cap H(\gamma, i) \neq \emptyset \text{ some } i < \omega\} \subseteq \Gamma_x$ . Since for all  $n, k < \omega$  if  $n \leq k$  then  $G_n \cap M_k = \emptyset$ , for all  $\gamma < \omega_1$

and for all  $n \leq i < \omega$ ,  $H(\gamma, i) \cap G_n = \phi$ . Thus  $\{(\gamma, i) \in \omega_1 \times \omega : H(\gamma, i) \cap O \cap G_n \neq \phi\} \subseteq \{(\gamma, i) : \gamma \in \Gamma_x \text{ and } i < n\}$ . Let  $O'$  be an open neighborhood of  $x$  meeting only finitely many members of  $\mathcal{W}$ . Then  $O \cap O' \cap G_n$  is an open neighborhood of  $x$  meeting only finitely many members of  $\mathcal{H} \cup \mathcal{W}$ . Hence the collection  $\mathcal{H} \cup \mathcal{W}$  is an open refinement of  $\mathcal{U}$  locally finite on  $C$ .  $\square$

The space of Example 12 has Lindelöf degree  $L(X) = \aleph_{\omega_0}$ . Theorem 15 suggests the following question.

**Question 2.** *Suppose that  $X$  is a space,  $L(X) < \aleph_{\omega_0}$  and  $C \subseteq X$  such that every monotone open cover of  $X$  has an open refinement point finite (locally finite) on  $C$ . Is  $C$  strongly meta-compact (1- paracompact) in  $X$ ?*

## 2. Connections to Other Relative Topological Properties

A subset  $C$  of a space  $X$  is said to be compact (Lindelöf) in  $X$  provided every open cover of  $X$  has a finite (countable) subcollection covering  $C$ , [AG1]. We say a subset  $C$  of a space  $X$  is strongly compact (Lindelöf) in  $X$  provided  $\overline{C}$  is compact (Lindelöf). The following theorem shows that strongly compact (Lindelöf) in a space  $X$  is a natural relative compact (Lindelöf) property corresponding to 1- paracompact in  $X$ .

**Theorem 16.** *A subset  $C$  of a space  $X$  is strongly compact (Lindelöf) in  $X$  if and only if every open cover  $\mathcal{U}$  of  $X$  has an open refinement  $\mathcal{V}$  such that  $\{V \in \mathcal{V} : V \cap C \neq \phi\}$  is finite (countable).*

For a regular space  $X$  a subset  $C$  of  $X$  is compact in  $X$  if and only if  $\overline{C}$  is compact, [Ra]. Thus for a  $T_3$  space  $X$ , compactness in  $X$  is equivalent to strong compactness in  $X$ . This is not the case for the corresponding relative Lindelöf properties. For example, suppose that  $X$  is any separable non Lindelöf space and  $D$  is any countable dense subset of  $X$ . Since  $D$  is countable it is Lindelöf in  $X$ . However since  $\overline{D} = X$  and  $X$  is not Lindelöf,  $D$  is not strongly Lindelöf in  $X$ .

Clearly for any space  $X$  if  $C \subseteq X$  is compact in  $X$  then  $C$  is 2- paracompact in  $X$  and for a regular space  $X$ , it is easily seen that if  $C \subseteq X$  is compact in  $X$  then  $C$  is 1- paracompact in  $X$ . For a regular space  $X$ , if  $C \subseteq X$  is Lindelöf in  $X$  then  $C$  is 2- paracompact in  $X$ . However, for example, let  $X = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$  be the Tychonoff plank and  $D = \{\omega_1\} \times \omega$ . Since  $D$  is a countable closed subset of  $X$  it is strongly Lindelöf in  $X$ , but  $C$  is not 1- paracompact in  $X$ , (see [Miy]).

**Theorem 17.** *Suppose that  $X$  is a normal space. If  $C$  is a closed Lindelöf subspace of  $X$  then  $C$  is 1- paracompact in  $X$ . That is if  $A \subseteq X$  is strongly Lindelöf in the normal space  $X$  then  $\overline{A}$  (and hence  $A$ ) is 1- paracompact in  $X$ .*

*Proof.* Let  $\mathcal{U}$  be an open cover of  $X$ . The space  $X$  being regular, there exist open refinements  $\mathcal{W}$  and  $\mathcal{V}$  such that for all  $W \in \mathcal{W}$  there is a  $V \in \mathcal{V}$  such that  $\overline{W} \subseteq V$ . Let  $\{W_n : n < \omega\}$  be a countable subset of  $\mathcal{W}$  covering  $\overline{A}$ . For each  $n < \omega$  let  $V_n \in \mathcal{V}$  such that  $\overline{W_n} \subseteq V_n$ . Let  $V^* = \cup\{V_n : n < \omega\}$ . Since  $X$  is normal there exist disjoint open sets  $O^*$  and  $O_A$  such that  $\overline{A} \subseteq O_A$  and  $X \setminus V^* \subseteq O^*$ . For each  $n < \omega$  let  $G_n = V_n \setminus \bigcup\{\overline{W_i} : i < n\}$  and let  $\mathcal{G} = \{G_n : n < \omega\} \cup \{U \cap O^* : U \in \mathcal{U}\}$ .

Suppose that  $x \in V^*$  and let  $k = \min\{n < \omega : x \in V_n\}$ . For all  $i < k$ , since  $\overline{W_i} \subseteq V_i$ ,  $x \notin \overline{W_i}$  and so  $x \in G_k$ . Thus  $\mathcal{G}$  is an open refinement of  $\mathcal{U}$ . Let  $x \in \overline{A}$  and  $m < \omega$  such that  $x \in W_m$ . Since  $W_m \cap G_n = \phi$  for all  $m < n < \omega$  and  $O_A \cap O^* = \phi$ , the set  $O_A \cap W_m$  is an open neighborhood of  $x$  meeting only finitely many members of  $\mathcal{G}$ . Thus  $\mathcal{G}$  is locally finite on  $\overline{A}$ .  $\square$

The product of a metacompact (paracompact) space and a compact space is metacompact (paracompact). The following give natural relative versions of this well known property of metacompact (paracompact) spaces.

**Theorem 18.** *Suppose that  $M$  is (strongly) metacompact in the space  $X$  and  $C$  is (strongly) compact in the space  $Y$ . Then  $M \times C$  is (strongly) metacompact in  $X \times Y$ .*

*Proof.* Suppose that  $M$  is metacompact in the space  $X$  and  $C$  is compact in the space  $Y$ . Let  $\{V_a^x : a \in A\}$  be a collection of open subsets of  $X$  and let  $\{V_a^y : a \in A\}$  be a collection of open subsets of  $Y$  such that the collection  $\mathcal{V} = \{V_a^x \times V_a^y : a \in A\}$  is a cover of  $X \times Y$ . For each  $z \in X$  let  $\mathcal{V}_z = \{V_a^y : z \in V_a^x, a \in A\}$  and  $A_z$  a finite subset of  $A$  such that for all  $a \in A_z$   $V_a^y \in \mathcal{V}_z$  and  $C \subseteq \cup\{V_a^y : a \in A_z\}$ . Let  $U_z = \cap\{V_a^x : a \in A_z\}$  and note that the collection  $\{U_z : z \in X\}$  is an open cover of  $X$ .

Let  $\mathcal{W}$  be an open partial refinement of  $\{U_z : z \in X\}$  point finite on  $M$ . For each  $W \in \mathcal{W}$  let  $z(W) \in X$  such that  $W \subseteq U_{z(W)}$ . Let  $\mathcal{G} = \cup\{W \times V_a^y : a \in A_{z(W)}\} : W \in \mathcal{W}\}$ . The collection  $\mathcal{G}$  is an open partial refinement of  $\mathcal{V}$  point finite on  $M \times C$ . Hence  $M \times C$  is metacompact in  $X \times Y$ .

For the strong version, proceed as above using the compact set  $\overline{C}$  in place of  $C$  and let  $\mathcal{W}$  be an open refinement of  $\{U_z : z \in X\}$  point finite on  $M$ . Then the collection  $\mathcal{G}$  is an open partial refinement of  $\mathcal{V}$  point finite on  $M \times \overline{C}$  covering the closed set  $X \times \overline{C}$ . Thus  $\mathcal{G} \cup \{(V_a^x \times V_a^y) \cap ((X \times Y) \setminus (X \times \overline{C})) : a \in A\}$  is an open refinement of  $\mathcal{V}$  point finite on  $M \times C$ .  $\square$

**Theorem 19.** *Suppose that  $M$  is 1- paracompact (2-,3- paracompact) in the space  $X$  and  $C$  is strongly compact (compact) in the space  $Y$ . Then  $M \times C$  is 1- paracompact (2-,3- paracompact) in  $X \times Y$ .*

Metacompact spaces are isocompact, i.e. closed countably compact subspaces are compact. In particular, countably compact, metacompact spaces are compact, [Du]. In [GGV] it is shown that nearly metacompact, countably compact  $T_3$  spaces are compact. The following is a direct consequence of this and Theorem 9.

**Theorem 20.** *For a countably compact  $T_3$  space  $X$  and a subset  $C$  of  $X$ , the following are equivalent*

1.  $C$  is strongly metacompact in  $X$ ,
2.  $\overline{C}$  is compact,
3.  $C$  is compact in  $X$

A space is said to be  $\aleph_1$ -compact provided that every closed discrete subset is countable. An  $\aleph_1$ -compact, metacompact space is Lindelöf. A subset  $C$  of an  $\aleph_1$ -compact space  $X$  strongly metacompact in  $X$  need not be strongly Lindelöf in  $X$  (Example 2.6 [GGV]).

However the following does hold.

**Theorem 21.** *Suppose that  $X$  is an  $\aleph_1$ -compact space and  $C \subseteq X$ . If  $C$  is strongly metacompact in  $X$  then  $C$  is Lindelöf in  $X$ .*

*Proof.* Suppose that  $X$  is an  $\aleph_1$ -compact space and  $C \subseteq X$  is strongly metacompact in  $X$ . Let  $\mathcal{U}$  be an open cover of  $X$  and  $\mathcal{V}$  be an open refinement of  $\mathcal{U}$  point finite on  $C$ . Let  $x(0) \in C$ . Suppose  $0 < \beta$  is an ordinal and for all  $\alpha < \beta$   $x(\alpha) \in C$  has been chosen such that if  $0 < \alpha < \beta$  then  $x(\alpha) \notin \cup\{st(x(\gamma), \mathcal{V}) : \gamma < \alpha\}$ . If  $C \subseteq \cup\{st(x(\alpha), \mathcal{V}) : \alpha < \beta\}$  then let  $A = \{x(\alpha) : \alpha < \beta\}$ . Otherwise let  $x(\beta) \in C \setminus \cup\{st(x(\alpha), \mathcal{V}) : \alpha < \beta\}$ . For some ordinal  $\lambda$  the set  $A = \{x(\alpha) : \alpha < \lambda\}$  is defined. Suppose  $x \in X$  and let  $V \in (\mathcal{V})_x$ . If  $V \cap A \neq \emptyset$  then, by the definition of  $A$ ,  $|V \cap A| = 1$ . Hence  $A$  is a closed discrete subset of  $X$  and since  $X$  is  $\aleph_1$ -compact  $A$  must be countable. Since  $\mathcal{V}$  is point finite on  $C$  and  $C \subseteq \cup\{st(x(\alpha), \mathcal{V}) : \alpha < \lambda\}$  the collection  $\cup\{(\mathcal{V})_{x(\alpha)} : \alpha < \lambda\}$  is a countable partial refinement of  $\mathcal{U}$  covering  $C$ .  $\square$

The following question remains.

**Question 3.** *Suppose that  $X$  is an  $\aleph_1$ -compact,  $T_3$  space and  $C$  is strongly metacompact in  $X$ . Is the subspace  $C$  strongly Lindelöf in  $X$ , i.e. is  $\overline{C}$  Lindelöf?*

Let  $C$  be a subspace of a space  $X$ . Then  $C$  is said to be countably compact in  $X$  provided every infinite subset  $A$  of  $C$  has an accumulation point in  $X$ , [A2]. We say that the subspace  $C$  is strongly countably compact in  $X$  provided  $\overline{C}$  is countably compact. Unlike the situation for subsets compact in a space  $X$ , in a completely regular space  $X$  a subspace  $C$  can be countably compact in  $X$  without  $\overline{C}$  being countably compact, [A2]. Like

the corresponding relative compactness properties both countably compact in a space  $X$  and strongly countably compact in  $X$  have natural open cover characterizations.

**Theorem 22.** *A subspace  $C$  of a space  $X$  is (strongly) countably compact in  $X$  if and only if every countable open cover of  $X$  contains a finite subcollection which covers  $(\overline{C})$ .*

*Proof.* Let us prove this for the countably compact in  $X$  case.

Suppose that  $C \subseteq X$  and  $D \subseteq C$  is a countable subset of  $C$  having no accumulation points in  $X$ . For each  $x \in D$ , let  $V_x = (X \setminus D) \cup \{x\}$  and note that  $V_x$  is an open neighborhood of  $x$  in  $X$ . Then  $\{V_x : x \in D\}$  is a countable open cover of  $X$  such that no finite subcollection covers  $C$ .

Suppose  $C \subseteq X$  and  $\mathcal{V} = \{V_n : n < \omega\}$  is a countable open cover of  $X$  such that no finite subcollection covers  $C$ . For each  $n < \omega$  let  $x_n \in C \setminus \bigcup \{V_k : k \leq n\}$ . Suppose  $x \in X$ . Let  $m < \omega$  such that  $x \in V_m$ . Since  $V_m \cap \{x_n : n < \omega\} \subseteq \{x_n : n < m\}$ , the point  $x$  is not an accumulation point of  $\{x_n : n < \omega\}$ . Hence  $\{x_n : n < \omega\}$  is an infinite subset of  $C$  having no accumulation points in  $X$  and so  $C$  is not countably compact in  $X$ .  $\square$

Closed countably compact subsets of a nearly metacompact space need not be compact. However we have the following relative version of isocompactness for regular spaces.

**Corollary 23.** *Suppose  $C$  and  $F$  are subspaces of the  $T_3$  space  $X$ . If  $C$  is strongly metacompact in  $X$  and  $F$  is strongly countably compact in  $X$  then  $C \cap F$  is compact in  $X$ .*

### 3. If $Y$ Is $\mathcal{P}$ In Every Larger Space Then What Is $Y$ ?

That is can topological properties of a space be characterized using relative topological properties. In 1996, Arhangel'skii and Tartir showed that a  $T_2$  space  $Y$  is regular in every larger  $T_2$  space if and only if  $Y$  is compact [AT]. The 3- paracompact version of the following is from [Miy].



**Theorem 24.** *The following are equivalent for a space  $Y$ .*

1.  $Y$  is 3– paracompact (metacompact) in every larger space  $X$ .
2.  $Y$  is 3– paracompact (metacompact) in every larger space  $X$  in which  $Y$  is closed.
3.  $Y$  is 3– paracompact (metacompact) in some larger space  $X$  in which  $Y$  is closed.
4.  $Y$  is paracompact (metacompact).

The following corollary is a direct consequence of Theorem 7.

**Corollary 25.** *The following are equivalent for a space  $Y$ .*

1.  $Y$  is strongly metacompact in every larger space  $X$  in which  $Y$  is closed.
2.  $Y$  is strongly metacompact in some larger space  $X$  in which  $Y$  is closed.
3.  $Y$  is metacompact.

**Theorem 26.** *[Miy]The following are equivalent for a regular space  $Y$ .*

1.  $Y$  is 1– paracompact in every larger regular space  $X$ .
2.  $Y$  is 1– paracompact in every larger regular space  $X$  in which  $Y$  is closed.
3.  $Y$  is compact.

**Corollary 27.** *A normal space  $Y$  is strongly metacompact in every larger regular space  $X$  if and only if  $Y$  is compact.*

*Proof.* Let  $Y$  be a normal space that is strongly metacompact in every larger regular space  $X$ . Suppose that  $Y$  is not compact. We proceed as in [Mi] and [Lu]. Since  $Y$  is not compact  $Y$  contains a countable closed discrete set  $D$ . Let  $j : D \rightarrow \mathbb{N}$  be a 1-to-1 onto mapping and  $Z = \beta\mathbb{N} \setminus \{x\}$  where  $x \in \beta\mathbb{N} \setminus \mathbb{N}$ .

Let  $X = Y \cup_j Z$  and  $D^* = \{\{d, j(d)\} : d \in D\} \subset X$ . Since  $Y$  is normal and  $D$  is closed in  $Y$ , the space  $X$  is regular. Since  $\mathbb{N}$  is not strongly metacompact in  $Z$ ,  $D^*$  and thus  $Y$  is not strongly metacompact in  $X$ , a contradiction. Hence  $Y$  is compact.  $\square$

**Theorem 28.** [Miy] *The following are equivalent for a regular space  $Y$ .*

1.  $Y$  is 2- paracompact in every larger regular space  $X$ .
2.  $Y$  is 2- paracompact in every larger regular space  $X$  in which  $Y$  is closed.
3.  $Y$  is Lindelöf.

**Theorem 29.** *A normal space  $Y$  is 1- paracompact in every larger normal space in which  $Y$  is closed if and only if  $Y$  is Lindelöf.*

*Proof.* Suppose that  $X$  is a normal space and  $Y$  is a closed Lindelöf subspace of  $X$ . Then by Theorem 17  $Y$  is 1- paracompact in  $X$ .

Suppose that  $Y$  is a normal space that is 1- paracompact in every larger normal space in which it is closed but that  $Y$  is not Lindelöf. We proceed as in the proof of (2)  $\rightarrow$  (3) of Theorem 28 in [Miy]. Since  $Y$  is paracompact but not Lindelöf, let  $D_y$  be a closed discrete subset of  $Y$  with  $|D_y| = \omega_1$ . Let  $F$  be Bing's Example G [B] where the set  $P$  has cardinality  $\omega_1$  and let  $D_f = \{x \in F : x \text{ is not isolated}\}$ . The set  $D_f$  is a closed discrete subspace of  $F$  with  $|D_f| = \omega_1$ . Let  $j : D_y \rightarrow D_f$  be a one-to-one onto mapping.

Let  $X = Y \cup_j Z$  and  $D^* = \{\{d, j(d)\} : d \in D_y\} \subset X$ . Since both  $Y$  and  $F$  are normal and  $D$  is closed in  $Y$  and  $F$ ,  $X$  is normal and  $Y$  is a closed subspace of  $X$ . Since  $D_f$  is not 1- paracompact in  $F$ ,  $D^*$  is not 1- paracompact in  $X$  and therefore  $Y$  is not 1- paracompact in  $X$ .  $\square$

#### 4. Having Property $\mathcal{P}$ in $X$ From Inside (Outside)

In [A1] Arkhangel'skii introduced the following types of relative topological properties. Let  $Y$  be a subspace of a space  $X$  and let  $\mathcal{P}$  be a topological property. We say that  $Y$  has property  $\mathcal{P}$  in  $X$  from inside, if every subspace of  $Y$  closed in  $X$  has property  $\mathcal{P}$ . If there is a subspace  $Z$  of  $X$  having property  $\mathcal{P}$  containing  $Y$  then we say that  $Y$  has property  $\mathcal{P}$  in  $X$  from outside. In [AG2] the following general question is posed:

Suppose  $C$  relatively  $\mathcal{P}$  in  $X$ . Is there a space  $Y$ ,  $C \subseteq Y \subseteq X$ , which has property  $\mathcal{P}$ ? That is, does  $C$  have property  $\mathcal{P}$  from outside?

For a  $T_3$  space  $X$ , a subspace  $C$  of  $X$  is compact in  $X$  if and only if  $\overline{C}$  is compact, [Ra] if and only if  $C$  is compact in  $X$  from outside, [A1]. In [AG1] they observe that there is a  $T_2$  space  $X$  having a dense subset  $Y$  such that  $Y$  is compact in  $X$  but no subspace of  $X$  containing  $Y$  is even Lindelöf.

**Theorem 30.** *For a  $T_2$  space  $X$  and a subset  $C$  of  $X$ ,  $C$  is strongly compact in  $X$  if and only if  $C$  is compact in  $X$  from outside.*

Naturally for a space  $X$  and a subset  $C$  of  $X$ , if  $C$  is strongly Lindelöf in  $X$  then  $C$  is Lindelöf in  $X$  from outside ( $\overline{C}$  is Lindelöf). In [DV] Dow and Vermeer give an example of a  $T_3$  space  $Z$  having a subspace  $X$  such that  $X$  is Lindelöf in  $Z$  but there is no Lindelöf subspace of  $Z$  containing  $X$ , i.e.  $X$  is Lindelöf in  $Z$  but not Lindelöf in  $Z$  from outside. If  $C$  is Lindelöf in the space  $X$  from outside, the subspace  $C$  is Lindelöf in  $X$  but it need not be strongly Lindelöf in  $X$ . For example, in Example 8 the set  $M = \{(x, y) : x, y \in \mathbb{Q}, y > 0\}$  is a Lindelöf (countable) subspace of  $X$  but  $M$  is not strongly Lindelöf in  $X$  (not even strongly metacompact in  $X$ ).

We know that if  $C \subseteq X$  is an  $F_\sigma$  set and  $C$  is metacompact in  $X$  then  $C$  is metacompact and hence  $C$  is metacompact in  $X$  from outside. A subset  $C$  being metacompact in a space  $X$  from outside does not imply that  $C$  is strongly metacompact in  $X$ . For example take  $X = \omega_1$  with the order topology and let  $C$  be the set of isolated points of  $X$ . Since  $C$  is metacompact,  $C$  is metacompact in  $X$  from outside but  $C$  is not strongly metacompact in  $X$ . However the following is obvious.

**Theorem 31.** *Suppose that  $C$  is a subset of the space  $X$  and  $C$  is metacompact (paracompact) in  $X$  from outside. Then  $C$  is metacompact ( $\beta$ -paracompact) in  $X$ .*

In [AG2] they give an example of a Tychonoff space  $X$  and a subspace  $C$  of  $X$  which is 1- paracompact in  $X$  but not paracompact in  $X$  from outside. We give an example of a Tychonoff space  $X$  and a subspace  $C$  of  $X$  which is 2- paracompact in  $X$  but which is not metacompact in  $X$  from outside.

**Example 32.** Let  $X$  be an non-metacompact 0-dimensional  $T_1$  space. Let  $Y$  be a compact 0-dimensional  $T_1$  space such that  $Y - \{p\}$  is not metacompact for some  $p \in Y$ . (For example  $X = \beta N - \{p\}$  and  $Y = \beta N$  where  $p$  is a remainder's point of  $\beta N$ .) Let  $Z = X \times Y$  and define a topology as follows:

1. For all  $x \in X$  and  $y \in Y \setminus \{p\}$  basic open neighborhoods of  $(x, y)$  are of the form  $\{x\} \times V$  where  $V$  is open neighborhood of  $y$  in  $Y$ .
2. For all  $x \in X$  basic open neighborhoods of  $(x, p)$  are of the form  $\cup\{\{x\} \times V_x : x \in U\}$  where  $U$  is an open neighborhood of  $x$  in  $X$  and  $\{V_x : x \in U\}$  is a collection of open neighborhoods of  $p$  in  $Y$ .

Then  $Z$  is a 0-dimensional  $T_1$  space and thus a Tychonoff space.

We denote  $X_p = X \times \{p\}$  and for each  $x \in X$ ,  $Y_x = \{x\} \times Y$ , and  $Y'_x = Y_x - \{(x, p)\}$ . Let  $C = \{(x, y) \in X \times Y : y \neq p\}$ . The subset  $C$  is an open subset of  $Z$  and is 2-paracompact in  $Z$ . Indeed  $C = \bigoplus\{Y'_x : x \in X_1\} \subset \bigoplus\{Y_x : x \in X_1\}$ . Since  $Y_x$  is compact,  $Y'_x$  is 1-paracompact in  $Y_x$  for every  $x \in X$ . Let  $\mathcal{U}$  be an open cover of  $Z$ . For every  $x \in X_1$ , let  $\mathcal{U}_x = \{U \cap Y_x : U \in \mathcal{U}\}$  and  $\mathcal{V}_x$  an open (in  $Y_x$ ) refinement of  $\mathcal{U}_x$  locally-finite on  $Y'_x$  (in  $Y_x$ ). Then for every  $x \in X$ ,  $\mathcal{W}_x = \{V \cap Y'_x : V \in \mathcal{V}_x\}$  is an open (in  $Z$ ) partial refinement of  $\mathcal{U}$  locally-finite on  $Y'_x$ . Put  $\mathcal{W} = \bigcup\{\mathcal{W}_x : x \in X_1\}$ . Then  $\mathcal{W}$  is an open partial refinement of  $\mathcal{U}$  and locally-finite on  $C$ . Thus  $C$  is 2-paracompact in  $Z$ .

To show that there is no metacompact space between  $C$  and  $Z$ . First note that the subspace  $C$  of  $Z$  is not metacompact because  $Y'_x$  is a non-metacompact closed subspace of  $C$  for every  $x \in X$ . Also  $X_p$  is a non-metacompact closed subspace of  $Z$ , so  $Z$  is not metacompact. Let  $W$  be a subset of  $Z$  containing  $C$  such that  $C \neq W \neq Z$ . Then there is a point  $x \in X$  such that  $Y'_x$  is a non-metacompact closed subspace of  $W$ . Hence  $W$  is not metacompact and thus we see that  $C$  is not metacompact in  $Z$  from outside.

Notice that in Example 32 the subspace  $C$  is not strongly metacompact in  $Z$ . To see this let  $\mathcal{U}$  be an open cover of  $X$  which doesn't have a point finite open refinement and let  $\mathcal{U}^* = \{U \times Y : U \in \mathcal{U}\}$ . Suppose that  $\mathcal{V}$  is an open refinement of  $\mathcal{U}^*$ . Then  $\{V \cap X_p : V \in \mathcal{V}\}$  is not point finite. Let  $x \in X$  such that  $(\mathcal{V})_{(x,p)}$  is infinite and for all  $n < \omega$  let  $V_n \in (\mathcal{V})_{(x,p)}$ . Note that since  $Y \setminus \{p\}$  is not metacompact it is not an  $F_\sigma$  subset of  $Y$  and thus the point  $p$  is not a  $G_\delta$  point of  $Y$ . Thus  $Y'_x \cap (\bigcap\{V_n : n < \omega\}) \neq \emptyset$ . Let  $y \in Y'_x \cap (\bigcap\{V_n : n < \omega\})$  and note that the point  $(x, y)$  of  $C$  is in infinitely many members of  $\mathcal{V}$ . The following relative metacompact versions of the general question posed in [AG2] remains.

**Question 4.** *Suppose  $C \subseteq X$  and  $C$  is strongly metacompact in  $X$ . Is  $C$  metacompact in  $X$  from outside?*

Since closed subsets of a metacompact space are metacompact, if  $Y$  is a subset of  $X$  and  $Y$  is metacompact in  $X$  from outside  $Y$  is metacompact in  $X$  from inside. We improve on this in the next theorem which follows directly from Theorem 7 and the comments after Example 8.

**Theorem 33.** *Suppose that  $C$  is a subset of the space  $X$  and  $C$  is metacompact ( $\mathcal{B}$ -paracompact) in  $X$ . Then  $C$  is metacompact (paracompact) in  $X$  from the inside.*

A subspace  $C$  being metacompact (paracompact) in a space  $X$  from inside does not imply that  $C$  is metacompact in  $X$ .

**Example 34.** Let  $X = \mathbb{R} \times [0, 1)$ ,  $Y = X \cup \{*\}$ ,  $L = \mathbb{R} \times \{0\}$ , and  $P = \mathbb{R} \times (0, 1)$ . For each  $z \in \mathbb{R}$  let  $B_z = \{(x, y) \in P : (x - z)^2 + (y - 1)^2 < 1\}$ , i.e. the portion of the disk of radius 1 tangent to  $(z, 0)$  in  $X$ . Points of  $X$  are given the usual tangent disk neighborhoods. While basic open neighborhoods of  $*$  have the form

$$\{*\} \cup (P \setminus \cup \{B_z : z \in F\}) \text{ for some finite } F \subset \mathbb{R}.$$

Notice that  $Y$  is Hausdorff but not regular.

Suppose that  $A \subseteq X$  is closed in  $Y$ . Then there is a finite  $F \subset \mathbb{R}$  such that  $A \subseteq L \cup (\cup \{B_z : z \in F\})$ . Clearly  $L \cup (\cup \{B_z : z \in F\})$  is hereditarily metacompact (paracompact) and so  $A$  must be metacompact. Thus  $X$  is metacompact in  $Y$  from inside. Since  $X$  is an  $F_\sigma$  subset of  $Y$  but is not metacompact,  $X$  is not metacompact in  $Y$  (see Corollary 6).

## 5. cp- metacompact (paracompact) in X

In 1957 Michael showed that a regular space is paracompact if and only if every open cover has a closed closure preserving refinement [Mi]. Junnila, in 1979, showed that a space is metacompact if and only if every directed open cover has a closed

closure preserving refinement [Ju]. For a space  $X$  and a subset  $A$  of  $X$ , a collection  $\mathcal{F}$  of closed subsets of  $X$  is said to be closure preserving with respect to  $A$  provided for all  $\mathcal{F}' \subseteq (\mathcal{F})_A$  either  $A \subseteq \cup \mathcal{F}'$  or  $\cup \mathcal{F}'$  is closed in  $X$ . The collection  $\mathcal{F}$  is said to be weakly closure preserving with respect to  $A$  provided for all  $\mathcal{F}' \subseteq (\mathcal{F})_A$ ,  $(\cup \mathcal{F}') \cap A = \overline{(\cup \mathcal{F}')} \cap A$ . We say that a subspace  $C$  of a space  $X$  is [weakly] cp- paracompact (metacompact) in  $X$  provided every (directed) open cover of  $X$  has a closed partial refinement covering  $C$  which is [weakly] closure preserving with respect to  $C$ . The following characterization of cp- paracompact in a  $T_3$  space  $X$  demonstrates that it is a natural property of relative paracompactness type along the lines of strongly compact, countably compact and Lindelöf in  $X$ .

**Theorem 35.** *Suppose  $X$  is a  $T_3$  space and  $A \subseteq X$ . The subspace  $A$  is cp- paracompact in  $X$  if and only if  $\overline{A}$  is a paracompact subspace of  $X$ .*

*Proof.* Suppose that  $A$  is a subset of the  $T_3$  space  $X$  and that  $\overline{A}$  is paracompact. Let  $\mathcal{U}$  be an open cover of  $X$ . Then  $\mathcal{U}^* = \{U \cap \overline{A} : U \in \mathcal{U}\}$  is an open (in  $\overline{A}$ ) cover of  $\overline{A}$ . Let  $\mathcal{F}$  be a closed closure preserving closed refinement of  $\mathcal{U}^*$ . Then  $\mathcal{F}$  is a closed partial refinement of  $\mathcal{U}$  closure preserving with respect to  $A$ .

Suppose that  $A \subseteq X$  is cp- paracompact in  $X$ . Suppose  $\mathcal{U}$  is an open (in  $\overline{A}$ ) cover of  $\overline{A}$ . For each  $U \in \mathcal{U}$  let  $U^*$  be an open subset of  $X$  such that  $U^* \cap \overline{A} = U$ . Assume  $|A| \geq 2$  and let  $x, y \in A$  with  $x \neq y$ . Choose  $U_x \in (\mathcal{U})_x$  and let  $\mathcal{U}^* = \{U_x^* \setminus \{y\}\} \cup \{U^* \setminus \{x\} : U \in \mathcal{U} \setminus \{U_x\}\} \cup \{X \setminus \overline{A}\}$ . Let  $\mathcal{F}$  be a closed refinement of  $\mathcal{U}^*$  closure preserving with respect to  $A$ . Then  $(\mathcal{F})_A$  is closure preserving and, since  $A \subseteq \cup (\mathcal{F})_A$ ,  $\overline{A} \subseteq \cup (\mathcal{F})_A$ . Thus  $\{F \cap \overline{A} : F \in (\mathcal{F})_A\}$  is a closed closure preserving refinement of  $\mathcal{U}$ . Hence  $\overline{A}$  is paracompact.  $\square$

**Corollary 36.** *Suppose  $X$  is a regular space and  $A \subseteq X$ . If the subspace  $A$  is cp- paracompact in  $X$  then  $A$  is paracompact in  $X$  from outside.*

It is easily seen that if  $X$  is a  $T_3$  space and  $A \subseteq X$  is 3- paracompact in  $X$  then  $A$  is weakly  $cp$ - paracompact in  $X$ . Thus in Example 32 the subspace  $C$  is weakly  $cp$ - paracompact in  $X$  but not metacompact in  $X$  from outside.

For a space  $X$  and a subset  $A$  of  $X$ ,  $A$  is said to be normal in  $X$  (nearly normal in  $X$ ) provided for each pair of disjoint closed subsets  $H$  and  $K$  of  $X$  there exist disjoint open (open in  $A$ ) subsets  $U$  and  $V$  of  $X$  such that  $A \cap H \subseteq U$  and  $A \cap K \subseteq V$ , [AG1]. If  $A$  is 2-paracompact in  $X$  (3-paracompact in  $X$ ) then  $A$  is normal (nearly normal) in  $X$ , [Go]. It follows from Theorem 35 that a subspace  $cp$ - paracompact in a space  $X$  is 3- paracompact in  $X$  and thus nearly normal in  $X$ . In the upper half real plane  $X$  with the tangent disk topology, Example 8, the subspace  $L = \mathbb{R} \times \{0\}$  is closed and discrete and thus  $cp$ -paracompact in  $X$ . However  $L$  is not normal in  $X$  [A1] and thus not 2- paracompact in  $X$ .

**Theorem 37.** *For a  $T_3$  space  $X$  and a closed  $A \subseteq X$  the following are equivalent:*

1.  $A$  is paracompact,
2.  $A$  is 3-paracompact in  $X$ ,
3.  $A$  is weakly  $cp$ - paracompact in  $X$ ,
4.  $A$  is  $cp$ - paracompact in  $X$ .

The metacompact version of this result extends Theorem 7.

**Theorem 38.** *For a space  $X$  and a closed  $A \subseteq X$  the following are equivalent:*

1.  $A$  is metacompact,
2.  $A$  is metacompact in  $X$ ,
3.  $A$  is weakly  $cp$ - metacompact in  $X$ ,
4.  $A$  is  $cp$ - metacompact in  $X$ .



**Corollary 39.** *For a  $(T_3)$  space  $X$  and a closed  $A \subseteq X$ , if  $A$  is weakly  $cp$ -metacompact (weakly  $cp$ -paracompact) in  $X$  then  $A$  is metacompact (paracompact) in  $X$  from inside.*

**Theorem 40.** *([K] and [PJ]) Every space  $X$  with a closure preserving closed cover by compact sets is metacompact.*

Since any cover of a space  $X$  consisting of compact sets will refine every directed open cover, we have the following relative version of this theorem.

**Corollary 41.** *Suppose  $C \subseteq X$  and that there is a collection of compact subsets of  $X$  covering  $C$  which is (weakly) closure preserving with respect to  $C$  then  $C$  is (weakly)  $cp$ -metacompact in  $X$ . In particular, if  $C$  is a countable subset of  $X$  then  $C$  is  $cp$ -metacompact in  $X$ .*

It follows that in Example 32 the subspace  $C$  is weakly  $cp$ -metacompact in  $Z$  but not metacompact in  $Z$  from outside, since  $\{Y_x : x \in X\}$  is a cover of  $Z$  consisting of compact sets which is weakly closure preserving with respect to  $C$ .

The characterization of  $cp$ -paracompact in a space  $X$  given in Theorem 35 does not have a  $cp$ -metacompact analog. That is in general for a space  $X$  a subspace  $A$  can be  $cp$ -metacompact in  $X$  while  $\overline{A}$  is not metacompact. Let  $X = \beta\mathbb{N} \setminus \{x\}$  where  $x \in \beta\mathbb{N} \setminus \mathbb{N}$ . Then  $X$  is a countably compact non compact space. The subspace  $\mathbb{N}$  is dense in  $X$  and  $cp$ -metacompact in  $X$ .

**Theorem 42.** *Suppose that  $X$  is a  $T_3$  space and  $A \subseteq X$  is metacompact in  $X$ . Then  $A$  is weakly  $cp$ -metacompact in  $X$ .*

*Proof.* Suppose that  $\mathcal{U}$  is a directed open cover of  $X$ . For each  $x \in X$  let  $U_x \in (\mathcal{U})_x$  and  $V_x$  an open neighborhood of  $x$  such that  $\overline{V_x} \subseteq U_x$ . Let  $\mathcal{W}$  be an open partial refinement of  $\{V_x : x \in X\}$ , point finite on  $A$ . Notice that  $\mathcal{W}^F = \{\cup \mathcal{W}' : \mathcal{W}' \text{ is a finite subset of } \mathcal{W}\}$  is a partial refinement of  $\mathcal{U}$  covering  $A$  and for all finite

$\mathcal{W}' \subseteq \mathcal{W}$ ,  $\overline{\cup \mathcal{W}'} \subseteq U$  some  $U \in \mathcal{U}$ . For each finite  $\mathcal{W}' \subseteq \mathcal{W}$  let  $F(\mathcal{W}') = \{x \in A : st(x, \mathcal{W}) \subseteq \cup \mathcal{W}'\}$ . Suppose  $\mathcal{W}'$  is a finite subset of  $\mathcal{W}$  and  $x \in A \setminus F(\mathcal{W}')$ . Then  $st(x, \mathcal{W}) \not\subseteq \cup \mathcal{W}'$ . Let  $W \in (\mathcal{W})_x$  such that  $W \not\subseteq \cup \mathcal{W}'$ . Then  $W \cap F(\mathcal{W}') = \phi$ . Thus for each finite  $\mathcal{W}' \subseteq \mathcal{W}$  the set  $F(\mathcal{W}')$  is closed in the subspace  $A$ .

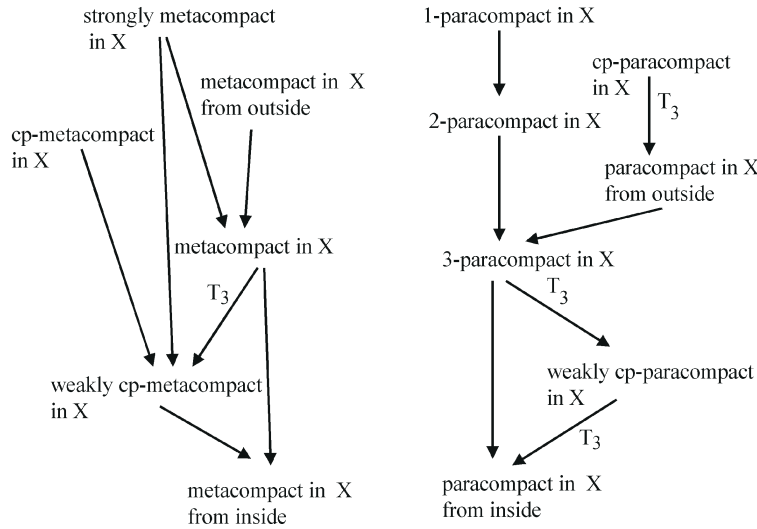
Let  $\mathcal{F} = \{\overline{F(\mathcal{W}')} : \mathcal{W}' \subseteq \mathcal{W} \text{ is finite}\}$ . Clearly  $\mathcal{F}$  is a closed partial refinement of  $\mathcal{U}^F$  and since  $\mathcal{W}$  is point finite on  $A$ ,  $\mathcal{F}$  covers  $A$ . Let  $\mathcal{F}' \subseteq \mathcal{F}$  such that  $A \not\subseteq \cup \mathcal{F}'$ . Let  $x \in A \setminus \cup \mathcal{F}'$  and suppose that  $x \in \overline{(\cup \mathcal{F}')}$ , i.e. suppose that the collection  $\mathcal{F}$  is not weakly closure preserving with respect to  $A$ . Let  $\mathcal{W}' \subseteq \mathcal{W}$  such that  $\overline{F(\mathcal{W}')} \in \mathcal{F}'$  and  $(\cap(\mathcal{W})_x) \cap \overline{F(\mathcal{W}')} \cap A \neq \phi$ . Since  $F(\mathcal{W}')$  is closed in  $A$ ,  $\overline{F(\mathcal{W}')} \cap A = F(\mathcal{W}')$ . Let  $y \in (\cap(\mathcal{W})_x) \cap F(\mathcal{W}') \cap A$ . Thus  $st(y, \mathcal{W}) \subseteq \cup \mathcal{W}'$ . However  $y \in \cap(\mathcal{W})_x$  implies  $(\mathcal{W})_x \subseteq (\mathcal{W})_y$  and so  $st(x, \mathcal{W}) \subseteq \cup \mathcal{W}'$ . Hence  $x \in F(\mathcal{W}') \subseteq \cup \mathcal{F}'$ , a contradiction. Thus the collection  $\mathcal{F}$  is weakly closure preserving with respect to  $A$ .  $\square$

We do not know if regularity is needed in Theorem 42. For the strongly metacompact analog it is not needed.

**Theorem 43.** *Suppose that  $X$  is space and  $A \subseteq X$  is strongly metacompact in  $X$ . Then  $A$  is weakly cp- metacompact in  $X$ .*

*Proof.* Suppose that  $\mathcal{U}$  is a directed open cover of  $X$ . Let  $\mathcal{W}$  be an open partial refinement of  $\mathcal{U}$ , point finite on  $A$ . For each finite  $\mathcal{W}' \subseteq \mathcal{W}$  let  $F(\mathcal{W}') = \{x \in X : st(x, \mathcal{W}) \subseteq \cup \mathcal{W}'\}$ . Proceeding as in the proof of Theorem 42 one can show that for each finite  $\mathcal{W}' \subseteq \mathcal{W}$  the set  $F(\mathcal{W}')$  is closed in  $X$ . Thus the collection  $\mathcal{F} = \{F(\mathcal{W}') : \mathcal{W}' \subseteq \mathcal{W} \text{ is finite}\}$  is a closed partial refinement of  $\mathcal{U}$  weakly closure preserving with respect to  $A$ .  $\square$

The following diagram gives the relationships among the properties of metacompactness and paracompactness types studied here.



The following questions remain.

**Question 5.** *Suppose  $C \subseteq X$  and  $C$  is cp- metacompact in  $X$ .*

- (a) *Is  $C$  metacompact in  $X$ ?*
- (b) *Is  $C$  metacompact in  $X$  from outside?*

### 6. Closed Mappings

The properties of the image and preimage of spaces having various covering properties under a variety of mappings have been well studied (see [Bu1] and [Bu2]). In particular, the perfect preimage of a metacompact (paracompact) space is metacompact (paracompact) and the closed image of a metacompact (paracompact) space is metacompact (paracompact) ,[E].

The proof of Theorem 5.1.35 of [E] is readily modified to prove the following lemma.

**Lemma 44.** *Suppose that  $f : X \rightarrow Y$  is a perfect mapping onto  $Y$  and  $D \subseteq Y$  is paracompact (metacompact). Then  $f^{-1}(D)$  is paracompact (metacompact).*

**Theorem 45.** *Suppose that  $f : X \rightarrow Y$  is a perfect mapping onto  $Y$  and  $C \subseteq Y$  is paracompact (metacompact) in  $Y$  from outside [inside]. Then  $f^{-1}(C)$  is paracompact (metacompact) in  $X$  from outside [inside].*

*Proof.* (outside) Suppose that  $f : X \rightarrow Y$  is a perfect mapping onto  $Y$ ,  $C \subseteq D \subseteq Y$  and  $D$  is paracompact (metacompact). By Lemma 44  $f^{-1}(D)$  is paracompact (metacompact). Since  $f^{-1}(C) \subseteq f^{-1}(D)$ ,  $f^{-1}(C)$  is paracompact (metacompact) in  $X$  from outside.

(inside) Suppose that  $f : X \rightarrow Y$  is a perfect mapping onto  $Y$ ,  $C \subseteq Y$  and  $C$  is paracompact (metacompact) in  $Y$  from inside. Let  $D \subseteq f^{-1}(C)$  be a closed subset of  $X$ . Then  $f(D) \subseteq C$  and since  $D$  is closed in  $X$  and  $f$  is a closed mapping  $f(D)$  is a closed subset of  $Y$ . Hence  $f(D)$  is paracompact (metacompact) and thus by Lemma 44  $f^{-1}(f(D))$  is paracompact (metacompact). Since  $D$  is a closed subset of the paracompact (metacompact) space  $f^{-1}(f(D))$ ,  $D$  is paracompact (metacompact). Hence  $f^{-1}(C)$  is paracompact (metacompact) in  $X$  from inside.  $\square$

The properties of being paracompact (metacompact) in a space from inside (outside) are not preserved by perfect mappings.

**Example 46.** Let  $X = \omega_1 \cup (\omega_1 \times \omega)$  and define a topology on  $X$  as follows:

1. points of  $\omega_1 \times \omega$  are isolated,
2. for each  $\beta < \omega_1$ ,  $\{[\alpha, \beta] \cup (([\alpha, \beta] \times \omega) \setminus F) : \alpha \text{ is a non-limit ordinal, } \alpha \leq \beta \text{ and } F \text{ a finite subset of } [\alpha, \beta] \times \omega\}$  form a local base at  $\beta$ .

The function  $f : X \rightarrow \omega_1$  defined by

1.  $f(\alpha) = \alpha$  for all  $\alpha < \omega_1$
2.  $f((\alpha, n)) = \alpha$  for all  $\alpha < \omega_1$  and  $n < \omega$

is a perfect mapping onto  $\omega_1$ . Notice that  $\omega_1 \times \omega$  is paracompact in  $X$  from both inside and outside but  $f(\omega_1 \times \omega) = \omega_1$  is not metacompact.

Gordienko observed in [Go] that for a perfect mapping  $f : X \rightarrow Y$  and  $C \subseteq Y$ , if  $Y$  is 1 – (2–, 3–) paracompact then  $f^{-1}(C)$  is 1 – (2–, 3–) paracompact in  $X$ .

**Theorem 47.** *Suppose  $f : X \rightarrow Y$  is a perfect mapping onto  $Y$  and  $C$  is a subset of  $Y$ .*

1. *If  $C$  is (strongly) metacompact in  $Y$  then  $f^{-1}(C)$  is (strongly) metacompact in  $X$ .*
2. *If  $C$  is (weakly) cp-metacompact [paracompact] in  $Y$  then  $f^{-1}(C)$  is (weakly) cp-metacompact [paracompact] in  $X$ .*

**Theorem 48.** *Suppose  $f : X \rightarrow Y$  is a closed mapping onto  $Y$  and  $C$  is a subset of  $X$ . If  $C$  is cp-paracompact (metacompact) in  $X$  then  $f(C)$  is cp-paracompact (metacompact) in  $Y$ .*

*Proof.* Suppose  $\mathcal{U}$  is an (a directed) open cover of  $Y$ . Then  $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$  is an (a directed) open cover of  $X$ . Let  $\mathcal{F}$  be a closed partial refinement of  $\mathcal{V}$  closure preserving with respect to  $C$ . Then  $\{f(F) : F \in \mathcal{F}\}$  is a closed refinement of  $\mathcal{U}$  closure preserving with respect to  $f(C)$ .  $\square$

There exists spaces  $X$  and  $Y$ , a perfect mapping  $f : X \rightarrow Y$  and a closed subset  $C$  of  $X$  which is 1– paracompact in  $X$  such that  $f(C)$  is not 2- paracompact in  $Y$ , (Example 2.4 [Miy]). The next example shows that the properties of being 2-, 3- paracompact, metacompact in, and weakly cp- paracompact (metacompact) in a space need not be preserved by a perfect mapping.

**Example 49.** Let  $X = \omega_1 \times \{0, 1\}$  and define a topology as follows:

1. the point  $(0, 0)$  and all points of the set  $A = \omega_1 \times \{1\}$  are isolated,
2. for every  $0 < \beta < \omega_1$  the collection  $\{((\alpha, \beta] \times \{0, 1\}) \setminus \{(\beta, 1)\} : \alpha < \beta\}$  forms a local base for the point  $(\beta, 0)$ . ( the Alexanderoff double of  $\omega_1$ )

The mapping  $f : X \rightarrow \omega_1$  defined by  $f(\alpha, k) = \alpha$  for all  $\alpha < \omega_1$  and  $k \in \{0, 1\}$  is a perfect mapping onto  $\omega_1$ . The set  $A$  is 2- paracompact in  $X$  but  $f(A) = \omega_1$  is not metacompact.

**Question 6.** *Suppose  $f : X \rightarrow Y$  is a closed (perfect) mapping onto  $Y$ ,  $C \subseteq X$  and  $C$  is strongly metacompact in  $X$ . Is  $f(C)$  strongly metacompact in  $Y$ ?*

Since the restriction of a closed mapping to a closed subset is still closed, the next result follow directly from Theorem 7.

**Theorem 50.** *Suppose  $f : X \rightarrow Y$  is a closed mapping onto the space  $Y$  and  $C$  is a closed subset of  $X$ . If  $C$  is (strongly) metacompact in  $X$  then  $f(C)$  is (strongly) metacompact in  $Y$ .*

A topological space  $X$  is said to be orthocompact provided every open cover of  $X$  has an interior preserving open refinement. Although in general orthocompactness is not preserved by perfect mappings, [Bu1], the following does hold.

**Corollary 51.** *Suppose  $f : X \rightarrow Y$  is a closed mapping from the orthocompact space  $X$  onto the space  $Y$  and  $C$  is a subset of  $X$  such that  $C$  is strongly metacompact in  $X$ . Then  $f(C)$  is strongly metacompact in  $Y$ . In fact  $f(\overline{C})$  is a metacompact subspace of  $Y$ .*

*Proof.* Since  $C$  is strongly metacompact in  $X$ ,  $\overline{C}$  is nearly metacompact on  $C$ . Since closed subsets of orthocompact spaces are orthocompact,  $\overline{C}$  is a nearly metacompact orthocompact space and therefore  $\overline{C}$  is metacompact, [HL]. Hence  $f(\overline{C})$  is a metacompact in  $Y$ . Since  $f(\overline{C})$  is closed, by Theorem 7,  $f(\overline{C})$  and thus  $f(C)$  is strongly metacompact in  $Y$ .  $\square$

The perfect preimage of a space  $Y$  nearly metacompact on a dense set  $C$  need not be nearly metacompact.

**Example 52.** Let  $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$  be a collection of non empty subsets of  $\omega_1$  having the finite intersection property such that for any infinite  $I \subseteq \omega_1$ ,  $\bigcap\{A_\alpha : \alpha \in I\} = \emptyset$ . (The existence of a regular ultrafilter on  $\omega_1$  guarantees the existence of such a collection, [CN]) For each  $\alpha < \omega_1$  let  $\mathcal{A}^\alpha = \{A \in \mathcal{A} : A \subseteq A_\alpha\}$ . Let  $Y = \omega_1 \cup (\omega_1 \times \omega_1)$  and define a topology on  $Y$  as follows:

1. points of  $\omega_1 \times \omega_1$  are isolated,
2. for each  $\beta < \omega_1$ ,  $\{[\alpha, \beta] \cup ([\alpha, \beta] \times A) : \alpha \text{ is a nonlimit ordinal, } \alpha \leq \beta \text{ and } A \in \mathcal{A}^\alpha\}$  form a local base at  $\beta$ .

Since  $\emptyset \notin \mathcal{A}$ , it is easily seen that the set  $\omega_1 \times \omega_1$  is dense in  $Y$  and that  $\omega_1$  is a closed subset of  $Y$  that as a subspace has the usual order topology. The space  $Y$  is nearly metacompact on  $\omega_1 \times \omega_1$ , [GGV].

Let  $Z = Y \cup (\omega_1 \times \{-1\})$  and define a topology on  $Z$  by defining basic neighborhoods of points of  $Y$  as above and for each  $\beta < \omega_1$ ,  $\{[\alpha, \beta] \times \{-1\} : \alpha \leq \beta \text{ is a nonlimit ordinal}\}$  form a local base at the point  $(\beta, -1)$ . Notice that  $\omega_1 \times \{-1\}$  is a clopen subset of  $Z$  isomorphic to  $\omega_1$ . Since  $\omega_1$  is not nearly metacompact, neither is  $Z$ . Consider the mapping  $f : Z \rightarrow Y$  such that  $f(y) = y$  for all  $y \in Y$  and  $f((\beta, -1)) = \beta$  for all  $\beta < \omega_1$ . The map  $f$  is a perfect map onto  $Y$ .

Notice that by Theorem 47, for the space  $Z$  in Example 52, the subspace  $\omega_1 \times \omega_1$  is strongly metacompact in  $Z$ . While  $\omega_1 \times \omega_1$  is dense in the space  $Y$  it is not dense in the space  $Z$ . In [G] Gavrusenko proves the paracompact version of the following. The proof is essentially identical to that of Gavrusenko.

**Theorem 53.** *Suppose  $f$  is an open perfect mapping from a space  $X$  onto a nearly metacompact (on the dense set  $C$ ) space  $Y$ . Then  $X$  is nearly metacompact (on the dense set  $f^{-1}(C)$ ).*

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Dept. Mathematics, Slippery Rock University, Slippery Rock  
PA, 16057 USA

*E-mail address:* `elise.grabner@sru.edu`

*E-mail address:* `gary.grabner@sru.edu`

Dept. Mathematics, Ehime University, Matsuyama 790-8577  
JAPAN

*E-mail address:* `bzq22206@nifty.ne.jp`