

# Topology Proceedings



**Web:** <http://topology.auburn.edu/tp/>  
**Mail:** Topology Proceedings  
Department of Mathematics & Statistics  
Auburn University, Alabama 36849, USA  
**E-mail:** [topolog@auburn.edu](mailto:topolog@auburn.edu)  
**ISSN:** 0146-4124

---

COPYRIGHT © by Topology Proceedings. All rights reserved.

## REMARKS ON HEREDITARILY INDECOMPOSABLE CONTINUA

Klaas Pieter Hart, Jan van Mill and Roman Pol

### Abstract

We recall a characterization of hereditary indecomposability originally obtained by Krasinkiewicz and Minc, and show how it may be used to give unified constructions of various hereditarily indecomposable continua. In particular we answer a question asked by Maćkowiak and Tymchatyn by showing that any continuum of arbitrary weight is a weakly confluent image of a hereditarily indecomposable continuum of the same weight.

We present two methods of constructing these preimages: (a) by model-theoretic means, using the compactness and completeness theorems from first-order logic to derive these results for continua of uncountable weight from their metric counterparts; and (b) by constructing essential mappings from hereditarily indecomposable continua onto Tychonoff cubes.

We finish by reviving an argument due to Kelley about hyperspaces of hereditarily indecomposable continua and show how it leads to a point-set argument that reduces Brouwer's Fixed-point theorem to its three-dimensional version.

---

*Mathematics Subject Classification:* Primary: 54F15. Secondary: 54F45 03B10 03C98 54D80.

*Key words:* Indecomposable continuum, hereditarily indecomposable continuum, dimension theory, model theory, lattice, Wallman space.

## 1. Preliminaries

### 1.1. Hereditary indecomposable spaces

A continuum is *decomposable* if it can be written as the union of two proper subcontinua; it is *indecomposable* otherwise. A *hereditarily indecomposable* continuum is one in which every subcontinuum is indecomposable. It is easily seen that this is equivalent to saying that whenever two continua in the space meet one is contained in the other.

This latter statement makes sense for arbitrary compact Hausdorff spaces, connected or not; we therefore extend this definition and call a compact Hausdorff space *hereditarily indecomposable* if it satisfies the statement above: whenever two continua in the space meet one is contained in the other. Thus, zero-dimensional spaces are hereditarily indecomposable too.

We shall mainly use a characterization of hereditary indecomposability that can be gleaned from [9, Theorem 3] and which was made explicit in [13, Theorem 2]. To formulate it we introduce some terminology.

Let  $X$  be compact Hausdorff and let  $C$  and  $D$  be disjoint closed subsets of  $X$ ; as in [9] we say that  $(X, C, D)$  is *crooked* between the neighbourhoods  $U$  of  $C$  and  $V$  of  $D$  if we can write  $X = X_0 \cup X_1 \cup X_2$ , where each  $X_i$  is closed and, moreover,  $C \subseteq X_0$ ,  $X_0 \cap X_1 \subseteq V$ ,  $X_0 \cap X_2 = \emptyset$ ,  $X_1 \cap X_2 \subseteq U$  and  $D \subseteq X_2$ . We say  $X$  is crooked between  $C$  and  $D$  if  $(X, C, D)$  is crooked between any pair of neighbourhoods of  $C$  and  $D$ ; no generality is lost if we consider pairs of *disjoint* neighbourhoods only, as crookedness between small neighbourhoods implies crookedness between larger neighbourhoods.

We can now state the characterization of hereditary indecomposability that we will use.

**Theorem 1.1** (Krasinkiewicz and Minc). *A compact Hausdorff space is hereditarily indecomposable if and only if it is crooked between every pair of disjoint closed (nonempty) subsets.*

This characterization can be translated into terms of closed sets only; we simply put  $F = X \setminus V$  and  $G = X \setminus U$ , and reformulate some of the premises and the conclusions. We get the following formulation.

**Theorem 1.2.** *A compact Hausdorff space  $X$  is hereditarily indecomposable if and only if whenever four closed sets  $C, D, F$  and  $G$  in  $X$  are given such that  $C \cap D = C \cap G = F \cap D = \emptyset$  one can write  $X$  as the union of three closed sets  $X_0, X_1$  and  $X_2$  such that  $C \subseteq X_0, D \subseteq X_2, X_0 \cap X_1 \cap F = \emptyset, X_0 \cap X_2 = \emptyset,$  and  $X_1 \cap X_2 \cap G = \emptyset.$*

To avoid having to write down many formulas we call a quadruple  $(C, D, F, G)$  with  $C \cap D = C \cap F = D \cap G = \emptyset$  a *plian* and *foursome* and we call a triple  $(X_0, X_1, X_2)$  with  $C \subseteq X_0, D \subseteq X_2, X_0 \cap X_1 \cap F = \emptyset, X_0 \cap X_2 = \emptyset,$  and  $X_1 \cap X_2 \cap G = \emptyset$  a *fold* for  $(C, D, F, G)$ . Thus, a compact Hausdorff space is hereditarily indecomposable if and only if there is a fold for every plian foursome. As above, no generality is lost if we consider foursomes with  $F \cup G = X$  only; there does not seem to be any need to add this assumption though.

This characterization can be improved by taking a base  $\mathcal{B}$  for the closed sets of the space  $X$  that is closed under finite intersections. The space is hereditarily indecomposable if and only if there is a fold for every plian foursome whose terms come from  $\mathcal{B}$ .

To prove the nontrivial implication let  $(C, D, F, G)$  be a plian foursome and let  $(O_C, O_D, O_F, O_G)$  be a swelling of it, i.e., every set  $O_P$  is an open around  $P$  and  $O_P \cap O_Q = \emptyset$  if and only if  $P \cap Q = \emptyset$ , where  $P$  and  $Q$  run through  $C, D, F$  and  $G$  (see [2, 7.1.4]). Now compactness and the fact that  $\mathcal{B}$  is closed under finite intersections guarantee that there are  $C', D', F'$  and  $G'$  in  $\mathcal{B}$  such that  $P \subseteq P' \subseteq O_P$  for  $P = C, D, F, G$ . Any fold for  $(C', D', F', G')$  is a fold for  $(C, D, F, G)$ .

**1.2. A crooked partition of the square**

Let  $P$  be the closure of crooked white strip in  $\mathbb{I}^2$ , depicted in Figure 1 below. The set  $\mathbb{I}^2 \setminus P$  is the union of the disjoint open sets  $M_0$  and  $M_1$  in the picture. Clearly,  $\text{cl } M_0 \cap \text{cl } M_1 = \emptyset$ . Observe that  $\{0\} \times \mathbb{I} \subseteq M_0$  and  $\{1\} \times \mathbb{I} \subseteq M_1$ . It follows that  $P$  is a partition between  $\{0\} \times \mathbb{I}$  and  $\{1\} \times \mathbb{I}$  in  $\mathbb{I}^2$ .

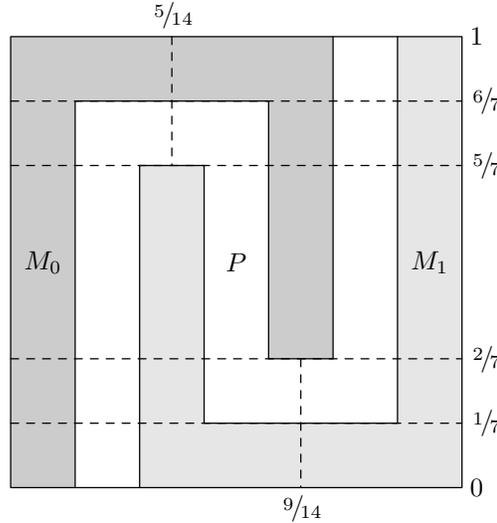


Fig. 1. The crooked partition

We shall use  $P$  to create folds for pliant foursomes. Here's how: given a pliant foursome  $(C, D, F, G)$  apply Urysohn's lemma to get a continuous function  $f : X \rightarrow \mathbb{I}$  such that  $f[C] = \{0\}$ ,  $f[G] \subseteq [0, 1/2]$ ,  $f[F] \subseteq [1/2, 1]$  and  $f[D] = \{1\}$ . One can then create a fold by creating a continuous function  $g : X \rightarrow \mathbb{I}$  such that  $(g \triangle f)[X] \subseteq P$  and setting  $X_j = (g \triangle f)^{-1}[P_j]$ , where  $P_0 = \{(x, y) : x \leq 5/14\}$ ,  $P_1 = \{(x, y) : 5/14 \leq x \leq 9/14\}$  and  $P_2 = \{(x, y) : 9/14 \leq x\}$ . We shall call a function as  $f$  a *Urysohn function* for the foursome  $(C, D, F, G)$ .

We summarize the foregoing discussion in the following lemmas.

**Lemma 1.3.** *Let  $X$  be a compact space of weight  $\kappa$ . Then there is a family  $\mathcal{F}$  of continuous functions, from  $X$  to  $\mathbb{I}$ , of size  $\kappa$  such that for every pliant foursome  $(C, D, F, G)$  there is an  $f$  in  $\mathcal{F}$  with  $f[C] = \{0\}$ ,  $f[G] \subseteq [0, 1/2]$ ,  $f[F] \subseteq [1/2, 1]$  and  $f[D] = \{0\}$ .*

We call a family  $\mathcal{F}$  as in Lemma 1.3 a *pliant family* for  $X$ .

**Lemma 1.4.** *Let  $X$  be a compact Hausdorff space and  $\mathcal{F}$  a pliant family of functions for  $X$ . If  $Y$  is a closed subspace of  $X$  with the property that for every  $f \in \mathcal{F}$  there is a continuous function  $g : X \rightarrow \mathbb{I}$  such that  $(g \Delta f)[Y] \subseteq P$  then  $Y$  is hereditarily indecomposable.*

*Proof.* Simply observe that every pliant foursome from  $Y$  is also a pliant foursome in  $X$  and hence that the restrictions of the elements of  $\mathcal{F}$  to  $Y$  form a pliant family for  $Y$ .  $\square$

### 1.3. Dimension and essential families

We adopt the contrapositive of the Theorem on Partitions [2, 7.2.15] as our definition of the covering dimension  $\dim$ . To this end we define a (finite or infinite) family  $\{(A_i, B_i) : i \in I\}$  of pairs of disjoint closed sets to be *essential* if whenever we take partitions  $L_i$  between  $A_i$  and  $B_i$  for all  $i$  the intersection  $\bigcap_{i \in I} L_i$  is nonempty.

If  $X$  is a compact Hausdorff (or even normal) space and  $n \in \mathbb{N}$  then we say that  $\dim X \geq n$  if  $X$  has an essential family of pairs of closed sets of cardinality  $n$ ; the *covering dimension* of  $X$  is the maximum  $n$  such that  $\dim X \geq n$ , if such an  $n$  exists; we write  $\dim X = \infty$  if  $\dim X \geq n$  for all  $n$ . Note that  $\dim X = \infty$  means that  $X$  must have arbitrarily large *finite* essential families; if  $X$  possesses an infinite essential family then we say that  $X$  is *strongly infinite dimensional*.

The following lemma will be used to verify that certain spaces have a large enough covering dimension.

**Lemma 1.5.** *Let  $\{(A_i, B_i) : i \in I\}$  be an essential family in a normal space  $X$  and assume  $I$  is split into two sets  $J$  and  $K$ . Let, for  $i \in J$ , a partition  $L_i$  between  $A_i$  and  $B_i$  be given and put  $L_J = \bigcap_{i \in J} L_i$ . Then  $\{(A_i \cap L_J, B_i \cap L_J) : i \in K\}$  is an essential family in  $L_J$ .*

*Proof.* By normality we can extend, for every  $i \in K$ , any partition in  $L_J$  between  $A_i \cap L_J$  and  $B_i \cap L_J$  to a partition in  $X$  between  $A_i$  and  $B_i$ . Now apply the assumption that the full family is essential.  $\square$

#### 1.4. Faces of cubes

Consider a Tychonoff cube  $\mathbb{I}^\kappa$ . For every  $\alpha \in \kappa$  we put  $A_\alpha = \{x : x_\alpha = 0\}$  and  $B_\alpha = \{x : x_\alpha = 1\}$ ; these are the  $\alpha$ th opposite faces of  $\mathbb{I}^\kappa$ . The following theorem is a fundamental fact about these faces.

**Theorem 1.6.** *The family  $\{(A_\alpha, B_\alpha) : \alpha \in \kappa\}$  is essential in  $\mathbb{I}^\kappa$ .*

For finite  $\kappa$  this follows from Brouwer's Fixed-Point Theorem [2, 7.3.19]. In case  $\kappa$  is infinite we put, for every finite subset  $a$  of  $\kappa$ ,  $F_a = \pi_a^{-1}[\bigcap_{\alpha \in a} \pi_a[L_\alpha]]$ , where  $\pi_a$  denotes the projection onto the subcube  $\mathbb{I}^a$ . By the finite case each  $F_a$  is nonempty and clearly  $a \subseteq b$  implies  $F_a \supseteq F_b$ , so  $\bigcap_a F_a \neq \emptyset$ . Now check that  $\bigcap_a F_a = \bigcap_\alpha L_\alpha$ .

With this fact in mind we call a continuous map  $f : X \rightarrow \mathbb{I}^\kappa$  *essential* if the family  $\{(f^{-1}[A_\alpha], f^{-1}[B_\alpha]) : \alpha < \kappa\}$  is essential. A routine application of Urysohn's lemma shows that  $X$  admits an essential map onto  $\mathbb{I}^\kappa$  iff  $X$  has an essential family of size  $\kappa$ .

#### 1.5. Hyperspaces

The *hyperspace* of a space  $X$  is the family  $2^X$  of nonempty closed subsets of  $X$  endowed with the *Vietoris topology*, which has the family of sets of the form  $\langle U \rangle = \{F : F \subseteq U\}$  and  $\langle X, U \rangle = \{F : F \cap U \neq \emptyset\}$ , where  $U$  is open, as a subbase. It is well-known that  $2^X$  is compact Hausdorff if  $X$  is and that

if  $X$  is compact metric, with metric  $d$ , then the corresponding Hausdorff metric  $d_H$  generates the Vietoris topology of  $2^X$ .

An important subspace of  $2^X$  is  $\mathcal{C}(X)$ , the space of all subcontinua of  $X$ ; it is a closed subset of  $2^X$ , hence it is also compact if  $X$  is.

## 2. Bing's Continua

We begin by constructing an infinite-dimensional hereditarily indecomposable compact subset of the Hilbert cube  $\mathbb{I}^\infty$ .

To this end we let  $\pi_i$  denote the projection of  $\mathbb{I}^\infty$  onto the  $i$ -th coordinate. Furthermore we fix a pliant family  $\{f_i : i \in \mathbb{N}\}$  of continuous functions for  $\mathbb{I}^\infty$ . For every  $i$  we let  $u_i = \pi_{2i} \Delta f_i$  be the diagonal map of  $\pi_{2i}$  and  $f_i$  from  $\mathbb{I}^\infty$  to  $\mathbb{I}^2$ . The letter  $P$  still refers to the partition of the square from Figure 1.

**Theorem 2.1.** *Let  $X = \bigcap_{i=1}^\infty u_i^{-1}[P]$ . Then  $X$  is an infinite-dimensional hereditarily indecomposable compact space.*

*Proof.* Lemma 1.4 implies immediately that  $X$  is hereditarily indecomposable: for every  $f_i$  the projection  $\pi_{2i}$  is as required.

To see that  $X$  is infinite-dimensional we observe that  $u_i^{-1}[P]$  is a partition between the even-numbered faces  $A_{2i}$  and  $B_{2i}$  of  $\mathbb{I}^\infty$  — indeed:

$$u_i^{-1}[P] \subseteq \{x \in \mathbb{I}^\infty : 1/7 \leq x_{2i} \leq 6/7\}.$$

By Theorem 1.6 and Lemma 1.5 this implies that

$$\{(A_{2i+1} \cap X, B_{2i+1} \cap X) : i \in \mathbb{N}\}$$

is an essential family in  $X$ ; so  $X$  is even strongly infinite-dimensional.  $\square$

**Corollary 2.2** (Bing). *For every  $n$  there is an  $n$ -dimensional hereditarily indecomposable continuum.*

*Proof.* As observed in the previous proof the traces of the odd-numbered faces of  $\mathbb{I}^\infty$  on  $X$  form an essential family in  $X$ . One can therefore find a component  $S$  of  $X$  such that the traces from that family on  $S$  also form an essential family. Now let  $\pi : S \rightarrow \mathbb{I}^{2n}$  be (the restriction of) the projection onto the first  $2n$  coordinates. Consider the monotone-light factorization of  $\pi$ , i.e., write  $\pi = \lambda \circ \mu$ , where  $\mu : S \rightarrow T$  is a monotone surjection and  $\lambda : T \rightarrow \mathbb{I}^{2n}$  is a light map, cf. [2, 6.2.22]. Since  $\lambda$  is light we have  $\dim T \leq 2n$ , cf. [2, 7.4.20].

For odd  $i < 2n$  let  $C_i = \lambda^{-1}[A_i]$  and  $D_i = \lambda^{-1}[B_i]$  and observe that  $\mu^{-1}[C_i] = S \cap A_i$  and  $\mu^{-1}[D_i] = S \cap B_i$ . From these last equalities it follows that the  $C_i$  and  $D_i$  form an essential family in  $T$  and so  $\dim T \geq n$ .

Because  $n \leq \dim T \leq 2n$  we may conclude that  $T$  contains an  $n$ -dimensional continuum  $B_n$ . Since  $\mu$  is monotone and  $S$  is hereditarily indecomposable, so is  $B_n$ .  $\square$

### 3. Bing's Partitions

We shall present a variation of the construction from the previous section to demonstrate the following well-known result.

**Theorem 3.1** (Bing). *Let  $X$  be a continuum and let  $F_0$  and  $F_1$  be disjoint closed sets in  $X$ . Then there are disjoint open neighborhoods  $W_0$  and  $W_1$  of  $F_0$  and  $F_1$ , respectively, such that  $X \setminus (W_0 \cup W_1)$  is hereditarily indecomposable.*

*Proof.* We use the partition  $P$  and the open sets  $M_0$  and  $M_1$  from Section 1.2 again. Let  $\{f_i : i \in \mathbb{N}\}$  be a pliant family for  $X$ .

Choose open neighbourhoods  $W_{0,0}$  and  $W_{1,0}$  of  $F_0$  and  $F_1$  respectively with disjoint closures. Whenever  $i \geq 0$  and the open sets  $W_{0,i}$  and  $W_{1,i}$  with disjoint closures are found apply Urysohn's lemma to get a continuous function  $g_i$  such that  $g_i[W_{0,i}] = \{0\}$  and  $g_i[W_{1,i}] = \{1\}$  and set

$$W_{0,i+1} = u_i^{-1}[M_0] \text{ and } W_{1,i+1} = u_i^{-1}[M_1],$$

where  $u_i = g_i \Delta f_i$ . Because the closures of  $M_0$  and  $M_1$  are disjoint the closures of  $W_{0,i+1}$  and  $W_{1,i+1}$  are disjoint as well. Furthermore, because  $u_i[W_{j,i}] \subseteq \{j\} \times \mathbb{I} \subseteq M_j$  we have  $\text{cl} W_{j,i} \subseteq W_{j,i+1}$  for  $j = 0, 1$ . In the end the sets  $W_0 = \bigcup_{i=0}^{\infty} W_{0,i}$  and  $W_1 = \bigcup_{i=0}^{\infty} W_{1,i}$  are disjoint open neighborhoods of  $F_0$  and  $F_1$ , respectively.

A direct application of Lemma 1.4 shows that  $L = X \setminus (W_0 \cup W_1)$  is hereditarily indecomposable: for every  $i$  the function  $g_i$  is a suitable partner for  $f_i$ .  $\square$

#### 4. Continua of Arbitrary Weight

This section contains some results on nonmetric continua.

##### 4.1. Bing's continua

We begin by showing that nonmetric hereditarily indecomposable continua of any prescribed weight exist.

**Theorem 4.1.** *For every infinite  $\kappa$  there is a hereditarily indecomposable continuum of weight  $\kappa$ . This continuum admits an essential map onto  $\mathbb{I}^\kappa$ .*

*Proof.* We use the proof of Theorem 2.1. Let  $\{f_\alpha : \alpha \in \kappa\}$  be a pliant family of continuous functions on the Tychonoff cube  $\mathbb{I}^\kappa$ . As before let  $X = \bigcap_{\alpha \in \kappa} u_\alpha^{-1}[P]$ , where  $u_\alpha = p_{2\alpha} \Delta f_\alpha$  and  $p_\alpha$  is the projection onto the  $\alpha$ th coordinate. The odd-numbered faces of  $\mathbb{I}^\kappa$  induce an essential family on  $X$ ; it is also essential on some component of  $X$ . This component is the required continuum.  $\square$

##### 4.2. Bing's partitions

There is no general analogue of Theorem 3.1; this follows from the following (well-known) fact.

**Proposition 4.2.** *Let  $\mathbb{L}$  denote the long segment and let  $L$  be any partition between  $\{0\} \times \mathbb{L}$  and  $\{1\} \times \mathbb{L}$  in the product  $\mathbb{I} \times \mathbb{L}$ ; then  $L$  contains a copy of  $\mathbb{I}$ .*

*Proof.* Let  $U$  and  $V$  be disjoint open sets around  $\{0\} \times \mathbb{L}$  and  $\{1\} \times \mathbb{L}$  respectively such that  $L = (\mathbb{I} \times \mathbb{L}) \setminus (U \cup V)$ . Let  $x_0 = \sup\{x : (x, \omega_1) \in U\}$  and  $x_1 = \inf\{x : x > x_0 \text{ and } (x, \omega_1) \in U\}$ . It is clear that  $0 < x_0 \leq x_1 < 1$  and that  $[x_0, x_1] \times \{\omega_1\} \subseteq L$ , so that we are done in case  $x_0 < x_1$ . If  $x_0 = x_1$  then we can find an increasing sequence  $\langle u_n \rangle_n$  and a decreasing sequence  $\langle v_n \rangle_n$  such that  $(u_n, \omega_1) \in U$  and  $(v_n, \omega_1) \in V$  for all  $n$ . Because the sets  $U$  and  $V$  are open and because  $\omega_1$  has uncountable cofinality we can find an  $\alpha < \omega_1$  such that  $\{u_n\} \times [\alpha, \omega_1] \subseteq U$  and  $\{v_n\} \times [\alpha, \omega_1] \subseteq V$  for all  $n$ .

$$\{x_0\} \times [\alpha, \omega_1] \subseteq \text{cl}U \cap \text{cl}V \subseteq L. \quad \square$$

Now consider  $\mathbb{I}^{\omega_1}$  and embed  $\mathbb{L}$  into  $\mathbb{I}^{[1, \omega_1]}$ ; this induces an embedding of  $\mathbb{I} \times \mathbb{L}$  into  $\mathbb{I}^{\omega_1}$  so that  $\{0\} \times \mathbb{L}$  is embedded in the face  $A_0$  and  $\{1\} \times \mathbb{L}$  is embedded in the face  $B_0$ . We see that every partition between  $A_0$  and  $B_0$  contains a copy of  $\mathbb{I}$ .

**Remark 4.3.** An easy modification of the proof of Theorem 3.1 will show that in compact  $F$ -spaces of weight  $\aleph_1$  there are hereditarily indecomposable partitions between any two disjoint closed sets. Under the Continuum Hypothesis this applies to many Čech-Stone remainders such as  $\beta\mathbb{R}^n \setminus \mathbb{R}^n$  and also  $\beta(\omega \times \mathbb{I}^{\omega_1}) \setminus (\omega \times \mathbb{I}^{\omega_1})$ .

### 4.3. Hereditarily indecomposable preimages

In [11, (19.3)] it is proven that every metric continuum is the weakly confluent image of some hereditarily indecomposable metric curve. A map is *weakly confluent* if every continuum in the range is the image of a continuum in the domain.

We shall show that this results holds in the nonmetric situation as well.

**Theorem 4.4.** *Every continuum is the continuous image of a one-dimensional hereditarily indecomposable continuum (of the same weight) by a weakly confluent map.*

For clarity of exposition we prove this theorem in stages; first we show that every continuum is the continuous image of some hereditarily indecomposable continuum of the same weight, then we modify the construction to get a weakly confluent map and finally we show how to make the domain one-dimensional.

*Proof.* [Creating a hereditarily indecomposable preimage] Let  $X$  be a continuum of weight  $\kappa$  and assume that  $X$  is embedded into the Tychonoff cube  $\mathbb{I}^\kappa$ . Let  $K$  be the hereditarily indecomposable continuum from Theorem 4.1 and let  $f : K \rightarrow \mathbb{I}^\kappa$  be an essential map.

For every finite subset  $a$  of  $\kappa$  we consider the map  $\pi_a \circ f : K \rightarrow \mathbb{I}^a$  and the continuum  $\pi_a[X]$ , where  $\pi_a$  is the projection of  $\mathbb{I}^\kappa$  onto  $\mathbb{I}^a$ . Because  $\pi_a \circ f$  is essential we may apply Theorem 4.3 from [5] to find a subcontinuum  $Y_a$  of  $K$  such that  $(\pi_a \circ f)[Y_a] = \pi_a[X]$ . Because  $K$  is compact the net  $\langle Y_a : a \in [\kappa]^{<\omega} \rangle$  has a convergent subnet in  $\mathcal{C}(K)$ ; its limit  $Y$  is a subcontinuum of  $K$  that maps onto  $X$ .  $\square$

To be able to improve this construction so as to make the map weakly confluent we need the following result, which follows from Theorem 3.5 of [4].

**Lemma 4.5.** *Let  $n \in \mathbb{N}$  and let  $X$  be a continuum in  $\mathbb{I}^n \times \{0\}$ . Then there is a copy  $H$  of the half line  $[0, \infty)$  in  $\mathbb{I}^n \times (0, 1]$  such that  $H \cup X = \text{cl } H$  is a continuum with the property that for every continuous surjection  $f : Z \rightarrow \text{cl } H$ , where  $Z$  is a continuum, there is a subcontinuum  $Y$  of  $Z$  such that  $f \upharpoonright Y : Y \rightarrow X$  is onto and weakly confluent.*

Using this lemma and a slightly more complicated proof we can ensure that we get a weakly confluent map from a hereditarily indecomposable continuum onto a given one.

*Proof.* [Creating a weakly confluent preimage] We now assume that our continuum  $X$  is embedded in  $\mathbb{I}^\kappa \times \{0\}$ , which we consider to be a subset of  $\mathbb{I}^\kappa \times \mathbb{I}$ . We take the continuum  $K$  from

Theorem 4.1 and let  $f : K \rightarrow \mathbb{I}^\kappa \times \mathbb{I}$  be an essential map. For every finite subset  $a$  of  $\kappa$  we let  $\pi_a$  denote the projection of  $\mathbb{I}^\kappa \times \mathbb{I}$  onto  $\mathbb{I}^a \times \mathbb{I}$ .

An application of Lemma 4.5 yields for every finite set  $a$  a subcontinuum  $Y_a$  of  $K$  such that  $(\pi_a \circ f)[Y_a] = \pi_a[X]$  and the restriction  $(\pi_a \circ f) \upharpoonright Y_a$  is weakly confluent. As before we take a convergent subnet  $\{Y_\alpha : \alpha \in A\}$  of  $\{Y_a : a \in [\kappa]^{<\omega}\}$  with limit  $Y$ ; then  $f[Y] = X$  and it remains to show that  $f \upharpoonright Y$  is weakly confluent. To this end let  $C$  be a subcontinuum of  $X$  and choose for every  $a$  a subcontinuum  $D_a$  of  $Y_a$  such that  $(\pi_a \circ f)[D_a] = \pi_a[C]$ . The subnet  $\{D_\alpha : \alpha \in A\}$  of  $\{D_a : a \in [\kappa]^{<\omega}\}$  has a convergent subnet  $\{D_\beta : \beta \in B\}$  with limit  $D$ ; it should be clear that  $D \subseteq Y$  and  $f[D] = C$ .  $\square$

Finally we indicate how to get a *one-dimensional* hereditarily indecomposable continuum  $Y_1$  and a weakly confluent map from  $Y_1$  onto  $X$ . For this we need the following lemma.

**Lemma 4.6.** *There are a one-dimensional subcontinuum  $U$  of  $\mathbb{I}^\kappa$  and a monotone surjection  $m : U \rightarrow \mathbb{I}^\kappa$ .*

*Proof.* This is a straightforward generalization of the proof of 19.1 in [11]. Let  $\mathbb{C}$  denote the standard Cantor set in  $\mathbb{I}$ . For  $\alpha \in \kappa$  put  $U_\alpha = \{x \in \mathbb{I}^\kappa : \text{if } \beta \neq \alpha \text{ then } x_\beta \in \mathbb{C}\}$  and set  $U = \bigcup_{\alpha \in \kappa} U_\alpha$ . Just as in [11] one verifies that  $U$  is a closed and connected subset of  $\mathbb{I}^\kappa$ ; to see that  $U$  is one-dimensional one only has to realize that every basic open cover lives on a finite subset of  $\kappa$  and hence that it can be given an open refinement of order 2. Finally, the map  $h^\kappa : U \rightarrow \mathbb{I}^\kappa$ , where  $h : \mathbb{I} \rightarrow \mathbb{I}$  is the Cantor step function, is a monotone map.  $\square$

*Proof.* [A one-dimensional preimage] By the previous lemma we can find a one-dimensional continuum  $X_1$  of the same weight as  $X$  and a monotone surjection  $m : X_1 \rightarrow X$ . Next find a hereditarily indecomposable continuum  $Y$  and a weakly confluent surjection  $f : Y \rightarrow X_1$ . As in the proof of Corollary 2.2

we take the monotone-light factorization of  $f$ , i.e., a space  $Y_1$ , a monotone map  $\mu : Y \rightarrow Y_1$  and a light map  $\lambda : Y_1 \rightarrow X_1$  so that  $f = \lambda \circ \mu$ . Because  $\mu$  is monotone the space  $Y_1$  is hereditarily indecomposable, because  $\lambda$  is light it is one-dimensional and because  $f$  is weakly confluent so is  $\lambda$  and because  $m$  is monotone the map  $m \circ \lambda$  is weakly confluent.  $\square$

#### 4.4. Hereditarily infinite-dimensional spaces

There are many constructions of hereditarily infinite-dimensional continua, i.e., continua with only infinite-dimensional (nontrivial) subcontinua. One such construction, due to Levin [10], admits a striking generalization to higher cardinals.

**Theorem 4.7.** *For every cardinal  $\kappa$  there is a hereditarily indecomposable space of weight  $\kappa$  such that every subcontinuum of it has an essential family of cardinality  $\kappa$ .*

*Proof.* Consider the continuum  $K$  constructed in Theorem 4.1 and its essential family  $E = \{(A_\beta, B_\beta) : \beta \leq \kappa\}$ . Partition  $\kappa$  into  $\kappa$  many sets  $T_\alpha$  of size  $\kappa$  and let  $\{C_\alpha : \alpha < \kappa\}$  be a base for  $K$ .

For each  $\alpha$  let  $W_\alpha$  be the union of all components of  $\text{cl} C_\alpha$  on which  $E_\alpha = \{(A_\beta, B_\beta) : \beta \in T_\alpha\}$  is not essential. Observe that  $W_\alpha$  is open in  $\text{cl} C_\alpha$  and that the family  $E_\alpha$  is not essential on any compact subset of  $W_\alpha$ .

Next let  $U_\alpha = W_\alpha \cap C_\alpha$  for each  $\alpha$  and put  $U = \bigcup_{\alpha < \kappa} U_\alpha$ . The set  $U$  is open in  $K$  and the family  $E$  is not essential on any compact subset of  $U$ : if  $C$  is such a set cover it by finitely many  $U_\alpha$  and use the disjointness of the sets  $T_\alpha$  to make a set of partitions whose intersection misses  $C$ .

It follows that every partition between  $A_\kappa$  and  $B_\kappa$  must meet  $K \setminus U$  and hence that  $K \setminus U$  contains a non-trivial continuum  $H$ . Let  $C$  be any subcontinuum of  $H$ , let  $p \in C$  and fix  $\alpha$  such that  $p \in C_\alpha$  and  $C \not\subseteq \text{cl} C_\alpha$ . Consider the component  $Z$  of  $p$  in  $\text{cl} C_\alpha$ ; because  $K$  is hereditarily indecomposable we have  $Z \subseteq C$ . But then  $Z \cap W_\alpha = \emptyset$  and so  $E_\alpha$  is essential on  $Z$  and hence on  $C$ .  $\square$

## 5. Model-theoretic Considerations

In this section we call attention to the curious fact that many results about compact spaces of uncountable weight can be derived by model-theoretic means; in fact, the Compactness Theorem and the Löwenheim-Skolem Theorem enable one to deduce the uncountable versions directly from the theorems in the metric case.

### 5.1. Wallman spaces

The basis for the model-theoretic approach is Wallman's generalization [14] of Stone's representation theorem for Boolean algebras to distributive lattices. If  $L$  is a distributive lattice (with  $\mathbf{0}$  and  $\mathbf{1}$ ) then there is a compact  $T_1$ -space  $wL$  with a base for its closed sets that is a homomorphic image of  $L$ . The homomorphism is an isomorphism if and only if  $L$  is *disjunctive*, which means: if  $a \not\leq b$  then there is  $c \in L$  such that  $c \leq a$  and  $c \wedge b = \mathbf{0}$ . Every compact  $T_1$ -space  $X$  can be obtained in this way:  $X$  is the Wallman space of its own family of closed sets. From this it is clear that  $wL$  is not automatically Hausdorff; in fact  $wL$  is Hausdorff if and only if  $L$  is *normal*, which is expressed as follows:

$$(5.1) \quad (\forall x)(\forall y)(\exists u)(\exists v)[(x \wedge y = \mathbf{0}) \\ \rightarrow ((x \wedge u = \mathbf{0}) \wedge (y \wedge v = \mathbf{0}) \wedge (u \vee v = \mathbf{1}))].$$

In a similar fashion we can express that  $wL$  is connected or hereditarily indecomposable. The following formula expressed the connectivity of  $wL$ :

$$(5.2) (\forall x)(\forall y)[((x \wedge y = \mathbf{0}) \wedge (x \vee y = \mathbf{1})) \rightarrow ((x = \mathbf{0}) \vee (x = \mathbf{1}))].$$

This suffices because every base for the closed sets of a compact space that is a lattice contains every clopen set of that space. We can read this formula as expressing a property of  $\mathbf{1}$ , to wit “ $\mathbf{1}$  is connected”; we therefore abbreviate it as  $\text{conn}(\mathbf{1})$  and we shall write  $\text{conn}(a)$  to denote Formula 5.2 with  $\mathbf{1}$  replaced by  $a$

and use it to express that  $a$  is connected (or better: the set represented by  $a$  is connected).

To ensure that  $wL$  is hereditarily indecomposable it suffices to have a fold for every pliant foursome from  $L$  and this is exactly what the following formula expresses.

$$(5.3) \quad (\forall x)(\forall y)(\forall u)(\forall v)(\exists z_1, z_2, z_3) \\ \left[ ((x \wedge y = \mathbf{0}) \wedge (x \wedge u = \mathbf{0}) \wedge (y \wedge v = \mathbf{0})) \right. \\ \rightarrow ((x \wedge (z_2 \vee z_3) = \mathbf{0}) \wedge (y \wedge (z_1 \vee z_2) = \mathbf{0}) \\ \wedge (z_1 \wedge z_3 = \mathbf{0}) \wedge (z_1 \wedge z_2 \wedge v = \mathbf{0}) \\ \left. \wedge (z_2 \wedge z_3 \wedge u = \mathbf{0}) \wedge (z_1 \vee z_2 \vee z_3 = \mathbf{1})) \right].$$

## 5.2. Existence of hereditarily indecomposable continua

The existence of the pseudoarc  $\mathbb{P}$  implies that there are one-dimensional hereditarily indecomposable continua of arbitrarily large weight. Indeed, the family of closed sets of  $\mathbb{P}$  is a distributive and disjunctive lattice that satisfies formulas 5.1, 5.2 and 5.3; it also satisfies

$$(5.4) (\forall x_0)(\forall y_0)(\forall x_1)(\forall y_1)(\exists u_0, v_0, u_1, v_1) \\ \left[ ((x_0 \wedge y_0 = \mathbf{0}) \wedge (x_1 \wedge y_1 = \mathbf{0})) \rightarrow ((x_0 \wedge u_0 = \mathbf{0}) \right. \\ \wedge (y_0 \wedge v_0 = \mathbf{0}) \wedge (x_1 \wedge u_1 = \mathbf{0}) \wedge (y_1 \wedge v_1 = \mathbf{0}) \\ \left. \wedge (u_0 \vee v_0 = \mathbf{1}) \wedge (u_1 \vee v_1 = \mathbf{1}) \wedge (u_0 \wedge v_0 \wedge u_1 \wedge v_1 = \mathbf{0})) \right].$$

This formula expresses  $\dim wL \leq 1$  in terms of closed sets, see Section 1.3. Therefore this combination of formulas is consistent and so, by the (upward) Löwenheim-Skolem theorem, it has models of every cardinality. Thus, given a cardinal  $\kappa$  there is a distributive and disjunctive lattice  $L$  of cardinality  $\kappa$  that satisfies 5.1, 5.2, 5.3 and 5.4. The space  $wL$  is compact Hausdorff, connected, hereditarily indecomposable, one-dimensional and of weight  $\kappa$  or less, but with at least  $\kappa$  closed sets. Thus, if  $\kappa \geq 2^\lambda$  then the weight of  $wL$  is at least  $\lambda$ .

To get a space of weight exactly  $\kappa$  we make sure that  $wL$  has at least  $2^\kappa$  many closed sets. To this end we introduce two sets

of  $\kappa$  many constants  $\{a_\alpha : \alpha < \kappa\}$  and  $\{b_\alpha : \alpha < \kappa\}$  and two sets of  $\kappa$  many formulas: for every  $\alpha$  the formula  $a_\alpha \wedge b_\alpha = \mathbf{0}$  and for any pair of disjoint finite subsets  $p$  and  $q$  of  $\kappa$  the formula  $\bigwedge_{\alpha \in p} a_\alpha \wedge \bigwedge_{\alpha \in q} b_\alpha \neq \mathbf{0}$ . Thus we have expanded the language of lattices by a number of constants and we have added a set of formulas to the formulas that we used above. This larger set  $\mathcal{T}_\kappa$  of formulas is still consistent.

Take a finite subset  $T$  of  $\mathcal{T}_\kappa$  and fix a finite subset  $t$  of  $\kappa$  such that whenever  $a_\alpha \wedge b_\alpha = \mathbf{0}$  or  $\bigwedge_{\alpha \in p} a_\alpha \wedge \bigwedge_{\alpha \in q} b_\alpha \neq \mathbf{0}$  belong to  $T$  we have  $\alpha \in t$  and  $p \cup q \subseteq t$ . Now take a map  $f$  from  $\mathbb{P}$  onto the cube  $\mathbb{I}^t$  and interpret  $a_\alpha$  by  $f^{-1}[A_\alpha]$  and  $b_\alpha$  by  $f^{-1}[B_\alpha]$ ; in this way we have ensured that every formula from  $T$  holds in the family of closed subsets of  $\mathbb{P}$ . Therefore  $T$  is a consistent set of formulas and so, because it was arbitrary and by the compactness theorem, the full set  $\mathcal{T}_\kappa$  is consistent.

Because  $\mathcal{T}_\kappa$  has cardinality  $\kappa$  it has a model  $L$  of cardinality  $\kappa$ . Now  $wL$  is as required: its weight is at most  $\kappa$  because  $L$  is a base of cardinality  $\kappa$ . On the other hand: for every subset  $S$  of  $\kappa$ , we have, by compactness, a nonempty closed set

$$F_S = \bigcap_{\alpha \in S} a_\alpha \cap \bigcap_{\alpha \notin S} b_\alpha$$

such that  $F_S \cap F_T = \emptyset$  whenever  $S \neq T$ .

**Remark 5.1.** The reader may enjoy modifying the above argument so as to ensure that  $\{(a_\alpha, b_\alpha) : \alpha < \kappa\}$  is an essential family in  $wL$ . To this end write down, for every finite subset  $a$  of  $\kappa$ , a formula  $\phi_a$  that expresses that  $\{(a_\alpha, b_\alpha) : \alpha \in a\}$  is essential. Theorem 2.1 more than ensures that the set of formulas consisting of 5.1, 5.2, 5.3 and the  $\phi_a$  is consistent.

### 5.3. Hereditarily indecomposable preimages

We can also give a model-theoretic proof of Theorem 4.4. For this we need to know how to ensure that  $wL$  maps onto the given continuum and how to get this surjection to be weakly confluent.

*Making a continuous surjection.* The following lemma tells us how to make continuous surjections.

**Lemma 5.2.** *Let  $X$  be compact Hausdorff and  $L$  some normal, distributive and disjunctive lattice. If  $X$  has a base  $\mathcal{B}$  for the closed sets that is a lattice and embeddable into  $L$  then  $wL$  admits a continuous surjection onto  $X$ .*

*Proof.* We only sketch the argument. Let  $\phi : \mathcal{B} \rightarrow L$  be an embedding and define  $f : wL \rightarrow X$  by “ $f(p)$  is the unique point in  $\bigcap \{C \in \mathcal{B} : p \in \phi(C)\}$ ”. It is straightforward to check that  $f$  is onto and that  $f^{-1}[C] = \phi(C)$  for all  $C$ .  $\square$

This tells us that to get a (one-dimensional) hereditarily indecomposable continuum that maps onto the given continuum  $X$  we need to construct a distributive, disjunctive and normal lattice  $L$  that satisfies formulas 5.2 and 5.3 (and 5.4), and an embedding  $\phi$  of some base  $\mathcal{B}$  for the closed sets of  $X$  into  $L$ .

Let a continuum  $X$  and a lattice-base  $\mathcal{B}$  for its closed sets be given. As before we start with the formulas that ensure that  $wL$  will be a hereditarily indecomposable continuum. To these formulas we add the diagram of  $\mathcal{B}$ ; this consists of  $\mathcal{B}$  itself, as a set of constants, and the ‘multiplication tables’ for  $\wedge$  and  $\vee$ , i.e.,  $A \wedge B = C$  whenever  $A \cap B = C$  and  $A \vee B = C$  whenever  $A \cup B = C$ .

Now, if  $L$  is to satisfy the diagram of  $\mathcal{B}$  it must contain elements  $x_A$  for every  $A \in \mathcal{B}$  so that  $x_{A \wedge B} = x_A \wedge x_B$  and  $x_{A \vee B} = x_A \vee x_B$  hold whenever appropriate; but this simply says that there is an embedding of  $\mathcal{B}$  into  $L$ .

We are left with the task of showing that the set  $\mathcal{T}$  of formulas that express distributivity, disjunctiveness, normality as well as formulas 5.2 and 5.3 (and 5.4), together with the diagram of  $\mathcal{B}$  is consistent. Let  $T$  be a finite subset of  $\mathcal{T}$  and, if necessary, add the first six formulas to it. Let  $\mathcal{B}'$  be a countable, normal and disjunctive sublattice of  $\mathcal{B}$  that contains the finitely many constants that occur in  $T$ . The Wallman space of  $\mathcal{B}'$ , call it  $Y$ ,

is a metric continuum and therefore the continuous image of a hereditarily indecomposable (one-dimensional) continuum  $K$ . The lattice of closed sets of  $K$  satisfies all the formulas from  $T$ : interpret  $A$  by its preimage in  $K$ .

It follows that  $\mathcal{T}$  is consistent and that it therefore has a model  $L$  of the same cardinality as  $\mathcal{T}$ , which is the same as the cardinality of  $\mathcal{B}$ . The lattice  $L$  satisfies all formulas from  $\mathcal{T}$ ; its Wallman space is a (one-dimensional) hereditarily indecomposable continuum that maps onto  $X$ . If  $\mathcal{B}$  is chosen to be of minimal size then  $wL$  is of the same weight as  $X$ .

This proof is much like the model-theoretic proof of the main theorem of [1, Section 2] which says that every continuum of weight  $\aleph_1$  is a continuous image of the Čech-Stone remainder of the real line.

*Making a weakly confluent map.* We now improve the foregoing construction so as to make the continuous surjection weakly confluent.

The following theorem — which is a souped-up version of the Mardešić factorization theorem — implies that it suffices to get some hereditarily indecomposable continuum  $Y$  that admits a weakly confluent map  $f$  onto our continuum  $X$ .

**Theorem 5.3.** *Let  $f : Y \rightarrow X$  be a continuous surjection between compact Hausdorff spaces. Then  $f$  can be factored as  $h \circ g$ , where  $Y \xrightarrow{g} Z \xrightarrow{h} X$  and  $Z$  has the same weight as  $X$  and shares many properties with  $Y$ .*

*Proof.* Let  $\mathcal{B}$  be a lattice-base for the closed sets of  $X$  (of minimal size) and identify it with its copy  $\{f^{-1}[B] : B \in \mathcal{B}\}$  in  $2^Y$ . By the Löwenheim-Skolem theorem [6, Corollary 3.1.5] there is a lattice  $\mathcal{D}$ , of the same cardinality as  $\mathcal{B}$ , such that  $\mathcal{B} \subseteq \mathcal{D} \subseteq 2^Y$  and  $\mathcal{D}$  is an elementary substructure of  $2^Y$ . The space  $Z = w\mathcal{D}$  is as required.  $\square$

Some comments on this theorem and its proof are in order, because they do not seem to say very much. However,

‘elementary substructure’ is an extremely powerful concept. In our context it means that the smaller structure is closed off under every possible finitary lattice-theoretic operation of interest.

For example, if  $Y$  is hereditarily indecomposable then  $2^Y$  is closed under the operation, implicit in formula 5.3, which assigns a fold to every plian and foursome. But then  $\mathcal{D}$  must be closed under this operation as well and hence  $Z$  is hereditarily indecomposable.

Likewise  $\dim Z = \dim Y$ , because if there is an essential family in  $2^Y$  of size  $n$  then there must be one in  $\mathcal{D}$  (use a constant operation that assigns an essential family of size  $n$  to everything) and, conversely, if there is an essential family of size  $n$  in  $\mathcal{D}$  then it is essential in  $2^Y$  as well:  $\mathcal{D}$  is closed under the operation of assigning sequences of partitions with empty intersection to inessential families.

We leave to the reader the verification that if  $f$  is weakly confluent then so is the map  $h$  in the factorization.

Now let  $X$  be a continuum. Our aim is of course to find a lattice  $L$  that contains the diagram of  $2^X$  — to get our continuous surjection  $f$  — and for every  $C \in \mathcal{C}(X)$  a continuum  $C'$  in  $wL$  such that  $f[C'] = C$ .

As before we add the diagram of  $2^X$  to the formulas that guarantee that  $wL$  will be a hereditarily indecomposable continuum. In addition we take a set of constants  $\{C' : C \in \mathcal{C}(X)\}$  and stipulate that  $C'$  will be a continuum that gets mapped onto  $C$ .

To make sure that every  $C'$  is connected we put  $\text{conn}(C')$  into our set of formulas, for every  $C$ . Next,  $f[C'] \subseteq C$  translates, via the embedding into  $L$ , into  $C' \leq C$  (or better  $C' = C' \wedge C$ ). Now, if it happens that  $f[C'] \subsetneq C$  then there is a closed set  $D$  in  $X$  (in fact it is  $f[C']$  but that is immaterial) such that  $C' \leq D$  and  $C \not\leq D$ . In order to avoid this we also add, for every  $C \in \mathcal{C}(X)$  and every  $D \in 2^X$ , the formula

$$(C' \leq D) \rightarrow (C \leq D)$$

to our set of formulas.

Again, the theorem in the metric case implies that this set of formulas is consistent — given a finite subset  $T$  of it make a metric continuum  $X_T$  as before, by expanding  $\{B \in 2^X : B \text{ occurs in } T\}$  to a countable normal sublattice  $\mathcal{B}$  of  $2^X$ ; then find a metric continuum  $Y_T$  of the desired type that admits a weakly confluent map  $f$  onto  $X_T$ ; finally choose for every  $C \in \mathcal{C}(X)$  that occurs in  $T$  a continuum in  $Y_T$  that maps onto  $C$  and assign it to  $C'$ ; this then makes the family of closed sets of  $Y_T$  a model of  $T$ .

As before we obtain a lattice  $L$  whose Wallman space is one-dimensional and hereditarily indecomposable, and which, in addition, admits a weakly confluent map onto  $X$ .

## 6. From Three to Infinity

In this section we shall show that Brouwer's Fixed-point Theorem in dimension three implies all of its higher-dimension versions, using only point-set arguments and a smattering of Linear Algebra. The point-set arguments can be culled from Kelley's proof, from [8], of his theorem that the hyperspace of a (at least) two-dimensional hereditarily indecomposable continuum is infinite-dimensional. To convince the reader that point-set arguments really suffice and to make Kelley's result better known we shall give the argument in full. In this section all continua under consideration are metrizable; we invariably use  $\rho$  to denote a compatible metric and  $\rho_H$  to denote the corresponding Hausdorff metric.

### 6.1. More on hyperspaces

Most of our arguments will take place in the hyperspace  $\mathcal{C}(X)$  of all subcontinua of a two-dimensional hereditarily indecomposable continuum  $X$ .

*Order arcs.* It is well-known that  $\mathcal{C}(X)$  is arcwise connected whenever  $X$  is a metric continuum; in fact if  $A \in \mathcal{C}(X)$  then there is a linearly ordered family  $\mathcal{C}$  of continua containing  $A$

and  $X$  and that is homeomorphic to  $\mathbb{I}$ . For hereditarily indecomposable continua we can give a completely elementary proof of this fact.

**Lemma 6.1.** *Let  $X$  be a hereditarily indecomposable continuum and for  $x \in X$  put  $\mathcal{C}_x = \{C \in \mathcal{C}(X) : x \in C\}$ . Then  $\mathcal{C}_x$  is a chain, whose subspace and order topologies coincide and make it homeomorphic to  $\mathbb{I}$ .*

*Proof.* That  $\mathcal{C}_x$  is a chain follows from hereditary indecomposability of  $X$ . It is clear that  $\mathcal{C}_x$  is complete: if  $\mathcal{F} \subseteq \mathcal{C}_x$  then  $\text{cl} \bigcup \mathcal{F}$  is the supremum of  $\mathcal{F}$  in  $\mathcal{C}_x$ . To see that  $\mathcal{C}_x$  has no jumps take  $C$  and  $D$  in  $\mathcal{C}$  with  $C \subsetneq D$  and fix an open set  $U$  such that  $C \subseteq U$  and  $D \not\subseteq \text{cl} U$ . Now the component  $E$  of  $\text{cl} U$  that contains  $C$  meets the boundary of  $U$ , so  $C \subsetneq E$ , and is contained in  $\text{cl} U$ , so  $E \subsetneq D$ .

The set  $\mathcal{C}_x$  is closed in  $\mathcal{C}(X)$ : its complement,  $\{C : C \subseteq X \setminus \{x\}\}$ , is a basic open set. Likewise the sets  $\{C : C \subseteq A\}$  and  $\{C : A \subseteq C\}$  are closed in  $\mathcal{C}(X)$ ; this shows that the order topology on  $\mathcal{C}_x$  is contained in the subspace topology. Because both topologies are compact Hausdorff they coincide; because this topology is metric we find that  $\mathcal{C}_x$  is isomorphic and homeomorphic to  $\mathbb{I}$ .  $\square$

*Whitney levels.* A Whitney map for  $2^X$  is a continuous function  $\mu : 2^X \rightarrow \mathbb{R}$  such that  $\mu(\{x\}) = 0$  for all  $x$  and  $\mu(C) < \mu(D)$  whenever  $C \subsetneq D$ . If  $X$  is compact metric then there are Whitney maps for  $2^X$ , see [12, 4.33].

We fix a hereditarily indecomposable continuum  $X$  and a Whitney map  $\mu : \mathcal{C}(X) \rightarrow \mathbb{R}$  (we shall work inside  $\mathcal{C}(X)$  only). The fibers  $\mu^{-1}(r)$  ( $0 \leq r \leq \mu(X)$ ) divide  $\mathcal{C}(X)$  into layers, referred to as Whitney levels. We list some properties of Whitney levels.

**Lemma 6.2.** *Every Whitney level is closed.*

**Lemma 6.3.** *Every Whitney level is a pairwise disjoint family of continua.*

*Proof.* Apply hereditary indecomposability. □

**Lemma 6.4.** *Every Whitney level covers  $X$ .*

*Proof.* The function  $\mu$  is continuous and, for every  $x$ , the set  $\mathcal{C}_x$  is an arc that connects  $\{x\}$  and  $X$ ; it follows that  $\mu[\mathcal{C}_x] = [0, \mu(X)]$ . □

One can easily show that  $\rho_H(A, B) < \delta$  implies  $|\text{diam}(A) - \text{diam}(B)| < 2\delta$ , so that  $\text{diam}$  is a continuous function on  $2^X$ . It follows that, for every  $r$ , the diameter function assumes a minimum on the Whitney level  $\mu^{-1}(r)$ . On the other hand, for every positive number  $\varepsilon$  the set  $\{A : \text{diam}(A) < \varepsilon\}$  is an open neighbourhood of the closed set  $\{\{x\} : x \in X\}$ ; it follows, by compactness, that there is a positive number  $s$  such that  $\mu^{-1}(r) \subseteq \{A : \text{diam}(A) < \varepsilon\}$  whenever  $r < s$ .

We now have all the ingredients we need to be able to present Kelley's argument.

## 6.2. Kelley's argument

For the remainder of this section we fix a hereditarily indecomposable continuum  $X$  that is at least two-dimensional and we fix an essential family  $\{(A_0, B_0), (A_1, B_1)\}$  witnessing this.

To begin fix  $\varepsilon > 0$  such that  $\rho(x, y) > \varepsilon$  whenever  $x \in A_i$  and  $y \in B_i$ , where  $i = 0, 1$ . We may assume, without loss of generality, that  $\varepsilon = 1$  (if necessary scale  $\rho$  by the factor  $\frac{1}{\varepsilon}$ ). The following lemma will be used toward the end of Kelley's argument.

**Lemma 6.5.** *Let  $\mathcal{N}$  be a finite disjoint collection of closed sets with diameter at most  $1/2$  in  $X$ . Then there is a continuum in  $X$  of diameter at least 1 that misses all elements from  $\mathcal{N}$ .*

*Proof.* Striving for a contradiction, assume that for  $\mathcal{N}$  such as in the formulation of the lemma, every continuum in  $X \setminus \bigcup \mathcal{N}$  has diameter less than 1. Since  $\mathcal{N}$  is finite, there clearly is a finite disjoint collection  $\mathcal{N}_0$  of closed subsets of  $X$  of mesh less than 1 such that  $\bigcup \mathcal{N}$  is contained in the interior  $W$  of  $\bigcup \mathcal{N}_0$ . Since by assumption each component of  $X \setminus W$  has diameter less than 1, the set  $X \setminus W$  can also be covered by a finite disjoint collection  $\mathcal{N}_1$  of closed sets with diameter less than 1. For  $i = 0$  and 1 let  $S_i$  be a closed set in  $X$  separating the disjoint closed sets

$$C_i = A_i \cup \bigcup \{N \in \mathcal{N}_i : N \cap B_i = \emptyset\} \text{ and}$$

$$D_i = B_i \cup \bigcup \{N \in \mathcal{N}_i : N \cap B_i = \emptyset\}.$$

Then, clearly,  $S_0 \cap S_1 = \emptyset$ , which in turn contradicts our assumption that the pairs  $(A_0, B_0)$  and  $(A_1, B_1)$  form an essential family.  $\square$

Let us now take a Whitney map  $\mu : \mathcal{C}(X) \rightarrow \mathbb{R}$  and fix  $s > 0$  such that the Whitney level  $\mu^{-1}(r)$  is contained in  $\{A : \text{diam}(A) < 1/2\}$ , whenever  $r < s$ . We shall show that  $\mu^{-1}(r)$  is infinite-dimensional whenever  $r < s$ . Fix such an  $r$  and put  $\eta = \min\{\text{diam}(A) : \mu(A) = r\}$ . The following proposition implies that the Whitney level  $\mu^{-1}(r)$  is infinite dimensional — we shall explain this later in Remark 6.8.

**Proposition 6.6.** *Every finite closed cover of  $\mu^{-1}(r)$  of mesh less than  $\eta/(4n)$  has an element that meets at least  $n$  other elements of the cover.*

*Proof.* Put  $\varepsilon = \eta/(4n)$  and assume that  $\mathfrak{F}$  is a finite closed cover of  $\mu^{-1}(r)$  with mesh less than  $\varepsilon$  such that each element of  $\mathfrak{F}$  meets at most  $n - 1$  other elements of  $\mathfrak{F}$ . We shall associate to each element  $\mathcal{A} \in \mathfrak{F}$  a compact subset  $\varphi(\mathcal{A})$  of  $\bigcup \mathcal{A}$  such that

- (1)  $\varphi(\mathcal{A})$  meets every element of  $\mathcal{A}$ ,
- (2)  $\text{diam } \varphi(\mathcal{A}) \leq 2\varepsilon$ ,

- (3)  $\varphi(\mathcal{A}) \cap \varphi(\mathcal{B}) = \emptyset$  whenever  $\mathcal{A}$  and  $\mathcal{B}$  are distinct elements of  $\mathfrak{F}$ .

Assume that  $\varphi$  is already defined on a subfamily  $\mathfrak{G}$  of  $\mathfrak{F}$ , and take  $\mathcal{A}$  in  $\mathfrak{F} \setminus \mathfrak{G}$ ; we show how to extend  $\varphi$  to  $\mathfrak{G} \cup \{\mathcal{A}\}$  (this then means that we can define  $\varphi$  on all of  $\mathfrak{F}$  in finitely many steps).

The set  $\mathfrak{H}$  of all elements of  $\mathfrak{G}$  that meet  $\mathcal{A}$  has, by assumption, cardinality less than  $n$ ; because  $\mathcal{A} \cap \mathcal{B} \neq \emptyset$  iff  $\bigcup \mathcal{A} \cap \bigcup \mathcal{B} \neq \emptyset$  it suffices sure that  $\varphi(\mathcal{A})$  does not meet  $\varphi(\mathcal{B})$  for any  $\mathcal{B}$  in  $\mathfrak{H}$ .

For each  $\mathcal{B}$  in  $\mathfrak{H}$  let  $B(\mathcal{B})$  be the closed  $\varepsilon$ -ball about  $\varphi(\mathcal{B})$  and fix an  $A \in \mathcal{A}$ . We shall show that  $\{B(\mathcal{B}) : \mathcal{B} \in \mathfrak{H}\}$  does not cover  $A$ . Indeed, otherwise, because  $A$  is connected, we could arrange this family into a sequence  $B_1, \dots, B_p$ ,  $p < n$ , with  $B_i \cap \bigcup_{j < i} B_j \neq \emptyset$  for each  $i \leq p$ . But then we could find an upper bound for  $\text{diam } A$ , thus:

$$\text{diam } A \leq (n - 1) \cdot \max_{i \leq p} \text{diam } B_i \leq (n - 1) \cdot (4\varepsilon) < \eta.$$

This would contradict our choice of  $\eta$  as the minimum diameter of the elements of  $\mu^{-1}(r)$ .

Take  $a \in A \setminus \bigcup \{B(\mathcal{B}) : \mathcal{B} \in \mathfrak{H}\}$  and let  $B$  be the closed  $\varepsilon$ -ball about  $a$ . We set  $\varphi(\mathcal{A}) = B \cap \bigcup \mathcal{A}$ .

If  $E \in \mathcal{A}$  then  $\varrho(a, E) \leq \varrho_H(A, E) < \varepsilon$  and so  $B$  meets  $E$ ; clearly  $\text{diam } B \leq 2\varepsilon$ , so  $\varphi(\mathcal{A})$  has the first two required properties. Finally, if  $\mathcal{B} \in \mathfrak{H}$  then  $\varphi(\mathcal{A}) \cap \varphi(\mathcal{B}) \subseteq B \cap \varphi(\mathcal{B}) = \emptyset$ .

The collection

$$\mathcal{N} = \{\varphi(\mathcal{A}) : \mathcal{A} \in \mathfrak{F}\}$$

is finite, disjoint and has mesh less than  $1/2$ . To reach our final contradiction consider any continuum  $C$  in  $X \setminus \bigcup \mathcal{N}$ , take  $E \in \mu^{-1}(r)$  that intersects  $C$  and fix  $\mathcal{A} \in \mathfrak{F}$  with  $E \in \mathcal{A}$ .

Now,  $E$  meets  $\varphi(\mathcal{A})$  and  $C$  does not so  $E \not\subseteq C$ ; but then  $C \subseteq E$ , because  $X$  is hereditarily indecomposable. This, however, means that

$$\text{diam } C \leq \text{diam } E \leq 1/2.$$

This contradicts Lemma 6.5. □

**Remark 6.7.** Although our argument took place in the hyperspace  $\mathcal{C}(X)$  it could have been presented as a decomposition result as well. We have already seen that  $\mu^{-1}(r)$  is a decomposition of  $X$ ; because  $\mu^{-1}(r)$  is a closed subset of  $\mathcal{C}(X)$  one can quite readily show that the decomposition map is actually closed and open. The latter condition implies that the Hausdorff metric defines a compatible metric on the decomposition space. We find that  $X$  admits an open continuous map onto an infinite-dimensional continuum.

### 6.3. From three to infinity

We now make good on our promise by showing that Brouwer's fixed-point theorem for  $\mathbb{I}^3$  implies the full version.

As remarked before, Brouwer's Fixed-Point theorem for  $\mathbb{I}^n$  implies that the pairs of faces of the cube  $\mathbb{I}^n$  form an essential family. Thus, from the version for  $\mathbb{I}^3$  we find that the pairs  $(A_0, B_0)$ ,  $(A_1, B_1)$  and  $(A_2, B_2)$  form an essential family. By Theorem 3.1 let  $L$  be any hereditarily indecomposable partition between  $A_2$  and  $B_2$ . By Lemma 1.5 the traces of the pairs  $(A_0, B_0)$ , and  $(A_1, B_1)$  form an essential family on  $L$ , whence  $\dim L \geq 2$ . There is a component  $X$  of  $S$  on which these traces also form an essential family. We find that Brouwer's theorem for  $\mathbb{I}^3$  implies the existence of an at least two-dimensional hereditarily indecomposable continuum  $X$ . We shall now prove, from this fact, that for every  $m$  the cube  $\mathbb{I}^m$  has the fixed-point property.

Working toward a contradiction we take the first  $m$  such that  $\mathbb{I}^m$  has a fixed-point free map  $f$ . Using  $f$  one can make fixed-point free maps on every  $\mathbb{I}^k$  with  $k \geq m$  and, as is well-known, for every  $k \geq m$  a retraction of  $\mathbb{I}^k$  onto its boundary.

We use the Whitney level  $\mu^{-1}(r)$  and the number  $\eta$  from Proposition 6.6. To begin we set  $n = 3^{2m+1} - 1$  and  $\varepsilon = \eta/(4n)$ . The compact space  $\mu^{-1}(r)$  has many finite open covers of mesh less than  $\varepsilon$ , each of which has a nerve, a polyhedron, associated with it, see [3, 1.10]. The canonical map onto this nerve is an  $\varepsilon$ -map, i.e., each fiber has diameter less than  $\varepsilon$ . We choose

a polyhedron  $P$  of minimal dimension, say  $k$ , such that there is an  $\varepsilon$ -map  $f : \mu^{-1}(r) \rightarrow P$ .

We use Proposition 6.6 to show that  $k \geq m$ . Indeed, assume  $k < m$  and apply Theorem 1.10.4 from [3] to see that  $P$  may be realized inside  $\mathbb{I}^{2k+1}$  (here is where the Linear Algebra is needed). For every  $l$  we can create a closed cover  $\mathcal{F}_l$  of  $\mathbb{I}^{2k+1}$  and hence of  $P$  by cutting along the hyperplanes  $x_i = j/l$ , where  $i < 2k+1$  and  $j = 0, \dots, l$ . Observe that every element of  $\mathcal{F}_l$  meets at most  $3^{2k+1} - 1$  other elements of  $\mathcal{F}_l$ . If  $l$  is taken large enough then the preimage under  $f$  of  $\mathcal{F}_l$  is a finite closed cover of  $\mu^{-1}(r)$  of mesh less than  $\varepsilon$  such that every element meets fewer than  $n$  other elements of the cover. This contradicts Proposition 6.6.

We find that  $k \geq m$ . To reach our final contradiction we consider the successive barycentric subdivisions of  $P$ . In each of these subdivisions we find retractions of the  $k$ -simplices onto their boundaries and combine these into a map  $r : P \rightarrow Q$ , where  $Q$  is the union of the at most  $(k-1)$ -dimensional simplices in the subdivision. For a fine enough subdivision the composition  $r \circ f$  is an  $\varepsilon$ -map from  $\mu^{-1}(r)$  onto a  $(k-1)$ -dimensional polyhedron  $Q$ . This contradicts the minimality of  $k$ .

**Remark 6.8.** The arguments given above imply in particular that  $\mu^{-1}(r)$  cannot be embedded into  $\mathbb{I}^n$  for any  $n$ . The Embedding Theorem ([3, 1.11.4]) now implies that  $\mu^{-1}(r)$  is infinite-dimensional.

This provides another route to Brouwer's Fixed-point theorem. The first step is to observe that  $\mu^{-1}(r)$  has arbitrarily large finite essential families of pairs of closed sets. The third step is to derive the fixed-point theorem for  $\mathbb{I}^n$  from the fact that the faces of  $\mathbb{I}^n$  form an essential family, see [3, 1.8.B]. The intermediate step is provided in the following proposition, which is related to a theorem of Holsztyński from [7].

**Proposition 6.9.** *If some normal space  $X$  has an essential family consisting of  $n$  pairs then the pairs of opposite faces of  $\mathbb{I}^n$  also form an essential family.*

*Proof.* Let  $\{(C_i, D_i) : i < n\}$  be an essential family in the normal space  $X$ . Apply Urysohn's lemma to get continuous functions  $f_i : X \rightarrow \mathbb{I}$  such that  $f_i[C_i] = \{0\}$  and  $f_i[D_i] = \{1\}$  for all  $i$  and take the diagonal map  $f = \Delta_{i < n} f_i$ . If  $L_i$  is a partition between the faces  $A_i$  and  $B_i$  of  $\mathbb{I}^n$  for each  $i$ , then  $f^{-1}[L_i]$  is a partition between  $C_i$  and  $D_i$  and so  $\bigcap_{i < n} f^{-1}[L_i] \neq \emptyset$ ; but then  $\bigcap_{i < n} L_i \neq \emptyset$  as well.  $\square$

## References

- [1] Alan Dow and Klaas Pieter Hart, *A universal continuum of weight  $\aleph$* , Transactions of the American Mathematical Society (2000), posted on June 20, 2000, PII S 0002-9947(00)02601-5 (to appear in print).
- [2] Ryszard Engelking, *General topology. revised and completed edition*, Sigma Series in Pure Mathematics, no. 6, Heldermann Verlag, Berlin, 1989. MR 91c:54001
- [3] ———, *Theory of dimensions, finite and infinite*, Sigma Series in Pure Mathematics, no. 10, Heldermann Verlag, Berlin, 1995. MR 97j:54033
- [4] J. Grispolakis and E. D. Tymchatyn, *Continua which admit only certain classes of onto mappings*, Topology Proceedings **3** (1978), 347–362. MR 80k:54063
- [5] ———, *On confluent mappings and essential mappings — a survey*, Rocky Mountain Journal of Mathematics **11** (1981), 131–153. MR 82k:54055
- [6] Wilfrid Hodges, *Model theory*, Encyclopedia of mathematics and its applications, no. 42, Cambridge University Press, Cambridge, 1993. MR 94e:03002
- [7] W. Holsztyński, *Une généralisation du théorème de Brouwer sur les points invariants*, Bulletin de L'Académie Polonaise des Sciences Série des sciences mathématiques, astronomiques et physiques **12** (1964), 603–606. MR 30 #4248
- [8] John L. Kelley, *Hyperspaces of a continuum*, Transactions of the American Mathematical Society **52** (1942), 22–36. MR 3,315b

- [9] J. Krasinkiewicz and P. Minc, *Mappings onto indecomposable continua*, Bulletin de L'Academie Polonaise des Sciences **25** (1977), 675–680. MR 57 #4119
- [10] Michael Levin, *A short construction of hereditarily infinite-dimensional compacta*, Topology and its Applications **65** (1995), no. 1, 97–99. MR 97b:54044
- [11] T. Maćkowiak and E. D. Tymchatyn, *Continuous mappings on continua. II*, Dissertationes Mathematicae **225** (1984), 57. MR 87a:54048
- [12] Sam B. Nadler, Jr, *Continuum theory*, Pure and Applied Mathematics, no. 158, Marcel Dekker, inc, New York, 1992. MR 93m:54002
- [13] Lex G. Oversteegen and E. D. Tymchatyn, *On hereditarily indecomposable compacta*, Geometric and Algebraic Topology, Banach Center Publications, no. 18, PWN — Polish Scientific Publishers, Warszawa, 1986. MR 88m:54044
- [14] Henry Wallman, *Lattices and topological spaces*, Annals of Mathematics **39** (1938), 112–126.

Faculty of Information Technology and Systems, TU Delft, Postbus 5031, 2600 GA Delft, the Netherlands

*E-mail address:* `k.p.hart@its.tudelft.nl`

URL:<http://aw.twi.tudelft.nl/~hart>

Faculty of Sciences, Division of Mathematics, Vrije Universiteit, De Boelelaan 1081<sup>a</sup>, 1081 HV Amsterdam, The Netherlands

*E-mail address:* `j.van.mill@cs.vu.nl`

Institute of Mathematics, University of Warsaw, Banacha 2, 02-197 Warszawa, Poland

*E-mail address:* `pol@mimuw.edu.pl`