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NON-PRODUCTIVE DUALITY PROPERTIES OF TOPOLOGICAL GROUPS

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Abstract

We address two properties of Abelian topological groups: "every closed subgroup is dually closed" and "every closed subgroup is dually embedded." We exhibit a pair of topological groups such that each has both of the properties and the product has neither, which refutes a remark of N. Noble.

These examples are the additive group of integers topologized with respect to a convergent sequence according to a method of E.G. Zelenyuk and I.V. Protasov. The proof for the product relies on a theorem on exponential Diophantine equations.

1. Introduction

Several duality properties are well-established for locally-compact Abelian groups. Although some are shared beyond the class ([3], [4], [6], [12]), the boundaries may be rather vague. Our concern is for two of them: "every closed subgroup is dually closed" and "every closed subgroup is dually embedded." We denote the former by X(1) and the latter by X(2) after [3].

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Some definitions and conventions follow. All topological groups treated here are Hausdorff and Abelian, and a character is a continuous homomorphism into the torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, unless otherwise stated. A subgroup H of a topological group G is dually closed if for each $g \in G \setminus H$, there exists a character χ of G that separates H and g, i.e., χ is identically zero on H and $\chi(g) \neq 0$. We say that H is dually embedded if each character of H extends to one of G.

That a locally-compact group has X(1) and X(2) is part of the celebrated Pontryagin-van Kampen duality theorem. S. Kaplan [4] shows the same for a product of locally-compact groups. Erroneously referring to the proofs, N. Noble claims in Section 3 of [6] that X(1) and X(2) are each preserved under arbitrary products.

Remark 1.1. Kaplan's arguments only show that a product $\prod_{i\in I} G_i$ of topological groups has X(1) (X(2), respectively) whenever each subproduct $\prod_{i\in J} G_i$ does for any finite $J \subseteq I$. So the problem is reduced to the preservation under finite products, which is obviously the case for locally-compact groups.

We exhibit a counterexample to Noble's assertion above. Let $\mathbf{Z}\{2^n\}$ denote the additive group of integers with the strongest group topology such that 2^n converges to 0, and $\mathbf{Z}\{3^n\}$ similarly (J.W. Nienhuys [5] and E.G. Zelenyuk and I.V. Protasov [13]). Each has both of X(1) and X(2) (Proposition 2.2). We show that the product $\mathbf{Z}\{2^n\} \times \mathbf{Z}\{3^n\}$ has neither (Theorem 2.4) as to the diagonal. The proof is dependent on the fact that the diagonal is closed and discrete (Theorem 4.4), which we get by invoking a theorem on exponential Diophantine equations (cf. [9]).

In Section 2, we investigate some basic properties of the topological group $\mathbf{Z}\{p^n\}$. Section 3 contains the results of number theory necessary for the rest of the argument. The proof of the non-preservation of X(1) and X(2) is completed in Section 4. In Section 5, we construct another counterexample with metrics which have close relation to certain family of exponential Diophantine equations.

2. Group Topologies with a Sequence Convergent

Let G be a group, Abelian or not, and $\langle a_n : n \in \mathbf{N}\rangle$ a sequence in G. Then there exists the strongest group topology on G such that $\langle a_n : n \in \mathbf{N}\rangle$ converges to the neutral element. We denote by $G\{a_n\}$ the topological group G with this topology, which need not be Hausdorff. Zelenyuk and Protasov [13] investigated such topological groups determined by a convergent filter in Abelian and Hausdorff case; unfortunately their recently published monograph [8] has not been available to the author.

The strongest topology thus defined has a useful characterization as follows.

Proposition 2.1. *Suppose that* G *and* $\langle a_n : n \in \mathbf{N}\rangle$ *are as above. Let* H *be a topological group and* $f : G \to H$ *a homomorphism between abstract groups. Then* f *is continuous on* $G\{a_n\}$ *if and only if* $f(a_n) \to 1$ *in* H.

Our main concern is for topological groups of the form $\mathbf{Z}\{p^n\}$ with p prime (cf. [5]). Their characters and closed subgroups are explicitly described in [5] and in [13].

Proposition 2.2.
1. *The closed subgroups of* $\mathbf{Z}\{p^n\}$ *are* $p^m\mathbf{Z}$ *(*$m = 0, 1, \ldots$*) and* $\{0\}$.
2. *The character group of* $\mathbf{Z}\{p^n\}$ *is (identified with)* $\mathbf{Z}[1/p]/\mathbf{R} \subset \mathbf{T}$*, where* $\mathbf{Z}[1/p]$ *is the ring generated by* $1/p$ *over* $\mathbf{Z}$.
3. *Both of* X(1) *and* X(2) *hold for* $\mathbf{Z}\{p^n\}$.

Let p and q be some distinct primes. The characters of the product $\mathbf{Z}\{p^n\} \times \mathbf{Z}\{q^n\}$ are induced by those of the factors, and hence we see how they behave with respect to the diagonal $\Delta = \{\langle u, u\rangle : u \in \mathbf{Z}\}$.

Lemma 2.3.
1. *The diagonal and each element lying outside cannot be separated by the characters.*
2. *The group of characters of* Δ *extendable to the whole product is a proper subgroup of* $\mathbf{T}$.

Proof. Each character χ of $\mathbf{Z}\{p^n\} \times \mathbf{Z}\{q^n\}$ is of form $\chi(u, v) = \chi(1, 0)u + \chi(0, 1)v$ with $\chi(1, 0) \in \mathbf{Z}[1/p]/\mathbf{R}$ and $\chi(0, 1) \in \mathbf{Z}[1/q]/\mathbf{R}$. □

Therefore assuming that the diagonal is closed and discrete (Theorem 4.4), we have the desired non-preservation result.

Theorem 2.4. *The product* $\mathbf{Z}\{p^n\} \times \mathbf{Z}\{q^n\}$ *has neither* X(1) *nor* X(2).

Remark 2.5. By [13, Theorem 6], $\mathbf{Z}\{p^n\}$ is not Fréchet-Urysohn, so not metrizable a fortiori. We observe that it is not α_4 (cf. [1], [2]) due to [13, Lemma 3]; Nyikos [7] proves that Fréchet-Urysohn group is α_4.

3. *S*-unit Equations

We recall a finiteness theorem for exponential Diophantine equations, a special case of [9, Ch. V, Theorem 2A]. Let S be a finite set of primes. A rational number is said to be an S-unit if it belongs to the multiplicative group generated by $S \cup \{-1\}$. The set of S-units is denoted by U_S.

Theorem 3.1. *Up to scalar multiplications, the equation* $x_1 + \cdots + x_k = 0$ *has only finitely many solutions* $\langle x_1, ..., x_k\rangle$ *in* S*-units whose non-trivial subsums do not vanish.*

As a corollary, we have that a certain subsum may attain only finitely many values in the solution set of equations with special form; this finiteness plays a crucial role in the next section.

Lemma 3.2. *Suppose that* S *and* T *are disjoint finite sets of primes. Let the tuple* $\langle x_1, ..., x_k, y_1, ..., y_l\rangle$ *run through the solutions of the equation* $x_1 + \cdots + x_k = y_1 + \cdots + y_l$ *with* $x_i \in U_S \cup \{0\}$ *and* $y_j \in U_T \cup \{0\}$ *for* $1 \le i \le k, 1 \le j \le l$*. Then the sum* $x_1 + \cdots + x_k$ *has only finitely many values.*

Proof. For notational convenience, we consider the equation $x_1+\cdots+x_k+x_{k+1}+\cdots+x_{k+l}=0$ replacing each y_j with $-x_{k+j}$. For $I \subseteq \{i : 1 \le i \le k+l\}$, we say a tuple $\langle x_i : i \in I\rangle$ to be an I-solution and $\sum_{1\le i\le k, i\in I} x_i$ an I-admissible sum if $\sum_{i\in I} x_i = 0$ with $\{x_i : 1 \le i \le k, i \in I\} \subseteq U_S \cup \{0\}$ and $\{x_i : k+1 \le i \le k+l, i \in I\} \subseteq U_T \cup \{0\}$. We shall show that there are only finitely many I-admissible sums.

Note that if I is contained in $\{i : 1 \le i \le k\}$ or $\{i : k+1 \le i \le k+l\}$, which we call a degenerate case, then the only I-admissible sum is 0.

We show the finiteness by the induction on the size of I. If $|I| = 1$, then it is degenerate.

Assume $|I| \ge 2$ and it is not degenerate. Then the equivalence among I-solutions up to scalar multiplications is just the identity. An I-admissible sum is either as to an I-solution whose non-trivial subsums do not vanish or of the form $u+u'$ such that u is a J-admissible sum and u' is an $I \setminus J$-admissible sum for a non-trivial subset J of I. By the theorem and by the induction hypothesis respectively, there are only finitely many such sums. □

4. Reduction to Number Theory

We establish the topological properties of $\mathbf{Z}\{p^n\} \times \mathbf{Z}\{q^n\}$ showing that the result in the previous section is applicable.

Lemma 4.1. *For topological groups of the form $G\{a_n\}, H\{b_n\}$, which need not be Abelian or Hausdorff, the product topology $G\{a_n\} \times H\{b_n\}$ is the same as $(G\times H)\{c_n\}$, where $c_{2n} = \langle a_n, 1\rangle$, $c_{2n+1} = \langle 1, b_n\rangle$.*

Proof. It is straightforward that $c_n \to \langle 1, 1\rangle$ in $G\{a_n\} \times H\{b_n\}$. So the product has at most as strong a topology as that determined by $\langle c_n : n \in \mathbf{N}\rangle$. Hence it is sufficient to show that the identity map $\mathrm{id} : G\{a_n\} \times H\{b_n\} \to (G \times H)\{c_n\}$ is continuous.

In general, the product topology of topological groups G and H has the following characterization. Let $i : G \to G \times H$ and $j : H \to G \times H$ be natural injections: $i(g) = \langle g, 1\rangle$, $j(h) = \langle 1, h\rangle$. Suppose that K is a topological group and $f : G \times H \to K$ is a homomorphism between abstract groups. Then f is continuous if and only if so are both of $f \circ i$ and $f \circ j$.

Due to Proposition 2.1, $\mathrm{id} \circ i : G\{a_n\} \to (G \times H)\{c_n\}$ and $\mathrm{id} \circ j : H\{b_n\} \to (G \times H)\{c_n\}$ are continuous. So we are done. □

We depend on the following two results in [13] concerning Hausdorff Abelian topologial groups of the form $G\{a_n\}$.

Lemma 4.2 ([13, Lemma 2]). *If $g_m \to 0$ in $G\{a_n\}$, then there exists a positive integer k such that $g_m \in \{x_1 + \cdots + x_k : (\forall i)(x_i \in \{\pm a_n : n \in \mathbf{N}\} \cup \{0\})\}$ for sufficiently large m.*

Theorem 4.3 ([13, Theorem 7]). *The topology of $G\{a_n\}$ is sequential.*

Now the last piece of the proof follows. Note that every discrete subgroup of a Hausdorff topological group is closed.

Theorem 4.4. *The diagonal Δ is discrete (and closed) in $\mathbf{Z}\{p^n\} \times \mathbf{Z}\{q^n\}$.*

Proof. Applying Theorem 4.3 to $\mathbf{Z}\{p^n\} \times \mathbf{Z}\{q^n\}$ (Lemma 4.1), we observe that the product is also sequential. So it is sufficient to show that Δ has no nontrivial convergent sequence.

Suppose that $\langle\langle u_i, u_i\rangle : i \in \mathbf{N}\rangle$ is a sequence in Δ converging to $\langle 0, 0\rangle$. By Lemma 4.2, there exists k such that u_i is a sum of less than k numbers in $\{\pm 2^n : n \in \mathbf{N}\}$ and in $\{\pm 3^n : n \in \mathbf{N}\}$, respectively, for sufficiently large i. Due to Lemma 3.2, there are only finitely many such sums. Therefore u_i is eventually equal to 0. □

Remark 4.5. Sequentiality need not be preserved under direct products even for topological groups ([11], [10]).

5. Metric Counterexample

Proposition 2.2 holds not only for $\mathbf{Z}\{p^n\}$ but also for a group topology such that each $p^m\mathbf{Z}$ is closed and $p^n \to 0$. Among those topologies the strongest is that of $\mathbf{Z}\{p^n\}$ and the weakest is the p-adic topology. They have the same closed (open) subgroups and characters.

If $\mathcal{T}$ is such a topology and $\mathcal{U}$ is one for another prime $q \neq p$, then Lemma 2.3 is true for $\langle \mathbf{Z}, \mathcal{T}\rangle \times \langle \mathbf{Z}, \mathcal{U}\rangle$ as well. So we have another counterexample, which we shall construct in this section, provided that the diagonal is discrete.

We utilize some metric group topologies on the integers as in [5]. Suppose that $\delta : \{p^n : n \in \mathbf{N}\} \to \mathbf{R}_{>0}$ is a non-increasing function with $\delta(p^n) \to 0$. We define a function $||\cdot||_\delta : \mathbf{Z} \to \mathbf{R}$ by

$$||u||_\delta = \inf \left\{ \sum_i \delta(p^{n_i}) : u = \sum_i e_i p^{n_i}, e_i \in \{1, -1\}, n_i \in \mathbf{N} \right\}.$$

We denote by $\mathbf{Z}_\delta$ the topological group $\mathbf{Z}$ with the metric induced by $||\cdot||_\delta$. This topology is strictly between the p-adic topology and the strongest topology.

For distinct primes p, q, we construct $\delta : \{p^n : n \in \mathbf{N}\} \to \mathbf{R}$ and $\epsilon : \{q^n : n \in \mathbf{N}\} \to \mathbf{R}$ with certain property. We need another corollary to Theorem 3.1.

Lemma 5.1. *We consider the equation $x_1 + \cdots + x_k = y_1 + \cdots + y_l$ in Lemma 3.2 under the restriction such that no (non-empty) subsum of $x_1 + \cdots + x_k$ or of $y_1 + \cdots + y_l$ vanishes, in particular $x_i \in U_S, y_j \in U_T$. Then the number of solutions is finite.*

Proof. We proceed similarly as in the proof of Lemma 3.2. We change the definition of I-solutions according to the restriction: they are tuples $\langle x_i : i \in I\rangle$ such that $\sum_{i\in I} x_i = 0$, $\{x_i : 1 \leq i \leq k, i \in I\} \subseteq U_S$, $\{x_i : k+1 \leq i \leq k+l, i \in I\} \subseteq U_T$ and no subsum of $\sum_{1\leq i\leq k, i\in I} x_i$ or of $\sum_{k+1\leq i\leq k+l, i\in I} x_i$ vanishes. So there is no I-solutions in the degenerate case.

By the induction on the size of I, we prove the finiteness of I-solutions. Assume that I is not degenerate. Then there are two possibilities for an I-solution: either it has no vanishing non-trivial sums or it is the union of a J-solution and an $I \setminus J$-solution for a non-trivial subset J of I. Hence Theorem 3.1 and the induction hypothesis yields the conclusion. □

Suppose that a non-zero integer u has representations as $u = \sum_{1\leq i\leq k} e_i p^{m_i} = \sum_{1\leq j\leq l} f_j q^{n_j}$ with $e_i, f_j \in \{1, -1\}$ and $m_i, n_j \in \mathbf{N}$. To estimate $||u||_\delta$ and $||u||_\epsilon$, we may assume that no subsum is zero. Due to the lemma above, for fixed k, l, there are only finitely many summands p^{m_i} which appear in such representations for some non-zero integer u. Let $F(p, q, k, l)$ denote the finite set of such p^{m_i}. For a positive integer s, set $F(p, q, s) = \bigcup_{k,l\leq s} F(p, q, k, l)$. Then it is a finite set non-decreasing with respect to s and $\bigcup_{s\in\mathbf{N}} F(p, q, s) = \{p^n : n \in \mathbf{N}\}$.

Now we define particular δ and ϵ by

$$\delta(p^n) = 1/\min\{s : p^n \leq \max F(p, q, s)\},$$

$$\epsilon(q^n) = 1/\min\{s : q^n \leq \max F(q, p, s)\}.$$

Then they are non-increasing and tend to 0. Moreover we have

$$p^n \in F(p, q, s) \Rightarrow \delta(p^n) \geq 1/s,$$

$$q^n \in F(q, p, s) \Rightarrow \epsilon(q^n) \geq 1/s.$$

We estimate $||u||_\delta$ and $||u||_\epsilon$ for a integer u with the representations above. If we set $s = \max\{k, l\}$, then $\delta(p^{m_i}), \epsilon(q^{n_j}) \geq 1/s$ for each summand. Without loss of generality, we may assume $k \geq l$. Then we have

$$\sum_{1\leq i\leq k} \delta(p^{m_i}) \geq k \cdot (1/s) = 1,$$

and hence $||u||_\delta \geq 1$.

Accordingly it does not occur that both $||u||_\delta$ and $||u||_\epsilon$ are less than 1 for a non-zero integer u. Therefore the diagonal is discrete in $\mathbf{Z}_\delta \times \mathbf{Z}_\epsilon$.

Theorem 5.2. *Neither* X(1) *nor* X(2) *is preserved under the product* $\mathbf{Z}_\delta \times \mathbf{Z}_\epsilon$ *for* δ *and* ϵ *decreasing slowly enough.*

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