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INFINITE PARTITION REGULAR MATRICES, II – EXTENDING THE FINITE RESULTS

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Abstract

A finite or infinite matrix A is *image partition regular* provided that whenever $\mathbb{N}$ is finitely colored, there must be some $\vec{x}$ with entries from $\mathbb{N}$ such that all entries of $A\vec{x}$ are in the same color class. Using the algebraic structure of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$, along with a good deal of elementary combinatorics, we investigate the degree to which the known characterizations of finite image partition regular matrices can be extended to infinite image partition regular matrices. We also describe new ways of constructing infinite image partition regular matrices.

1. Introduction

In 1993 several characterizations of finite *image partition regular* matrices were obtained [5]. A $u \times v$ matrix A is image partition regular if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in {C_i}^u$. (Here and elsewhere we use the $\vec{x}$ notation for both column and row vectors, expecting the reader to rely on the context to

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tell which is intended.) More recently, in [6], several additional characterizations of finite image partition regular matrices were obtained.

Image partition regular matrices are of special interest because many of the classical theorems of Ramsey Theory are naturally stated as statements about image partition regular matrices. For example, Schur's Theorem [11] and the length 4 version of van der Waerden's Theorem [12] amount to the assertions that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

are image partition regular.

The notion of image partition regular matrices extends naturally to infinite $\omega \times \omega$ matrices, provided the matrix has only finitely many nonzero entries in each row. (Here ω, the first infinite cardinal, is also the set of nonnegative integers. We take $\mathbb{N}$ to be the set of positive integers.) These matrices also occur naturally in Ramsey Theory. For example, the Finite Sums Theorem (see [4, Theorem 3.15] or [9, Corollary 5.9]) is the assertion that the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 1 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ 0 & 1 & 1 & \ldots \\ 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

(whose rows are all vectors with entries from $\{0, 1\}$ with only finitely many 1's and not all 0's) is image partition regular.

Previous results in [2] and [7] have shown that none of the simple characterizations of finite image partition regular matrices apply to infinite matrices. In this paper we shall be concerned with the extent to which the known results about finite image partition regular matrices can be extended.

Several characterizations of finite image partition regular matrices involve the notion of a "first entries matrix", a concept based on Deuber's (m, p, c) sets. We follow here, and elsewhere, the custom of denoting the entries of a matrix by the lower case letter corresponding to the upper case letter denoting the matrix.

Definition 1.1. Let A be a $u \times v$ matrix with rational entries. Then A is a *first entries matrix* if and only if no row of A is $\vec{0}$ and there exist $d_1, d_2, \ldots, d_v \in \{x \in \mathbb{Q} : x > 0\}$ such that, whenever $i \in \{1, 2, \ldots, u\}$ and $l = \min\{j \in \{1, 2, \ldots, v\} : a_{i,j} \neq 0\}$, one has $a_{i,l} = d_l$. If there exists $i \in \{1, 2, \ldots, u\}$ such that $l = \min\{j \in \{1, 2, \ldots, v\} : a_{i,j} \neq 0\}$, then d_l is a *first entry* of A.

A $u \times v$ matrix A is *kernel partition regular* if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in {C_i}^v$ such that $A\vec{x} = \vec{0}$. In 1933 R. Rado showed that A is kernel partition regular if and only if A satisfies a computable property called the *columns condition*. One of the characterizations of finite image partition regular matrices converts the problem into the determination of whether a certain matrix is kernel partition regular, thereby allowing the use of Rado's Theorem.

Definition 1.2. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be the columns of A. The matrix A satisfies the *columns condition* if and only if there exist $m \in \mathbb{N}$ and $I_1, I_2, \ldots, I_m$ such that

(1) $\{I_1, I_2, \ldots, I_m\}$ is a partition of $\{1, 2, \ldots, v\}$,

(2) $\sum_{i \in I_1} \vec{c_i} = \vec{0}$, and

(3) if $m > 1$ and $t \in \{2, 3, \ldots, m\}$, then $\sum_{i \in I_t} \vec{c_i}$ is a linear combination of $\{\vec{c_i} : i \in \bigcup_{j=1}^{t-1} I_j\}$.

It is not hard to show that the $u \times v$ matrix A satisfies the columns condition if and only if there exist m and a $v \times m$ first entries matrix B such that $AB = \mathbf{O}$, where $\mathbf{O}$ is the $u \times m$ matrix with all zero entries.

Theorem 1.3 (Rado). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The matrix A is kernel partition regular if and only if A satisfies the columns condition.*

Proof. [10]. Or see [4, Theorem 3.5] or [9, Theorem 15.20]. □

Some of the known characterizations of finite image partition regular matrices involve the notion of *central* sets. Central sets were introduced by Furstenberg [3] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [3, Proposition 8.21] or [9, Chapter 14].) They have a nice characterization in terms of the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. We shall present this characterization below, after introducing the necessary background information.

Let $(S, +)$ be an infinite discrete semigroup. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $+$ of S to βS making βS a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q + p$ and $\lambda_x(q) = x + q$. We are denoting the operation by $+$ because we shall be primarily concerned with the semigroup $(\mathbb{N}, +)$. However, the reader should be cautioned that, even if the operation on S is commutative, it is very unlikely to be commutative on βS. See [9] for an elementary introduction to the semigroup βS.

Any compact Hausdorff right topological semigroup $(T, +)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed [9, Theorem 2.8], and any compact right topological semigroup contains idempotents. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p + q = q + p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [9, Theorem 1.59]. Such an idempotent is called simply "minimal"

Definition 1.4. Let $(S, +)$ be an infinite discrete semigroup. A set $A \subseteq S$ is *central* if and only if there is some minimal idempotent p such that $A \in p$.

See [9, Theorem 19.27] for a proof of the equivalence of the definition above with the original dynamical definition.

We present now most of the known characterizations of finite image partition regular matrices. We write $\mathbb{Q}^+$ for $\{x \in \mathbb{Q} : x > 0\}$.

Theorem 1.5. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) A is image partition regular.

(b) For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.

(c) For every central set C in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v$: such that $A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.

(d) There exist $t_1, t_2, \ldots, t_v \in \mathbb{Q}^+$ such that the matrix

$$M = \begin{pmatrix} t_1a_{1,1} & t_2a_{1,2} & t_3a_{1,3} & \ldots & t_va_{1,v} & -1 & 0 & 0 & \ldots & 0 \\ t_1a_{2,1} & t_2a_{2,2} & t_3a_{2,3} & \ldots & t_va_{2,v} & 0 & -1 & 0 & \ldots & 0 \\ t_1a_{3,1} & t_2a_{3,2} & t_3a_{3,3} & \ldots & t_va_{3,v} & 0 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1a_{u,1} & t_2a_{u,2} & t_3a_{u,3} & \ldots & t_va_{u,v} & 0 & 0 & 0 & \ldots & -1 \end{pmatrix}$$

is kernel partition regular.

(e) *There exist* $b_1, b_2, \ldots, b_v \in \mathbb{Q}^+$ *such that the matrix*

$$N = \begin{pmatrix} b_1 & 0 & 0 & \ldots & 0 \\ 0 & b_2 & 0 & \ldots & 0 \\ 0 & 0 & b_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & b_v \\ & & A & & \end{pmatrix}$$

is image partition regular.

(f) *There exist* $t_1, t_2, \ldots, t_v \in \mathbb{Q}^+$ *such that the matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_1 a_{1,1} & t_2 a_{1,2} & t_3 a_{1,3} & \ldots & t_v a_{1,v} \\ t_1 a_{2,1} & t_2 a_{2,2} & t_3 a_{2,3} & \ldots & t_v a_{2,v} \\ t_1 a_{3,1} & t_2 a_{3,2} & t_3 a_{3,3} & \ldots & t_v a_{3,v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 a_{u,1} & t_2 a_{u,2} & t_3 a_{u,3} & \ldots & t_v a_{u,v} \end{pmatrix}$$

is image partition regular.

(g) *There exist* $m \in \mathbb{N}$ *and a* $u \times m$ *first entries matrix* B *such that for each* $\vec{y} \in \mathbb{N}^m$ *there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $A\vec{x} = B\vec{y}$.

(h) *There exist* $m \in \mathbb{N}$, *a* $u \times m$ *first entries matrix* B *with all entries from* ω, *and* $c \in \mathbb{N}$ *such that* c *is the only first entry of* B *and for each* $\vec{y} \in \mathbb{N}^m$ *there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $A\vec{x} = B\vec{y}$.

(i) *There exist* $m \in \mathbb{N}$, *a* $v \times m$ *matrix* G *with entries from* ω *and no row equal to* $\vec{0}$, *and a* $u \times m$ *first entries matrix* B *with entries from* ω *such that* $AG = B$.

(j) For each $\vec{r} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ *there exists* $b \in \mathbb{Q} \setminus \{0\}$ *such that*

$$\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$$

is image partition regular.

(k) Whenever $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ *are non zero linear mappings from* $\mathbb{Q}^v$ *to* $\mathbb{Q}$, *and* C *is central in* $\mathbb{N}$, *there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $A\vec{x} \in C^u$ *and, for each* $i \in \{1, 2, \ldots, m\}$, $\phi_i(\vec{x}) \neq 0$.

(l) For every central set C *in* $\mathbb{N}$, *there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $\vec{y} = A\vec{x} \in C^u$, *all entries of* $\vec{x}$ *are distinct, and for all* $i, j \in \{1, 2, \ldots, u\}$, *if rows* i *and* j *of* A *are unequal, then* $y_i \neq y_j$.

Proof. [6, Theorem 2.3]. □

It is an immediate consequence of Theorem 1.5(b) that whenever A and B are finite image partition regular matrices, so is $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$, where $\mathbf{O}$ represents a matrix of the appropriate size with all zero entries. (This fact was first established by W. Deuber in [1].) However, it is a consequence of [2, Theorem 3.14] that the corresponding result is not true for infinite image partition regular matrices. (Technically, if A and B are $\omega \times \omega$ matrices, then $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$ is an $(\omega + \omega) \times (\omega + \omega)$ matrix. This is not a substantive distinction, and we shall ignore it.)

Inspired by this distinction and by the condition of Theorem 1.5(l), we introduced in [7] the notions of "centrally image partition regular" matrices and "strongly centrally image partition regular" matrices.

Definition 1.6. Let A be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$.

(a) The matrix A is *centrally image partition regular* if and only if for every central subset C of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^\omega$ such that $A\vec{x} \in C^\omega$.

(b) The matrix A is *strongly centrally image partition regular* if and only if for every central subset C of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^\omega$ such that $A\vec{x} \in C^\omega$ and entries of $A\vec{x}$ corresponding to distinct rows of A are distinct.

In Section 2 we show that there are severe limitations on the combinations of row patterns that occur often in infinite image partition regular matrices.

In Section 3 we investigate an infinite analogue of the notion of first entries matrix and the extent to which analogues of the implications in Theorem 1.5 hold.

In Section 4 we present some classes of matrices that are to be image partition regular, as well as methods of constructing such matrices from known image partition regular matrices.

2. Digit Patterns in Rows

In [2] it was shown that matrices with positive digits occurring in a fixed pattern are image partition regular, and in [7] this result was extended to allow negative entries. (See Theorem 2.2 below.) In this section we show that there are severe restrictions on the combinations of digit patterns that can occur in image partition regular matrices.

Definition 2.1. Let $\vec{x} \in \mathbb{Z}^\omega$. Then

(a) $d(\vec{x})$ is the sequence obtained by deleting all occurrences of 0 from $\vec{x}$;

(b) $c(\vec{x})$ is the sequence obtained by deleting every digit in $d(\vec{x})$ which is equal to its predecessor; and

(c) $\vec{x}$ is a *compressed* sequence if and only if $\vec{x} = c(\vec{x})$.

For example, if $\vec{x} = \langle -1, 0, -1, 3, 0, 2, 2, 0, 2, 0, 0, \ldots \rangle$, then $d(\vec{x}) = \langle -1, -1, 3, 2, 2, 2 \rangle$ and $c(\vec{x}) = \langle -1, 3, 2 \rangle$.

Theorem 2.2. *Let $\vec{a}$ be a (finite) sequence in $\mathbb{Z} \setminus \{0\}$ such that $c(\vec{a}) = \vec{a}$ and the last entry of $\vec{a}$ is positive. Let A be an $\omega \times \omega$ matrix such that for each row $\vec{r}$ of A, $c(\vec{r}) = \vec{a}$. Then A is image partition regular.*

Proof. This is [2, Theorem 2.5] and [7, Corollary 3.6]. □

Lemma 2.3. *Let $k, m \in \omega$, let $\langle a_0, a_1, \ldots, a_k \rangle \in (\mathbb{Z}\setminus\{0\})^{k+1}$, let $\langle b_0, b_1, \ldots, b_m \rangle \in (\mathbb{Z}\setminus\{0\})^{m+1}$, and let A be an $\omega \times \omega$ matrix with the property that every $\vec{z} \in \mathbb{Z}^\omega$ such that $d(\vec{z}) = \vec{a}$ or $d(\vec{z}) = \vec{b}$ occurs as a row of A.*

(i) If there exists $\vec{x} \in \mathbb{N}^\omega$ such that all entries of $A\vec{x}$ are in $\mathbb{N}$ and $\{x_n : n \in \omega\}$ is bounded, then $\sum_{i=0}^{k} a_i > 0$ and $\sum_{i=0}^{m} b_i > 0$.

(ii) If there exists $\vec{x} \in \mathbb{N}^\omega$ such that all entries of $A\vec{x}$ are in $\mathbb{N}$ and $\{x_n : n \in \omega\}$ is unbounded, then $a_k > 0$ and $b_m > 0$.

(iii) If $a_k \neq b_m$, then there is a coloring of $\mathbb{N}$ (with at most 3 colors) such that there is no $\vec{x} \in \mathbb{N}^\omega$ with all entries of $A\vec{x}$ monochrome and $\{x_n : n \in \omega\}$ unbounded.

(iv) If $\sum_{i=0}^{k} a_i \neq \sum_{i=0}^{m} b_i$, then there is a coloring of $\mathbb{N}$ (with at most 2 colors) such that there is no $\vec{x} \in \mathbb{N}^\omega$ with all entries of $A\vec{x}$ monochrome and $\{x_n : n \in \omega\}$ bounded.

Proof. Let $l = \max\{k, m\}$.

(i). Pick $n_0 < n_1 < \ldots < n_l$ and v with $x_{n_0} = x_{n_1} = \ldots = x_{n_l} = v$. Then $\sum_{i=0}^{k} a_i x_{n_i} = (\sum_{i=0}^{k} a_i) \cdot v$ and $\sum_{i=0}^{m} b_i x_{n_i} = (\sum_{i=0}^{m} b_i) \cdot v$ are entries of $A\vec{x}$ and so are in $\mathbb{N}$.

(ii). Pick $n \geq l$ such that

$$x_n > \max\{a_0 x_0 + a_1 x_1 + \ldots + a_{k-1} x_{k-1}, b_0 x_0 + b_1 x_1 + \ldots + b_{m-1} x_{m-1}\}.$$

Then $a_0x_0+a_1x_1+\ldots+a_{k-1}x_{k-1}+a_kx_n$ and $b_0x_0+b_1x_1+\ldots+b_{m-1}x_{k-1}+b_mx_n$ are entries of $A\vec{x}$ and so are in $\mathbb{N}$.

(iii). If either $a_k<0$ or $b_m<0$, then by (ii) a single color will do. Therefore we assume that $a_k>0$ and $b_m>0$. We assume without loss of generality that $a_k<b_m$ and pick $\alpha\in\mathbb{R}$ such that $1<\alpha<\frac{b_m}{a_k}<\alpha^2$ and define $\varphi:\mathbb{N}\to\{0,1,2\}$ so that $\varphi(x)\equiv\lfloor\log_\alpha x\rfloor \pmod 3$. Suppose that we have $\vec{x}\in\mathbb{N}^\omega$ such that the entries of $A\vec{x}$ are monochrome and $\{x_n:n\in\omega\}$ is unbounded.

Let $w=\max\{|a_0x_0+a_1x_1+\ldots+a_{k-1}x_{k-1}|,|b_0x_0+b_1x_1+\ldots+b_{m-1}x_{m-1}|\}$. (Of course, if say $k=0$, then $a_0x_0+a_1x_1+\ldots+a_{k-1}x_{k-1}=0$.) Pick $n\geq l$ such that

$$x_n>\max\left\{w,\frac{(\alpha+1)\cdot w}{b_m-\alpha a_k},\frac{(1+\alpha^2)\cdot w}{a_k\alpha^2-b_m}\right\}.$$

We have that $a_0x_0+a_1x_1+\ldots+a_{k-1}x_{k-1}+a_kx_n$ and $b_0x_0+b_1x_1+\ldots+b_{m-1}x_{k-1}+b_mx_n$ are entries of $A\vec{x}$. Pick $t,s\in\mathbb{N}$ such that $\alpha^t\leq a_0x_0+a_1x_1+\ldots+a_{k-1}x_{k-1}+a_kx_n<\alpha^{t+1}$ and $\alpha^s\leq b_0x_0+b_1x_1+\ldots+b_{m-1}x_{k-1}+b_mx_n<\alpha^{s+1}$. Then $t\equiv s \pmod 3$. Now by the choice of x_n,

$$\begin{aligned}\alpha &< \frac{-w+b_mx_n}{w+a_kx_n}\\ &\leq \frac{b_0x_0+b_1x_1+\ldots+b_{m-1}x_{k-1}+b_mx_n}{a_0x_0+a_1x_1+\ldots+a_{k-1}x_{k-1}+a_kx_n}\\ &\leq \frac{w+b_mx_n}{-w+a_kx_n}<\alpha^2\end{aligned}$$

and so $\alpha^{t+1}\leq b_0x_0+b_1x_1+\ldots+b_{m-1}x_{k-1}+b_mx_n<\alpha^{t+3}$. Thus $s=t+1$ or $s=t+2$, a contradiction.

(iv). If either $\sum_{i=0}^k a_i\leq 0$ or $\sum_{i=0}^m b_i\leq 0$, then by (i) a single color will do. So assume that $\sum_{i=0}^k a_i>0$ and $\sum_{i=0}^m b_i>0$.

Assume without loss of generality that $\sum_{i=0}^{k} a_i < \sum_{i=0}^{m} b_i$ and let

$$\alpha = \frac{\sum_{i=0}^{m} b_i}{\sum_{i=0}^{k} a_i}.$$

Define $\varphi : \mathbb{N} \to \{0,1\}$ so that $\varphi(x) \equiv \lfloor \log_\alpha x \rfloor \pmod 2$. Suppose that we have $\vec{x} \in \mathbb{N}^\omega$ such that the entries of $A\vec{x}$ are monochrome and $\{x_n : n \in \omega\}$ is bounded. Pick $n_0 < n_1 < \ldots < n_l$ and v such that $x_{n_0} = x_{n_1} = \ldots = x_{n_l} = v$. Then $\sum_{i=0}^{k} a_i x_{n_i} = v \cdot \sum_{i=0}^{k} a_i$ and $\sum_{i=0}^{m} b_i x_{n_i} = v \cdot \sum_{i=0}^{m} b_i$ are both entries of $A\vec{x}$ and $\varphi(v \cdot \sum_{i=0}^{m} b_i) = \varphi(v \cdot \sum_{i=0}^{k} a_i) + 1$, a contradiction. □

Theorem 2.4. *Let $k, m \in \omega$, let $\langle a_0, a_1, \ldots, a_k \rangle \in (\mathbb{Z} \setminus \{0\})^{k+1}$, let $\langle b_0, b_1, \ldots, b_m \rangle \in (\mathbb{Z} \setminus \{0\})^{m+1}$, and let A be an $\omega \times \omega$ matrix with the property that every $\vec{z} \in \mathbb{Z}^\omega$ such that $d(\vec{z}) = \vec{a}$ or $d(\vec{z}) = \vec{b}$ occurs as a row of A. If A is image partition regular, then either $a_k = b_m > 0$ or $\sum_{i=0}^{k} a_i = \sum_{i=0}^{m} b_i > 0$.*

Proof. By Lemma 2.3(iii) and (iv), either $a_k = b_m$ or $\sum_{i=0}^{k} a_i = \sum_{i=0}^{m} b_i$.

Assume first that $a_k = b_m < 0$. Then by Lemma 2.3(i) and (ii), $\sum_{i=0}^{k} a_i > 0$ and $\sum_{i=0}^{m} b_i > 0$. Suppose that $\sum_{i=0}^{k} a_i \neq \sum_{i=0}^{m} b_i > 0$ and pick a coloring of $\mathbb{N}$ as guaranteed by Lemma 2.3(iv). Pick $\vec{x} \in \mathbb{N}^\omega$ such that the entries of $A\vec{x}$ are monochrome. Then by Lemma 2.3(iv), $\{x_n : n \in \omega\}$ is unbounded so by Lemma 2.3(ii) $a_k > 0$ and $b_k > 0$, a contradiction.

The assertion that $\sum_{i=0}^{k} a_i = \sum_{i=0}^{m} b_i \leq 0$ is handled similarly. □

Theorem 2.5. *Let $k, m \in \omega$, let $\langle a_0, a_1, \ldots, a_k \rangle \in (\mathbb{Z} \setminus \{0\})^{k+1}$ with $\sum_{i=0}^{k} a_i \neq 0$ and let $\langle b_0, b_1, \ldots, b_m \rangle \in (\mathbb{Z} \setminus \{0\})^{m+1}$ with $\sum_{i=0}^{m} b_i \neq 0$. Let A be an $\omega \times \omega$ matrix with the property that every $\vec{z} \in \mathbb{Z}^\omega$ such that $d(\vec{z}) = \vec{a}$ or $d(\vec{z}) = \vec{b}$ occurs as a row of A. If A is image partition regular, then either $a_0 = b_0$ or $\sum_{i=0}^{k} a_i = \sum_{i=0}^{m} b_i$.*

Proof. Let $l = \max\{k, m\}$. Pick a prime

$$r > \max\left\{|a_0|, |b_0|, |a_0 - b_0|, |\sum_{i=0}^{k} a_i|, |\sum_{i=0}^{m} b_i|, |\sum_{i=0}^{k} a_i - \sum_{i=0}^{m} b_i|\right\}.$$

For each $x \in \mathbb{Z}\setminus\{0\}$, let $\gamma(x) = \max\{t \in \omega : x \in r^t\mathbb{Z}\}$ and pick $f(x) \in \{1, 2, \ldots, r-1\}$ such that $\dfrac{x}{r^{\gamma(x)}} \equiv f(x) \pmod r$. (Thus $f(x) = j \in \{1, 2, \ldots, r-1\}$ if and only if there is some $z \in \mathbb{Z}$ such that $x = z \cdot r^{\gamma(x)+1} + j \cdot r^{\gamma(x)}$.) Observe that for any $x \in \mathbb{Z}\setminus\{0\}$ and any $c \in \{1, 2, \ldots, r-1\}$, $f(c \cdot x) \equiv c \cdot f(x) \pmod r$.

Pick $\vec{x} \in \mathbb{N}^\omega$ such that the entries of $A\vec{x}$ are monochrome with respect to f. Assume first that $\{\gamma(x_n) : n \in \omega\}$ is unbounded. Let $t = \gamma(x_0)$ and pick $n_1 < n_2 < \ldots < n_l$ such that $\gamma(x_{n_i}) > t$ for each $i \in \{1, 2, \ldots, l\}$. Let $j = f(x_0)$. Then $f(a_0x_0 + a_1x_{n_1} + \ldots + a_kx_{n_k}) \equiv a_0 \cdot j \pmod r$ and $f(b_0x_0 + b_1x_{n_1} + \ldots + b_mx_{n_m}) \equiv b_0 \cdot j \pmod r$. Since $a_0x_0 + a_1x_{n_1} + \ldots + a_kx_{n_k}$ and $b_0x_0 + b_1x_{n_1} + \ldots + b_mx_{n_m}$ are entries of $A\vec{x}$, we have that $a_0 \cdot j \equiv b_0 \cdot j \pmod r$. Since $r > |a_0 - b_0|$, we have that $a_0 = b_0$.

Now assume that $\{\gamma(x_n) : n \in \omega\}$ is bounded. Pick by the pigeon hole principle $t \in \omega$, $j \in \{1, 2, \ldots, r-1\}$, and $n_0 < n_1 < \ldots < n_l$ such that $\gamma(x_{n_0}) = \gamma(x_{n_1}) = \ldots = \gamma(x_{n_l}) = t$ and $f(x_{n_0}) = f(x_{n_1}) = \ldots = f(x_{n_l}) = j$. For $i \in \{0, 1, \ldots, l\}$ pick $y_i \in \mathbb{Z}$ such that $x_{n_i} = y_i \cdot r^{t+1} + j \cdot r^t$. Then

$$\sum_{i=0}^{k} a_i x_{n_1} = (\sum_{i=0}^{k} a_i y_i) \cdot r^{t+1} + (\sum_{i=0}^{k} a_i) \cdot j \cdot r^t .$$

Since $\sum_{i=0}^{k} a_i \neq 0$ and $r > |\sum_{i=0}^{k} a_i|$ we have that $f(\sum_{i=0}^{k} a_i x_{n_i}) \equiv (\sum_{i=0}^{k} a_i) \cdot j \pmod r$. Similarly $f(\sum_{i=0}^{m} b_i x_{n_i}) \equiv (\sum_{i=0}^{m} b_i) \cdot j \pmod r$. Since $\sum_{i=0}^{k} a_i x_{n_i}$ and $\sum_{i=0}^{m} b_i x_{n_i}$ are entries of $A\vec{x}$, we have that $(\sum_{i=0}^{k} a_i) \cdot j \equiv (\sum_{i=0}^{m} b_i) \cdot j \pmod r$. Since $r > |\sum_{i=0}^{k} a_i - \sum_{i=0}^{m} b_i|$, we have that $\sum_{i=0}^{k} a_i = \sum_{i=0}^{m} b_i$. □

Corollary 2.6. *Let $k, m \in \omega$, let $\langle a_0, a_1, \ldots, a_k\rangle \in (\mathbb{Z}\setminus\{0\})^{k+1}$ with $\sum_{i=0}^{k} a_i \neq 0$ and let $\langle b_0, b_1, \ldots, b_m\rangle \in (\mathbb{Z}\setminus\{0\})^{m+1}$ with $\sum_{i=0}^{m} b_i \neq 0$. Let A be an $\omega \times \omega$ matrix with the property that every $\vec{z} \in \mathbb{Z}^\omega$ such that $d(\vec{z}) = \vec{a}$ or $d(\vec{z}) = \vec{b}$ occurs as a row of A. If A is image partition regular, then either* *(i)* $a_0 = b_0$ *and* $a_k = b_m > 0$ *or*

(ii) $\sum_{i=0}^{k} a_i = \sum_{i=0}^{m} b_i$.

Proof. This follows immediately from Theorems 2.4 and 2.5. □

3. Segmented Image Partition Regular Matrices

We know that a verbatim extension of "first entries matrix" to infinite matrices will not necessarily produce even image partition regular matrices. (If $\vec{r} = \langle 1, 0, 0, \ldots \rangle$ and A is a matrix whose rows are all rows $\vec{a} \in \mathbb{Q}^\omega$ with only finitely many nonzero entries such that $c(\vec{a}) = \langle 1, 2 \rangle$, then $\begin{pmatrix} \vec{r} \\ A \end{pmatrix}$ is a first entries matrix while by [7, Theorem 2.1], $\begin{pmatrix} \vec{r} \\ A \end{pmatrix}$ is not image partition regular.) However, a restricted version of the notion of first entries matrix does turn out to be useful.

Definition 3.1. Let A be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$. Then A is a *segmented image partition regular matrix* if and only if

(1) no row of A is $\vec{0}$;

(2) for each $i \in \omega$, $\{j \in \omega : a_{i,j} \neq 0\}$ is finite; and

(3) there is an increasing sequence $\langle \alpha_n \rangle_{n=0}^{\infty}$ in ω such that $\alpha_0 = 0$ and for each $n \in \omega$,

$$\{\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1} \rangle : i \in \omega\} \setminus \{\vec{0}\}$$

is empty or is the set of rows of a finite image partition regular matrix.

If each of these finite image partition regular matrices is a first entries matrix, we shall say that A is a *segmented first entries matrix.* If also the first nonzero entry of each

$$\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1}\rangle,$$

if any, is 1, then A is a *monic segmented first entries matrix*.

Any finite sums matrix is an example of a segmented first entries matrix. Other examples are the matrices generating the $(\mathcal{M},\mathcal{P},\mathcal{C})$-systems of [8].

Theorem 3.2. *Let A be a segmented image partition regular matrix. Then A is strongly centrally image partition regular.*

Proof. Let $\vec{c_0}, \vec{c_1}, \vec{c_2}, \ldots$ denote the columns of A. Let $\langle \alpha_n\rangle_{n=0}^{\infty}$ be as in the definition of a segmented image partition regular matrix. For each $n \in \omega$, let A_n be the matrix whose columns are $\vec{c}_{\alpha_n}, \vec{c}_{\alpha_n+1}, \ldots, \vec{c}_{\alpha_{n+1}-1}$. Then the set of non-zero rows of A_n is finite and, if non-empty, is the set of rows of a finite image partition regular matrix. (Notice that we are not saying that A_n has only finitely many nonzero rows; just that there are only finitely many *distinct* rows of A_n.) Let $B_n = (\, A_0 \; A_1 \; \ldots \; A_n\,)$.

Let C be a central subset of $\mathbb{N}$ and let p be a minimal idempotent in $\beta\mathbb{N}$ such that $C \in p$. Let $C^\star = \{n \in C : -n + C \in p\}$. Then $C^\star \in p$ and, for every $n \in C^\star$, $-n + C^\star \in p$ by [9, Lemma 4.14].

By Theorem 1.5(l), we can choose $\vec{x}^{\,(0)} \in \mathbb{N}^{\alpha_1-\alpha_0}$ such that, if $\vec{y} = A_0\vec{x}^{\,(0)}$, then $y_i \in C^\star$ for every $i \in \omega$ for which the $i^{\rm th}$ row of A_0 is non-zero, and entries of $\vec{y}$ which correspond to unequal rows of A_0 are distinct.

We now make the inductive assumption that, for some $m \in \omega$, we have chosen $\vec{x}^{\,(0)}, \vec{x}^{\,(1)}, \ldots, \vec{x}^{\,(m)}$ such that $\vec{x}^{\,(i)} \in \mathbb{N}^{\alpha_{i+1}-\alpha_i}$ for every $i \in \{0,1,2,\ldots,m\}$, and, if $\vec{y} = B_m \begin{pmatrix} \vec{x}^{\,(0)} \\ \vec{x}^{\,(1)} \\ \vdots \\ \vec{x}^{\,(m)} \end{pmatrix}$, then $y_j \in C^\star$ for every $j \in \omega$ for which the $j^{\rm th}$ row of B_m is non-zero. We also suppose that entries of $\vec{y}$ which correspond to unequal rows of B_m are distinct.

Let $D = \{j \in \omega : \text{row } j \text{ of } B_{m+1} \text{ is not } \vec{0}\}$ and note that for each $j \in \omega$, $-y_j + C^\star \in p$. (Either $y_j = 0$ or $y_j \in C^\star$.) Let $l = \max\{y_i : i \in \omega\} + 1$ and note that $\mathbb{N}l \in p$ by [9, Lemma 6.6]. Thus by Theorem 1.5(l) we can choose $\vec{x}^{(m+1)} \in \mathbb{N}^{\alpha_{m+2}-\alpha_{m+1}}$ such that, if $\vec{z} = A_{m+1}\vec{x}^{(m+1)}$, then $z_j \in \mathbb{N}l \cap \bigcap_{t\in D}(-y_t + C^\star)$ for every $j \in D$, and $z_j \neq z_k$ whenever rows j and k of A_{m+1} are distinct and not equal to $\vec{0}$. Because each $z_j \in \mathbb{N}l$, we also have that $y_j + z_j \neq y_k + z_k$ whenever $j, k \in D$ and rows j and k of B_{m+1} are distinct.

Thus we can choose an infinite sequence $\langle \vec{x}^{\,(i)} \rangle_{i\in\omega}$ such that, for every $i \in \omega$, $\vec{x}^{\,(i)} \in \mathbb{N}^{\alpha_{i+1}-\alpha_i}$, and, if $\vec{y} = B_i \begin{pmatrix} \vec{x}^{\,(0)} \\ \vec{x}^{\,(1)} \\ \vdots \\ \vec{x}^{\,(i)} \end{pmatrix}$, then $y_j \in C^\star$ for every $j \in \omega$ for which the j^{th} row of B_i is non-zero. Furthermore, entries of $\vec{y}$ which correspond to distinct rows of B_i are distinct.

Let $\vec{x} = \begin{pmatrix} \vec{x}^{\,(0)} \\ \vec{x}^{\,(1)} \\ \vec{x}^{\,(2)} \\ \vdots \end{pmatrix}$ and let $\vec{y} = A\vec{x}$. We note that, for every $j \in \omega$, there exists $m \in \omega$ such that y_j is the j^{th} entry of $B_i \begin{pmatrix} \vec{x}^{\,(0)} \\ \vec{x}^{\,(1)} \\ \vdots \\ \vec{x}^{\,(i)} \end{pmatrix}$ whenever $i > m$. Thus all the entries of $\vec{y}$ are in $C^\star$ and entries which correspond to distinct rows are distinct. □

We set out to show that analogues of some of the implications of Theorem 1.5 can be established. The next result shows that the analogue of Theorem 1.5(f) is valid for segmented image partition regular matrices.

Theorem 3.3. *Let A be a segmented image partition regular matrix with columns $\vec{c}_0, \vec{c}_1, \vec{c}_2, \ldots$. Then there exist a sequence*

$\langle s_n\rangle_{n=0}^{\infty}$ in $\mathbb{Q}^+$ such that the matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ s_0 a_{0,0} & s_1 a_{0,1} & s_2 a_{0,2} & \cdots \\ 0 & 1 & 0 & \cdots \\ s_0 a_{1,0} & s_1 a_{1,1} & s_2 a_{1,2} & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a segmented image partition regular matrix. If A is a segmented first entries matrix, then R can be chosen to be a monic segmented first entries matrix. If, in addition, A is monic, then this occurs with $s_n = 1$ for every n.

Proof. The conclusions in the last two sentences of the theorem are immediate.

Now assume that A is simply a segmented image partition regular matrix and let $\langle \alpha_n\rangle_{n=0}^{\infty}$ be as guaranteed by Definition 3.1. For each n, let B_n be a $u(n) \times v(n)$ image partition regular matrix such that

$$\{\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1}\rangle : i \in \omega\} \setminus \{\vec{0}\}$$

is contained in the set of rows of B_n, where $v(n) = \alpha_{n+1} - \alpha_n$. Denote the entry in row i and column j of B_n by $b_{i,j}^{(n)}$. By Theorem 1.5(f), pick for each n a sequence $\langle t_j^{(n)}\rangle_{j=0}^{v(n)-1}$ such that the matrix

$$C_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_0^{(n)} b_{0,0}^{(n)} & t_1^{(n)} b_{0,1}^{(n)} & t_2^{(n)} b_{0,2}^{(n)} & \ldots & t_{v(n)}^{(n)} b_{0,v(n)}^{(n)} \\ t_0^{(n)} b_{1,0}^{(n)} & t_1^{(n)} b_{1,1}^{(n)} & t_2^{(n)} b_{1,2}^{(n)} & \ldots & t_{v(n)}^{(n)} b_{1,v(n)}^{(n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_0^{(n)} b_{u(n),0}^{(n)} & t_1^{(n)} b_{u(n),1}^{(n)} & t_2^{(n)} b_{u(n),2}^{(n)} & \ldots & t_{v(n)}^{(n)} b_{u(n),v(n)}^{(n)} \end{pmatrix}$$

is image partition regular.

For each $n \in \omega$ and each $j \in \{0, 1, \ldots, v(n) - 1\}$, let $s_{\alpha_n + j} = t_j^{(n)}$. Then for each n,

$$\{\langle r_{i,\alpha_n}, r_{i,\alpha_n+1}, r_{i,\alpha_n+2}, \ldots, r_{i,\alpha_{n+1}-1}\rangle : i \in \omega\} \setminus \{\vec{0}\}$$

is contained in the set of rows of C_n. □

The following lemma is not quite as trivial as its finite version.

Lemma 3.4. *Let A be a segmented first entries matrix and let $\vec{r} \in \mathbb{Z}^\omega \setminus \{\vec{0}\}$ with finitely many nonzero entries. Then there exists $b \in \mathbb{Q} \setminus \{0\}$ such that $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$ is a segmented first entries matrix.*

Proof. Let $\langle \alpha_n \rangle_{n=0}^{\infty}$ be as guaranteed by Definition 3.1. Pick $m \in \mathbb{N}$ such that for all $t \geq \alpha_m$, $r_t = 0$. Let $l = \min\{j : r_j \neq 0\}$ and pick $d \in \mathbb{Q}^+$ such that for all $i \in \omega$, if $l = \min\{j : a_{i,j} \neq 0\}$, then $a_{i,l} = d$. Let $b = \frac{d}{r_l}$. Let $\delta_0 = 0$ and for $n > 0$, let $\delta_n = \alpha_{m+n-1}$. Then the sequence $\langle \delta_n \rangle_{n=0}^{\infty}$ is as required by Definition 3.1 for $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$. □

Definition 3.5. Let A be an $\omega \times \omega$ matrix.

(a) The matrix A is *kernel partition regular* if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in {C_i}^\omega$ such that $A\vec{x} = \vec{0}$.

(b) The matrix A is *centrally kernel partition regular* if and only if for every central subset C of $\mathbb{N}$, there exists $\vec{x} \in C^\omega$ such that $A\vec{x} = \vec{0}$.

Theorem 3.6. *Let A be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$ and columns $\vec{c}_0, \vec{c}_1, \ldots$. Consider the following statements.*

(a) A is a segmented image partition regular matrix.

(b) There exists a sequence $\langle s_n\rangle_{n=0}^{\infty}$ *in* $\mathbb{Q}^+$ *such that the matrix*

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ s_0a_{0,0} & s_1a_{0,1} & s_2a_{0,2} & \cdots \\ 0 & 1 & 0 & \cdots \\ s_0a_{1,0} & s_1a_{1,1} & s_2a_{1,2} & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a segmented image partition regular matrix. In particular R *is strongly centrally image partition regular.*

(c) There exists a sequence $\langle s_n\rangle_{n=0}^{\infty}$ *in* $\mathbb{Q}^+$ *such that the matrix*

$$P = (\, s_0\vec{c}_0 \quad -\vec{e}_0 \quad s_1\vec{c}_1 \quad -\vec{e}_1 \quad s_2\vec{c}_2 \quad -\vec{e}_2 \quad \cdots\,)$$

is centrally kernel partition regular, where $\vec{e}_n$ *denotes the* $n^{\rm th}$ $\omega\times 1$ *unit vector.*

(d) There exist a segmented first entries matrix D *with entries from* ω *and a matrix* G *with entries from* ω *and no row equal to* $\vec{0}$ *such that* $AG = D$.

(e) There exists a matrix G *with entries from* ω *and no row equal to* $\vec{0}$ *such that for every* $\vec{r}\in\omega^\omega\setminus\{\vec{0}\}$ *with finitely many nonzero entries, there exist* $b\in\mathbb{Q}^+$ *and a segmented first entries matrix* D*, with all entries nonnegative and all entries except possibly those in the first row from* ω*, such that* $\begin{pmatrix} b\vec{r} \\ A\end{pmatrix} G = D$.

(f) There exists a segmented first entries matrix D *with entries from* ω *such that*

(i) for every $y\in\omega^\omega$ *there exists* $x\in\omega^\omega$ *such that* $A\vec{x} = D\vec{y}$ *and*

(ii) for every $y\in\mathbb{N}^\omega$ *there exists* $x\in\mathbb{N}^\omega$ *such that* $A\vec{x} = D\vec{y}$.

(g) There exists a segmented first entries matrix D with entries from ω such that for every $y \in \mathbb{N}^\omega$ there exists $x \in \mathbb{N}^\omega$ such that $A\vec{x} = D\vec{y}$.

(h) A is centrally image partition regular.

Then:

(1) Statement (a) implies statement (b) which implies statement (c). If A is a monic segmented first entries matrix, then the sequence $\langle s_n \rangle_{n=0}^\infty$ can be chosen constantly equal to 1.

(2) Statement (a) implies statement (d).

(3) Statements (d) and (e) are equivalent and imply statement (f).

(4) Statement (f)(i) implies the weaker version of statement (d) which does not demand that no row of G be $\vec{0}$.

(5) Statement (f) implies statement (g) which implies statement (h).

Proof. That statement (a) implies statement (b) was proved in Theorems 3.2 and 3.3, as was the second sentence of conclusion (1). To see that statement (b) implies statement (c), note that (with the same sequence $\langle s_n \rangle_{n=0}^\infty$), $PR = \mathbf{O}$. Let a central subset C of $\mathbb{N}$ be given. Pick $\vec{x} \in \mathbb{N}^\omega$ such that $\vec{y} = R\vec{x} \in C^\omega$. Then $P\vec{y} = \vec{0}$.

To verify conclusion (2), let $\langle \alpha_n \rangle_{n=0}^\infty$ be as guaranteed by Definition 3.1. To simplify the discussion, we shall assume that for each n,

$$\{\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1} \rangle : i \in \omega\} \setminus \{\vec{0}\} \neq \emptyset\,.$$

(If it happens that this set is empty, simply add a new row with a 1 in position α_n and all other entries equal to 0.)

For each $n \in \omega$, choose $u(n)$ and $v(n)$ in $\mathbb{N}$ and a image partition regular $u(n) \times v(n)$ matrix A_n such that

$$\{\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1} \rangle : i \in \omega\} \setminus \{\vec{0}\}$$

is the set of rows of A_n. (Necessarily $v(n) = \alpha_{n+1} - \alpha_n$.)

By Theorem 1.5(i), pick for each $n \in \omega$ some $m(n) \in \mathbb{N}$ and a $v(n) \times m(n)$ matrix G_n with entries from ω and no row equal to $\vec{0}$ and $u(n) \times m(n)$ first entries matrix D_n with entries from ω such that $A_n G_n = D_n$. Let

$$G = \begin{pmatrix} G_0 & \mathbf{O} & \mathbf{O} & \cdots \\ \mathbf{O} & G_1 & \mathbf{O} & \cdots \\ \mathbf{O} & \mathbf{O} & G_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and let $D = AG$.

Trivially G has no row equal to $\vec{0}$ and has all entries from ω. Let $\alpha'(0) = 0$ and for $n \in \mathbb{N}$, let $\alpha'(n) = \sum_{t=0}^{n-1} m(t)$. It is a routine exercise to verify that D is a segmented first entries matrix with entries from ω where $\langle \alpha'(n) \rangle_{n=0}^{\infty}$ is as required by Definition 3.1(3).

Now we verify conclusion (3). That (e) implies (d) is trivial. To see that (d) implies (e), let $\vec{r} \in \omega^\omega \setminus \{\vec{0}\}$ with finitely many nonzero entries, and let $\vec{s} = \vec{r}G$. Notice that $\vec{s} \in \omega^\omega \setminus \{\vec{0}\}$ and $\vec{s}$ has only finitely many nonzero entries. (Some $r_i \neq 0$ and for this i some $g_{i,j} \neq 0$ so $s_i \geq r_i \cdot g_{i,j} > 0$. Also, if $k \in \mathbb{N}$ and $r_i = 0$ for all $i > k$, then pick $l \in \mathbb{N}$ such that for all $i \in \{0, 1, \ldots, k\}$ and all $j > l$, $g_{i,j} = 0$. Then for all $j > l$, $s_j = 0$.) By Lemma 3.4, pick $b \in \mathbb{Q}^+$ such that $\begin{pmatrix} b\vec{s} \\ D \end{pmatrix}$ is a segmented first entries matrix. Then $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix} G = \begin{pmatrix} b\vec{s} \\ D \end{pmatrix}$.

To see that (d) implies (f), given $\vec{y}$ in ω^ω or in $\mathbb{N}^\omega$, let $\vec{x} = G\vec{y}$.

To verify conclusion (4), assme that (f)(i) holds. For each $i \in \omega$, let $\vec{e_i}$ be column i of the $\omega \times \omega$ identity matrix and pick $\vec{x_i} \in \omega^\omega$ such that $A\vec{x_i} = D\vec{e_i}$. Let $G = (\vec{x_0} \quad \vec{x_1} \quad \vec{x_2} \quad \ldots)$. Then $AG = D$.

For conclusion (5), the fact that (f) implies (g) is trivial. To see that (g) implies (h), let C be a central set and pick by Theorem 3.2 some $\vec{y} \in \mathbb{N}^\omega$ such that $D\vec{y} \in C^\omega$. Pick $\vec{x}$ such that $A\vec{x} = D\vec{y}$. □

We note that conclusion (4) in the above theorem cannot be strengthened to having statement (f)(i) imply the entirety of statement (d).

Theorem 3.7. *There is a matrix A with entries from ω which satisfies statement (f)(i) of Theorem 3.6 but is not image partition regular. In particular, statement (f)(i) does not imply either of statements (f)(ii) or (d).*

Proof. The "in particular" conclusion follows from the fact that statements (f) and (d) each imply statement (h) in Theorem 3.6.

Choose a matrix A such that a row $\vec{r}$ is a row of A if and only if

(a) either $r_0 = 1$ or $r_0 = 2$ and

(b) for some finite nonempty subset F of $\mathbb{N}$, $r_i = 0$ if $i \in \mathbb{N} \setminus F$ and $r_i = 1$ if $i \in F$.

Let D be the matrix obtained by deleting the first column of A. Then D is a segmented first entries matrix. Given $\vec{y} \in \omega^\omega$, define $\vec{x} \in \omega^\omega$ by $x_0 = 0$ and $x_n = y_{n-1}$ for $n \in \mathbb{N}$. Then $A\vec{x} = D\vec{y}$.

Now we show that A is not image partition regular. Given $x \in \mathbb{N}$, let $m(x) = \max\{t \in \omega : x \in 2^t\mathbb{N}\}$. For $i \in \{0,1\}$, let $C_i = \{x \in \mathbb{N} : m(x) \equiv i \pmod 2\}$. Suppose that we have $i \in \{0,1\}$ and $\vec{x} \in \mathbb{N}^\omega$ such that $A\vec{x} \in {C_i}^\omega$. Pick a finite subset F of $\mathbb{N}$ such that $2^{m(x_0)+2}$ divides $\sum_{n\in F} x_n$ (by choosing $2^{m(x_0)+2}$ terms x_n congruent to each other mod $2^{m(x_0)+2}$). Then $x_0 + \sum_{n\in F} x_n$ and $2x_0 + \sum_{n\in F} x_n$ are both entries of $A\vec{x}$, while $m(2x_0 + \sum_{n\in F} x_n) = m(x_0 + \sum_{n\in F} x_n) + 1$, a contradiction. □

We now see that some of the conclusions of Theorem 3.6 need not hold for all strongly centrally image partition regular matrices.

Theorem 3.8 . *There is a strongly centrally image partition regular matrix M which fails to satisfy statement (c) of Theorem 3.6.*

Proof. Let

$$M = \begin{pmatrix} -2 & 1 & 0 & \dots \\ -2 & 0 & 1 & \dots \\ 0 & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

That is, M is a matrix whose rows have somewhere a single -2 followed somewhere by a single 1. By [7, Corollary 3.7], M is strongly centrally image partition regular.

Let $\vec{c}_0, \vec{c}_1, \vec{c}_2, \dots$ be the columns of M and suppose that we have a sequence $\langle s_n \rangle_{n=0}^{\infty}$ in $\mathbb{Q}^+$ so that

$$P = (\, s_0\vec{c}_0 \quad -\vec{e}_0 \quad s_1\vec{c}_1 \quad -\vec{e}_1 \quad s_2\vec{c}_2 \quad -\vec{e}_2 \quad \cdots)$$

is kernel partition regular. For each $i \in \omega$, pick $k_i, l_i \in \mathbb{N}$ such that $s_i = \dfrac{k_i}{l_i}$ and $(k_i, l_i) = 1$.

Pick a prime $r \geq 5$. If there exist $u < v < w < z$ in ω such that for each $i \in \{u, v, w, z\}$, $s_i \neq 1$, then choose such u, v, w, and z and require also that $r > 2k_i l_j + k_j l_i + l_i l_j$ for all $i, j \in \{u, v, w, z\}$. For each $x \in \mathbb{N}$, let $m(x) = \max\{t \in \omega : x \in r^t\mathbb{N}\}$ and pick $a(x) \in \omega$ and $f(x) \in \{1, 2, \dots, r-1\}$ such that $x = a(x) \cdot r^{m(x)+1} + f(x) \cdot r^{m(x)}$. Choose $\vec{x} \in \mathbb{N}^\omega$ and $c \in \{1, 2, \dots, r-1\}$ such that $P\vec{x} = \vec{0}$ and for each $n \in \omega$, $f(x_n) = c$.

We show first that there do not exist $u < v < w < z$ in ω such that for each $i \in \{u, v, w, z\}$, $s_i \neq 1$. Suppose instead we have chosen such u, v, w, and z. We claim that for $i, j \in \{u, v, w, z\}$ with $i < j$, $-2s_i + s_j \in \{0, 1\}$. We have some $t \in 2\omega + 1$ such that $x_t = -2s_i x_{2i} + s_j x_{2j}$ and thus

$$l_i l_j a(x_t) r^{m(x_t)+1} + l_i l_j c r^{m(x_t)} + 2k_i l_j a(x_{2i}) r^{m(x_{2i})+1} + 2k_i l_j c r^{m(x_{2i})} = k_j l_i a(x_{2j}) r^{m(x_{2j})+1} + k_j l_i c r^{m(x_{2j})}\,.$$

Suppose first that $m(x_{2i}) < m(x_{2j})$. Since $r^{m(x_{2j})}$ divides the right side of the above equation, we must have that $m(x_{2i}) = m(x_t)$ and $l_i l_j c + 2k_i l_j c \equiv 0 \pmod{r}$. But then $2s_i + 1 = 0$, contradicting the fact that $s_i > 0$.

Now suppose that $m(x_{2i}) > m(x_{2j})$. Then we have that $m(x_{2j}) = m(x_t)$ and $l_i l_j c \equiv k_j l_i c \pmod{r}$, and thus that $s_j = 1$, a contradiction. Thus we must have that $m(x_{2i}) = m(x_{2j})$ and $m(x_t) \geq m(x_{2i})$. If $m(x_t) > m(x_{2i})$ we have that $2k_i l_j c \equiv k_j l_i c \pmod{r}$ and so $2s_i = s_j$. If $m(x_t) = m(x_{2i})$ we have that $l_i l_j c + 2k_i l_j c \equiv k_j l_i c \pmod{r}$ and so $1 + 2s_i = s_j$. Thus we have established that $-2s_i + s_j \in \{0, 1\}$ as claimed.

Since $-2s_i + s_z \in \{0, 1\}$ for $i \in \{u, v, w\}$ we have some $i < j$ in $\{u, v, w\}$ with $s_i = s_j$. But then $-2s_i + s_j \notin \{0, 1\}$, a contradiction.

We therefore have that $T = \{i \in \omega : s_i = 1\}$ is infinite. Given $i < j$ in T we have

$$a\,(x_t) r^{m(x_t)+1} + cr^{m(x_t)} + 2a(x_{2i}) r^{m(x_{2i})+1} + 2cr^{m(x_{2i})} = a\,(x_{2j}) r^{m(x_{2j})+1} + cr^{m(x_{2j})} .$$

If we had any $i < j$ in T with $m(x_{2i}) < m(x_{2j})$ we would have $m(x_t) = m(x_{2i})$ and $c + 2c \equiv 0 \pmod{r}$, which is a contradiction since $r \geq 5$. Thus we can pick $i < j$ in T with $m(x_{2i}) = m(x_{2j})$. But then, if $m(x_t) = m(x_{2i})$ we have $3c \equiv c \pmod{r}$, and if $m(x_t) > m(x_{2i})$ we have $2c \equiv c \pmod{r}$. □

Notice that the presence of negative entries in the matrix M in Theorem 3.8 is essential. In fact, if A is any matrix with nonnegative entries and row sums constantly equal to m, then taking $s_n = \frac{1}{m}$ for each n, one has that P is centrally kernel partition regular. (Given a central set C and any $a \in C$, let each $x_n = a$.) We see now, however, that even with positive entries and constant row sums, statement (b) of Theorem 3.6 need not be satisfied.

Theorem 3.9. *Let A be a matrix consisting of all rows $\vec{z}$ with $d(\vec{z}) = \langle 2, 1 \rangle$. Then A is strongly centrally image partition regular but there does not exist a sequence $\langle s_n \rangle_{n=0}^{\infty}$ such that the matrix R of statement (b) of Theorem 3.6 is strongly centrally image partition regular.*

Proof. By [7, Theorem 3.7] A is strongly centrally image partition regular.

Suppose that we have a sequence $\langle s_n\rangle_{n=0}^{\infty}$ so that the matrix R is strongly centrally image partition regular. For each $i \in \omega$, pick k_i and l_i in $\mathbb{N}$ such that $s_i = \dfrac{k_i}{l_i}$.

Pick a prime $r \geq 5$. If there exist $u < v < w$ in ω such that for each $i \in \{u, v, w\}$, $s_i \notin \{1, \frac{1}{2}, \frac{1}{3}\}$, then choose such u, v, and w and require also that $r > 2k_il_j + k_jl_i + l_il_j$ for all $i, j \in \{u, v, w\}$. For each $x \in \mathbb{N}$, let $m(x) = \max\{t \in \omega : x \in r^t\mathbb{N}\}$ and pick $a(x) \in \omega$ and $f(x) \in \{1, 2, \ldots, r-1\}$ such that $x = a(x) \cdot r^{m(x)+1} + f(x) \cdot r^{m(x)}$. Choose $c \in \{1, 2, \ldots, r-1\}$ such that $\{x \in \mathbb{N} : f(x) = c\}$ is central. Choose $\vec{x} \in \mathbb{N}^{\omega}$ such that for every entry y of $R\vec{x}$, $f(y) = c$ and entries corresponding to distinct rows of R are distinct.

We show first that there do not exist $u < v < w$ in ω such that for each $i \in \{u, v, w\}$, $s_i \notin \{1, \frac{1}{2}, \frac{1}{3}\}$. Suppose instead we have chosen such u, v, and w. Then for $i < j$ in $\{u, v, w\}$, we have x_i, x_j, and $2s_ix_i + s_jx_j$ are entries of $R\vec{x}$ so we have $x_i = a(x_i)r^{m(x_i)+1} + cr^{m(x_i)}$, $x_j = a(x_j)r^{m(x_j)+1} + cr^{m(x_j)}$, and $2s_ix_i + s_jx_j = dr^{n+1} + cr^n$ where $d = a(2s_ix_i + s_jx_j)$ and $n = m(2s_ix_i + s_jx_j)$. Thus we have

$$2k_il_ja(x_i)r^{m(x_i)+1} + 2k_il_jcr^{m(x_i)} + k_jl_ia(x_j)r^{m(x_j)+1} + k_jl_icr^{m(x_j)} = l_il_jdr^{n+1} + l_il_jcr^n\,.$$

If we had $m(x_i) > m(x_j)$, we would have $k_jl_ic \equiv l_il_jc \pmod r$ so that $k_jl_i = l_il_j$ and thus $s_j = 1$. If we had $m(x_i) < m(x_j)$, we would have $2k_il_jc \equiv l_il_jc \pmod r$ so that $2k_il_j = l_il_j$ and thus $s_i = \frac{1}{2}$. Thus we must have $m(x_i) = m(x_j)$ for any choice of $i < j$ in $\{u, v, w\}$ and thus that $2k_il_jc + k_jl_ic \equiv l_il_jc \pmod r$ so that $2s_i + s_j = 1$. But the equations $2s_u + s_v = 1$, $2s_u + s_w = 1$, and $2s_v + s_w = 1$ imply that $s_u = s_v = s_w = \frac{1}{3}$.

Now we claim that we do not have infinitely many i's for which $s_i = 1$. Suppose instead that we do and pick $i < j$ such that $s_i = s_j = 1$ and $m(x_i) \leq m(x_j)$. Then as above we have

$$2a(x_i)r^{m(x_i)+1} + 2cr^{m(x_i)} + a(x_j)r^{m(x_j)+1} + cr^{m(x_j)} = dr^{n+1} + cr^n .$$

If $m(x_i) = m(x_j)$ we conclude that $3c \equiv c \pmod{r}$ and if $m(x_i) < m(x_j)$ we conclude that $2c \equiv c \pmod{r}$.

Thus there exist an infinite set $J \subseteq \omega$ and $s \in \{\frac{1}{2}, \frac{1}{3}\}$ such that for all $i \in J$, $s_i = s$. Let B be the matrix which results from deleting all columns j of R for which $j \notin J$ and deleting all rows i of R for which there is some $j \notin J$ with $r_{i,j} \neq 0$. Then B is also strongly centrally image partition regular and B has the property that every $\vec{z} \in \mathbb{Z}^\omega$ with $d(\vec{z}) = \langle 2s, s\rangle$ or $d(\vec{z}) = \langle 1 \rangle$ occurs as a row of B. Thus by Lemma 2.3(iii) there is a finite coloring of $\mathbb{N}$ such that no $\vec{x} \in \mathbb{N}^\omega$ with the entries of $B\vec{x}$ monochrome has $\{x_n : n \in \omega\}$ unbounded. Since for $n \neq m$, x_n and x_m are entries of $B\vec{x}$ corresponding to different rows, this contradicts the claim that B is strongly centrally image partition regular. □

4. Additional Classes of Image Partition Regular Matrices

In this section we introduce the *restricted triangular matrices*, a class of strongly centrally image partition regular matrices, and investigate ways to construct new image partition regular or centrally image partition regular matrices, based on existing ones.

Definition 4.1. Let A be an $\omega \times \omega$ matrix. Then A is a *restricted triangular* matrix if and only if all entries of A are from $\mathbb{Z}$ and there exist $d \in \mathbb{N}$ and an increasing function $j : \omega \to \omega$ such that for all $i \in \omega$,

(1) $a_{i,j(i)} \in \{1, 2, \ldots, d\}$,

(2) for all $l > j(i)$, $a_{i,l} = 0$, and

(3) for all $k > i$ and all $t \in \{1, 2, \ldots, d\}$, $t | a_{k,j(i)}$.

Theorem 4.2. *Let A be a restricted triangular matrix. Then A is strongly centrally image partition regular. In fact, if $p \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}_{\beta\mathbb{N}}(n\mathbb{N})$ and $P \in p$, then there exixts $\vec{x} \in \mathbb{N}^\omega$ such that the entries of $A\vec{x}$ are distinct elements of P.*

Proof. Since $\mathbb{N}d! \in p$, we may assume that $P \subseteq \mathbb{N}d!$.

If $j(0) = 0$, pick $y_0 \in P$ and let $x_0 = y_0/a_{0,j(0)}$. Otherwise, for $t \in \{0, 1, \ldots, j(0)-1\}$, let $x_t = d!$ and pick $y_0 \in P$ such that $y_0 > \Sigma_{t=0}^{j(0)-1} a_{0,t}x_t$. Let $x_{j(0)} = (y_0 - \Sigma_{t=0}^{j(0)-1} a_{0,t}x_t)/a_{0,j(0)}$.

Inductively, given $i \in \mathbb{N}$, for $t \in \{j(i-1)+1, j(i-1)+2, \ldots, j(i)-1\}$, if any, let $x_t = d!$. Pick $y_i \in P$ such that $y_i > \Sigma_{t=0}^{j(i)-1} a_{i,t}x_t$ and $y_i \notin \{y_0, y_1, \ldots, y_{i-1}\}$. Then $d!|y_i$ and if $t \notin \{j(0), j(1), \ldots, j(i-1)\}$, then $x_t = d!$. If $t \in \{j(0), j(1), \ldots, j(i-1)\}$, then by conditions (1) and (3) of Definition 4.1, $a_{i,j(i)}|a_{i,t}$. Thus $a_{i,j(i)}$ divides $y_i - \Sigma_{t=0}^{j(i)-1} a_{i,t}x_t$. Let $x_{j(i)} = (y_i - \Sigma_{t=0}^{j(i)-1} a_{i,t}x_t)/a_{i,j(i)}$.

The induction being complete, one has $A\vec{x} = \vec{y} \in P^\omega$. □

Corollary 4.3. *Let A be an $\omega \times \omega$ matrix with entries from $\mathbb{Z}$. If there exists an increasing function $j : \omega \to \omega$ such that for all $i \in \omega$,*

(1) $a_{i,j(i)} = 1$ and

(2) for all $l > j(i)$, $a_{i,l} = 0$,

then A is strongly centrally image partition regular.

Proof. One has that A is a restricted triangular matrix with $d = 1$. □

Corollary 4.4. *Let A be an $\omega \times \omega$ matrix with entries from $\mathbb{Z}$ and only finitely many nonzero entries in each row. If there exist $d \in \mathbb{N}$ and a function $j : \omega \to \omega$ such that for all $i \in \omega$,*

(1) $a_{i,j(i)} \in \{1, 2, \ldots, d\}$ and

(2) for all $k \neq i$, $a_{k,j(i)} = 0$,

then A is strongly centrally image partition regular.

Proof. By rearranging columns, one may presume that for each i and each $l > j(i)$, $a_{i,l} = 0$. (This can be done inductively by rows. By condition (2), the assignment at stage i cannot affect the conclusion for earlier rows.) Then by rearranging rows, one may presume that j is an increasing function. Thus A is a restricted triangular matrix. □

Theorem 4.5. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 3 & 2 & 1 & 0 & \dots \\ 4 & 3 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A satisfies statement (h) of Theorem 3.6 *but does not satisfy statement (d).*

Proof. By Corollary 4.3, A is strongly centrally image partition regular. Suppose that there do exist a matrix G with entries from ω and no row equal to $\vec{0}$ and a segmented first entries matrix D with entries from ω such that $AG = D$.

Some column, say column j, of D is not $\vec{0}$. Then column j of G is not $\vec{0}$ and thus the entries of column j of AG are unbounded while the entries of column j of D are bounded, a contradiction. □

We know by [7, Theorem 3.9] that strongly centrally image partition regular matrices do not in general have the property that they can be extended by some multiple of an arbitrary row with the resulting matrix being image partition regular. But we saw in Lemma 3.4 that segmented first entries matrices do have the property that they can be extended with the resulting matrix being strongly centrally image partition regular. We show now that the same statement applies to restricted triangular matrices.

Theorem 4.6. *Let A be a restricted triangular matrix and let $\vec{r} \in \mathbb{Z}^\omega \setminus \{\vec{0}\}$ with finitely many nonzero entries. Then there exist $b \in \mathbb{Q} \setminus \{0\}$ such that $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$ is strongly centrally image partition regular.*

Proof. Pick $d \in \mathbb{N}$ and $j : \omega \to \omega$ as guaranteed by Definition 4.1. Pick $l \geq j(0)$ such that $r_i = 0$ for all $i > l$ and pick $\gamma \in \omega$ such that $j(\gamma) \leq l < j(\gamma + 1)$.

Let B be the upper left $(\gamma + 1) \times (l + 1)$ corner of A. By Theorem 4.2, A is centrally image partition regular and thus B is image partition regular. Applying Theorem 1.5(j) $l + 2$ times, pick $b_0, b_1, \ldots, b_l, b$ in $\mathbb{Q}$ such that

$$D = \begin{pmatrix} br_0 & br_1 & br_2 & \ldots & br_l \\ b_0 & 0 & 0 & \ldots & 0 \\ 0 & b_1 & 0 & \ldots & 0 \\ 0 & 0 & b_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & b_l \\ & & B & & \end{pmatrix}$$

is image partition regular. To see that $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$ is centrally image partition regular, let C be a central set. Let c be a common multiple of the numerators of $b_0, b_1, \ldots, b_l$. Then $C \cap \mathbb{N}cd!$ is central. By Theorem 1.5(l) pick $x_0, x_1, \ldots, x_l$ such that all entries of $D \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_l \end{pmatrix}$ are in $C \cap \mathbb{N}cd!$ and are distinct. For $t \in \{0, 1, \ldots, l\}$, one has in particular that $b_t x_t \in \mathbb{N}cd!$ and thus $x_t \in \mathbb{N}d!$. For $t > l$, choose x_t exactly as in the proof of Theorem 4.2. One concludes immediately that all entries of $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix} \vec{x}$ are in C and are distinct. $\square$

Now we turn our attention to methods of constructing new image partition regular or centrally image partition regular matrices based on existing ones.

Theorem 4.7. *Let A be a centrally image partition regular matrix and let $\langle b_n\rangle_{n=0}^{\infty}$ be a sequence in $\mathbb{N}$. Let*

$$B = \begin{pmatrix} b_0 & 0 & 0 & \cdots \\ 0 & b_1 & 0 & \cdots \\ 0 & 0 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \text{ Then } \begin{pmatrix} \mathbf{O} & B \\ A & \mathbf{O} \\ A & B \end{pmatrix}$$

is centrally image partition regular.

Proof. Let C be a central subset of $\mathbb{N}$. Pick a minimal idempotent p in $\beta\mathbb{N}$ such that $C \in p$. Let $D = \{x \in C : -x + C \in p\}$. Then by [9, Lemma 4.14] $D \in p$ and thus D is central. So pick $\vec{x} \in \mathbb{N}^{\omega}$ such that $A\vec{x} \in D^{\omega}$.

Given $n \in \omega$, let $c_n = \Sigma_{t=0}^{\infty} a_{n,t} \cdot x_t$. Then $C \cap (-c_n + C) \in p$, so pick $z_n \in C \cap (-c_n + C) \cap \mathbb{N}b_n$ and let $y_n = z_n/b_n$. Then

$$\begin{pmatrix} \mathbf{O} & B \\ A & \mathbf{O} \\ A & B \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \in C^{\omega+\omega+\omega}.$$

□

Our remaining examples are based on one method of construction.

Definition 4.8. Let $\gamma, \delta \in \omega \cup \{\omega\}$ and let C be a $\gamma \times \delta$ matrix with finitely many nonzero entries in each row. For each $t < \delta$, let B_t be a finite matrix of dimension $u_t \times v_t$. Let $R = \{(i,j) : i < \gamma \text{ and } j \in \times_{t<\delta}\{0,1,\ldots,u_t-1\}\}$. Given $t < \delta$ and $k \in \{0,1,\ldots,u_t-1\}$, denote by $\vec{b}_k^{(t)}$ the k^{th} row of B_t. Then D is an *insertion matrix of $\langle B_t\rangle_{t<\delta}$ into C* if and only if the rows of D are all rows of the form

$$c_{i,0} \cdot \vec{b}_{j(0)}^{(0)} {}^\frown c_{i,1} \cdot \vec{b}_{j(1)}^{(1)} {}^\frown \ldots$$

where $(i,j) \in R$.

For example, if $C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B_0 = \begin{pmatrix} 1 & 1 \\ 5 & 7 \end{pmatrix}$, and $B_1 = \begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}$, then

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 5 & 7 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ 10 & 14 & 0 & 1 \\ 10 & 14 & 3 & 3 \end{pmatrix}$$

is an insertion matrix of $\langle B_t \rangle_{t<2}$ into C.

Lemma 4.9 . *Let C be a segmented first entries matrix and for each $t < \omega$, let B_t be a $u_t \times v_t$ (finite) first entries matrix. Then any insertion matrix of $\langle B_t \rangle_{t<\omega}$ into C is a segmented first entries matrix.*

Proof. Let $\langle \alpha_n \rangle_{n=0}^{\infty}$ be as guaranteed by Definition 3.1 for C. Let $\delta_0 = 0$ and inductively given $n \in \omega$, let $\delta_{n+1} = \delta_n + \Sigma_{t=\alpha_n}^{\alpha_{n+1}-1} v_t$. Then $\langle \delta_n \rangle_{n=0}^{\infty}$ is as required by Definition 3.1 for the insertion matrix. □

Theorem 4.10. *Let C be a segmented first entries matrix and for each $t < \omega$, let B_t be a $u_t \times v_t$ (finite) image partition regular matrix. Then any insertion matrix of $\langle B_t \rangle_{t<\omega}$ into C is centrally image partition regular.*

Proof. Let A be an insertion matrix of $\langle B_t \rangle_{t<\omega}$ into C. For each $t \in \omega$, pick by Theorem 1.5(g), some $m_t \in \mathbb{N}$ and a $u_t \times m_t$ first entries matrix D_t such that for all $\vec{y} \in \mathbb{N}^{m_t}$ there exists $\vec{x} \in \mathbb{N}^{v_t}$ such that $B_t \vec{x} = D_t \vec{y}$. Let E be an insertion matrix of $\langle D_t \rangle_{t<\omega}$ into C where the rows occur in the corresponding position to those of A. That is, if $i < \omega$ and $j \in \times_{t<\omega}\{0, 1, \ldots, u_t - 1\}\}$ and

$$c_{i,0} \cdot \vec{b}^{(0)}_{j(0)} {}^\frown c_{i,1} \cdot \vec{b}^{(1)}_{j(1)} {}^\frown \ldots$$

is row k of A, then

$$c_{i,0} \cdot \vec{d}^{(0)}_{j(0)} \frown c_{i,1} \cdot \vec{d}^{(1)}_{j(1)} \frown \ldots$$

is row k of E.

Let H be a central subset of $\mathbb{N}$. By Lemma 4.9, E is a segmented first entries matrix so pick $\vec{y} \in \mathbb{N}^\omega$ such that all entries of $E\vec{y}$ are in H. Let $\delta_0 = \gamma_0 = 0$ and for $n \in \mathbb{N}$ let $\delta_n = \Sigma_{t=0}^{n-1} v_t$ and let $\gamma_n = \Sigma_{t=0}^{n-1} m_t$. For each $n \in \omega$, pick

$$\begin{pmatrix} x_{\delta_n} \\ x_{\delta_n+1} \\ \vdots \\ x_{\delta_{n+1}-1} \end{pmatrix} \in \mathbb{N}^{v_n} \text{ such that } B_t \begin{pmatrix} x_{\delta_n} \\ x_{\delta_n+1} \\ \vdots \\ x_{\delta_{n+1}-1} \end{pmatrix} = D_t \begin{pmatrix} y_{\gamma_n} \\ y_{\gamma_n+1} \\ \vdots \\ y_{\gamma_{n+1}-1} \end{pmatrix}.$$

Then $A\vec{x} = E\vec{y}$. □

We see next that for certain image partition regular matrices, an analogue of Theorem 4.10 is valid.

Definition 4.11. Let $\vec{a} = \langle a_0, a_1, \ldots, a_l \rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_l > 0$. Then C is a *Milliken-Taylor matrix for* $\vec{a}$ if and only if C consists of all rows $\vec{r} \in \mathbb{Z}^\omega$ such that $c(\vec{r}) = \vec{a}$.

If C is a Milliken-Taylor matrix for $\vec{a}$, where all entries of $\vec{a}$ are positive, and $\vec{x} \in \mathbb{N}^\omega$, then the set of entries of $C\vec{x}$ is the Milliken-Taylor system $MT(\vec{a}, \vec{x})$ as defined in [2, Definition 2.3]. It is a consequence of [2, Theorem 2.5] (when the entries of $\vec{a}$ are positive) and [7, Corollary 3.6] (for the general case) that any Milliken-Taylor matrix is image partition regular. On the other hand, it is a consequence of [2, Theorem 3.14] that if $\vec{a} = \langle a_0, a_1, \ldots, a_l \rangle$ is a compressed sequence with entries from $\mathbb{N}$ and $l > 0$, then any Milliken-Taylor matrix for $\vec{a}$ is not centrally image partition regular.

Theorem 4.12 . *Let $\vec{a} = \langle a_0, a_1, \ldots, a_l \rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_l > 0$, let C be a Milliken-Taylor matrix for $\vec{a}$, and for each $t < \omega$, let B_t be a $u_t \times v_t$ (finite) image partition regular matrix. Then any insertion matrix of $\langle B_t \rangle_{t<\omega}$ into C is image partition regular.*

Proof. Assume first that $l = 0$. Then C is in fact a segmented first entries matrix so the result follows from Theorem 4.10.

Assume then that $l > 0$. Let $\alpha_0 = 0$ and inductively let $\alpha_{n+1} = \alpha_n + v_n$. Pick by [9, Corollary 2.6] some $p \in K(\beta\mathbb{N})$ such that $p = p + p$. Recall from [9, Lemma 4.14] that if $A \in p$ and $A^\star = \{x \in A : -x + A \in p\}$, then $A^\star \in p$ and for all $x \in A^\star$, $-x + A^\star \in p$.

Note that $a_i p$ is the product of a_i with p in the semigroup $(\beta\mathbb{Z}, \cdot)$. (If $a_i > 0$, then $a_i p$ is not the sum of p with itself a_i times, which is just p.) Also, by [9, Exercise 4.3.5], $a_0 p + a_1 p + \ldots + a_l p \in \beta\mathbb{N}$, since $a_l > 0$. Let $\mathcal{G}$ be a finite partition of $\mathbb{N}$ and pick $A \in \mathcal{G}$ such that $A \in a_0 p + a_1 p + \ldots + a_l p$.

Now $\{x \in \mathbb{Z} : -x + A \in a_1 p + a_2 p + \ldots + a_l p\} \in a_0 p$ so that $D_0 = \{x \in \mathbb{Z} : -a_0 x + A \in a_1 p + a_2 p + \ldots + a_l p\} \in p$. Then ${D_0}^\star \in p$ so pick by Theorem 1.5(b), $x_0, x_1, \ldots, x_{\alpha_1 - 1} \in \mathbb{N}$ such that

$$B_0 \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\alpha_1 - 1} \end{pmatrix} \in ({D_0}^\star)^{u_0} .$$

Let H_0 be the set of entries of

$$B_0 \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\alpha_1 - 1} \end{pmatrix} .$$

Inductively, let $n \in \mathbb{N}$ and assume that we have chosen $\langle x_t \rangle_{t=0}^{\alpha_n - 1}$ in $\mathbb{N}$, $\langle D_k \rangle_{k=0}^{n-1}$ in p, and $\langle H_k \rangle_{k=0}^{n-1}$ in the set $\mathcal{P}_f(\mathbb{N})$ of finite nonempty subsets of $\mathbb{N}$ such that for $r \in \{0, 1, \ldots, n-1\}$,

(I) H_r is the set of entries of

$$B_r \begin{pmatrix} x_{\alpha_r} \\ x_{\alpha_r+1} \\ \vdots \\ x_{\alpha_{r+1}-1} \end{pmatrix} ;$$

(II) if $\emptyset \neq F \subseteq \{0, 1, \ldots, r\}$, $k = \min F$, and for each $t \in F$, $y_t \in H_t$, then $\Sigma_{t \in F}\, y_t \in {D_k}^{\star}$;

(III) if $r < n - 1$, then $D_{r+1} \subseteq D_r$;

(IV) if $m \in \{0, 1, \ldots, l-1\}$, $F_0, F_1, \ldots, F_m$ are nonempty subsets of $\{0, 1, \ldots, r\}$, for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^{m} F_i$, $y_t \in H_t$, then $-\Sigma_{i=0}^{m} a_i \Sigma_{t \in F_i}\, y_t + A \in a_{m+1}p + a_{m+2}p + \ldots + a_l p$;

(V) if $r < n - 1$, $F_0, F_1, \ldots, F_{l-1}$ are nonempty subsets of $\{0, 1, \ldots, r\}$, for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^{m} F_i$, $y_t \in H_t$, then

$$D_{r+1} \subseteq {a_l}^{-1}(-\Sigma_{i=0}^{l-1} a_i \Sigma_{t \in F_i}\, y_t + A); \text{ and}$$

(VI) if $r < n - 1$, $m \in \{0, 1, \ldots, l-2\}$, $F_0, F_1, \ldots, F_m$ are nonempty subsets of $\{0, 1, \ldots, r\}$, for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^{m} F_i$, $y_t \in H_t$, then $D_{r+1} \subseteq \{x \in \mathbb{Z} : -a_{m+1}x + (-\Sigma_{i=0}^{m} a_i \Sigma_{t \in F_i}\, y_t + A) \in a_{m+2}p + a_{m+3}p + \ldots + a_l p\}$.

At $n = 1$, hypotheses (I), (II), and (IV) hold directly while (III), (V), and (VI) are vacuous.

For $m \in \{0, 1, \ldots, l-1\}$, let

$$G_m = \{\Sigma_{i=0}^{m} a_i \Sigma_{t \in F_i}\, y_t : \; F_0, F_1, \ldots, F_m \text{ are nonempty subsets of } \{0, 1, \ldots, n-1\}, \text{ for each } i \in \{0, 1, \ldots, m-1\}, \ \max F_i < \min F_{i+1}, \text{ and for each } t \in \bigcup_{i=0}^{m-1} F_i, \ y_t \in H_t\}.$$

For $k \in \{0, 1, \ldots, n-1\}$, let

$$E_k = \{\Sigma_{t \in F}\ y_t : \emptyset \neq F \subseteq \{0, 1, \ldots, n-1\}\,,\ \min F = k\,,\\ \text{and for each } t \in F\,,\ y_t \in H_t\}\,.$$

Given $b \in E_k$, we have that $b \in {D_k}^\star$ by hypothesis (II) and so $-b + {D_k}^\star \in p$. If $d \in G_{l-1}$, then by (IV), $-d + A \in a_l p$ so that ${a_l}^{-1}(-d + A) \in p$. If $m \in \{0, 1, \ldots, l-2\}$ and $d \in G_m$, then by (IV), $-d + A \in a_{m+1}p + a_{m+2}p + \ldots + a_l p$ so that

$$\{x \in \mathbb{Z} : -a_{m+1}x + (-d + A) \in a_{m+2}p + \ldots + a_l p\} \in p\,.$$

Thus we have that $D_n \in p$, where
$D_n =$

$$D_{n-1} \cap \bigcap_{k=0}^{n-1} \bigcap_{b \in E_k} (-b + {D_k}^\star) \cap \bigcap_{d \in G_{l-1}}\ {a_l}^{-1}(-d + A)\\ \cap \bigcap_{m=0}^{l-2} \bigcap_{d \in G_m} \{x \in \mathbb{Z} : -a_{m+1}x + (-d + A) \in a_{m+2}p + \ldots + a_l p\}.$$

(Here, if say $l = 1$ or $n < l$, we are using the convention that $\bigcap \emptyset = \mathbb{Z}$.)

Pick, again by Theorem 1.5(b), $x_{\alpha_n}, x_{\alpha_n+1}, \ldots, x_{\alpha_{n+1}-1} \in \mathbb{N}$ such that

$$B_n \begin{pmatrix} x_{\alpha_n} \\ x_{\alpha_n+1} \\ \vdots \\ x_{\alpha_{n+1}-1} \end{pmatrix} \in ({D_n}^\star)^{u_n}\,.$$

Let H_n be the set of entries of

$$B_n \begin{pmatrix} x_{\alpha_n} \\ x_{\alpha_n+1} \\ \vdots \\ x_{\alpha_{n+1}-1} \end{pmatrix}\,.$$

Then hypotheses (I), (III), (V), and (VI) hold directly.

To verify hypothesis (II), let $\emptyset \neq F \subseteq \{0, 1, \ldots, n\}$, let $k = \min F$, and for $t \in F$, let $y_t \in H_t$. If $n \notin F$, then $\Sigma_{t \in F}\ y_t \in {D_k}^\star$ by hypothesis (II) at $n-1$, so assume that $n \in F$. If $F = \{n\}$, then we have that $y_n \in D_n{}^\star$ directly so assume that $F \neq \{n\}$. Let $b = \Sigma_{t\in F\setminus\{n\}}\ y_t$. Then $b \in E_k$ and so $y_n \in -b + D_k{}^\star$ and thus $b + y_n \in D_k{}^\star$ as required.

To verify hypothesis (IV), let $m \in \{0, 1, \ldots, l-1\}$ and let $F_0, F_1, \ldots, F_m$ be nonempty subsets of $\{0, 1, \ldots, n\}$ such that for each $i \in \{0, 1, \ldots, m-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, let $y_t \in H_t$. If $m = 0$, then $\Sigma_{t\in F_0}\ y_t \in D_0{}^\star$ by (II) and (III) so that $-a_0\Sigma_{t\in F_0}\ y_t + A \in a_1p + a_2p + \ldots + a_lp$ as required.

So assume that $m > 0$. Let $k = \min F_m$ and $j = \max F_{m-1}$. Then

$$\begin{array}{llll} \Sigma_{t\in F_m}\ y_t & \in & D_k{}^\star & \text{by (II)} \\ & \subseteq & D_{j+1} & \text{by (III)} \\ & \subseteq & \{x \in \mathbb{Z} : \ -a_m x + (-\Sigma_{i=0}^{m-1} a_i \Sigma_{t\in F_i}\ y_t + A) & \\ & & \qquad \in a_{m+1}p + a_{m+2}p + \ldots + a_lp\} & \text{by (VI)} \end{array}$$

as required.

The induction being complete, we claim that whenever $F_0, F_1, \ldots, F_l$ are nonempty subsets of ω such that for each $i \in \{0, 1, \ldots, l-1\}$, $\max F_i < \min F_{i+1}$, and for each $t \in \bigcup_{i=0}^l F_i$, $y_t \in H_t$, then $\Sigma_{i=0}^l a_i \Sigma_{t\in F_i}\ y_t \in A$. To see this, let $k = \min F_l$ and let $j = \max F_{l-1}$. Then $\Sigma_{t\in F_l}\ y_t \in D_k{}^\star \subseteq D_{j+1} \subseteq a_l{}^{-1}(-\Sigma_{i=0}^{l-1} a_i \Sigma_{t\in F_i}\ y_t + A)$ by hypothesis (V), and so $\Sigma_{i=0}^l a_i \Sigma_{t\in F_i}\ y_t \in A$ as claimed.

Let Q be an insertion matrix of $\langle B_t\rangle_{t<\omega}$ into C. We claim that all entries of $Q\vec{x}$ are in A. To see this, let $\gamma < \omega$ be given and let $j \in \times_{t<\omega}\{0, 1, \ldots, u_t - 1\}$, so that

$$c_{\gamma,0} \cdot \vec{b}^{(0)}_{j(0)} \frown c_{\gamma,1} \cdot \vec{b}^{(1)}_{j(1)} \frown \ldots$$

is a typical row of Q, say row δ. For each $t \in \{0, 1, \ldots, m\}$, let $y_t = \Sigma_{k=0}^{v_t-1} b^{(t)}_{j(t),k} \cdot x_{\alpha_t+k}$ (so that $y_t \in H_t$). Then $\Sigma_{s=0}^\infty q_{\delta,s} \cdot x_s = \Sigma_{t=0}^m c_{\gamma,t} \cdot y_t$. Choose nonempty subsets $F_0, F_1, \ldots, F_l$ of $\{0, 1, \ldots, m\}$ such that for each $i \in \{0, 1, \ldots, l-1\}$, $\max F_i < \min F_{i+1}$ and for each $i \in \{0, 1, \ldots, l\}$ and each $t \in F_i$, $c_{\gamma,t} = a_i$. (One can do this because C is a Milliken-Taylor matrix for $\vec{a}$.) Then $\Sigma_{t=0}^m c_{\gamma,t} \cdot y_t = \Sigma_{i=0}^l a_i \Sigma_{t\in F_i}\ y_t \in A$. □

It is natural to expect that one could let C be any image partition regular matrix in Theorem 4.12. We see now that this fails badly, even in the finite case and even with the sequence $\langle B_t\rangle_{t<\delta}$ taken to be constant.

Theorem 4.13. *Let* $C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ *and let*

$$B_0 = B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then C is image partition regular, $B_0 = B_1$ is a first entries matrix, and any insertion matrix of $\langle B_t\rangle_{t<2}$ into C is not image partition regular.

Proof. Trivially C is image partition regular and $B_0 = B_1$ is a first entries matrix.

Let A be an insertion matrix of $\langle B_t\rangle_{t<2}$ into C. The rows of the matrix

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$$

consist of some of the rows of A and one of the columns of this matrix is $\vec{0}$ and so it suffices to show that the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

is not image partition regular.

It is an easy exercise to show that there do not exist $t_0, t_1, t_2 \in \mathbb{Q}^+$ such that the matrix

$$\begin{pmatrix} t_0 & 2t_1 & 0 & -1 & 0 & 0 & 0 \\ t_0 & 2t_1 & 2t_2 & 0 & -1 & 0 & 0 \\ 2t_0 & t_1 & 0 & 0 & 0 & -1 & 0 \\ 2t_0 & 0 & t_2 & 0 & 0 & 0 & -1 \end{pmatrix}$$

satisfies the columns condition. Theorems 1.3 and 1.5(d) then yield the desired conclusion. □

We do see that by taking the sequence $\langle B_t\rangle_{t<\delta}$ to be constantly equal to B and not allowing choices of different rows from B, one is guaranteed a new image partition regular matrix.

Theorem 4.14. *Let C be an infinite image partition regular matrix and let B be a $u \times v$ (finite) image partition regular matrix. Let A be a matrix with all rows of the form*

$$c_{i,0} \cdot \vec{b}^\frown c_{i,1} \cdot \vec{b}^\frown c_{i,2} \cdot \vec{b} \ldots ,$$

where $i \in \omega$ and $\vec{b}$ is a row of B, is image partition regular.

Proof. Let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Let n be large enough so that whenever $\{1, 2, \ldots, n\}$ is r-colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $B\vec{x}$ are monochrome. (This is possible by a standard compactness argument. See, for example, [9, Section 5.5].) Color $\mathbb{N}$ with r^n colors via ψ where $\psi(x) = \psi(y)$ if and only if $\varphi(tx) = \varphi(ty)$ for all $t \in \{1, 2, \ldots, n\}$. Pick $\vec{y} \in \mathbb{N}^\omega$ such that the entries of $C\vec{y}$ are monochrome with respect to ψ.

Choose an entry a of $C\vec{y}$ and define $\gamma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, r\}$ by $\gamma(i) = \varphi(ia)$. Pick $\vec{x} \in \mathbb{N}^v$ such that the entries of $B\vec{x}$ are monochrome with respect to γ. Define $\vec{z} \in \mathbb{N}^\omega$ by specifying that for $l \in \omega$ and $j \in \{0, 1, \ldots, v-1\}$, $z_{lv+j} = y_l \cdot x_j$. Pick an entry d of $B\vec{x}$. We show that for any entry g of $A\vec{z}$, $\varphi(g) = \varphi(da)$, so let an entry g be given. Then for some $i \in \omega$, $s \in \{1, 2, \ldots, u\}$, and $m \in \mathbb{N}$,

$$g = \textstyle\sum_{l=0}^{m} \sum_{j=0}^{v-1} c_{i,l} \cdot b_{s,j} \cdot y_l \cdot x_j = \alpha \cdot \delta$$

where $\alpha = \sum_{l=0}^{m} c_{i,l} \cdot y_l$ and $\delta = \sum_{j=0}^{v-1} b_{s,j} \cdot x_j$. Then δ and d are entries of $B\vec{x}$ and so $\varphi(\delta a) = \varphi(da)$. Also α and a are entries of $C\vec{y}$ and so $\varphi(\delta\alpha) = \varphi(\delta a)$. □

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