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ON A CLASS OF HEREDITARILY PARACOMPACT SPACES

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Abstract

A topological space (X, τ) is called upholstered provided that for any quasi-pseudometric q on X such that $\tau_q \subseteq \tau$ there is a pseudometric p on X such that $\tau_q \subseteq \tau_p \subseteq \tau$. Each upholstered space is shown to be a perfect paracompact regular space and every perfect compact regular space is shown to be upholstered.

Each semi-stratifiable paracompact regular space is upholstered and each quasi-metrizable upholstered space is metrizable. The property of upholsteredness is preserved under closed continuous surjections.

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1. Introduction

The paper is an attempt to improve our understanding of nonsymmetric distance functions. We define and study a class of topological spaces that are so intrinsically symmetric that their topology is — in some sense — completely determined by symmetric distance functions. Nonsymmetric distance functions do not make any essential contribution to the description of the internal structure of their topology.

Let us formulate this idea more precisely. It is well known that those topologies that can be generated by a family of pseudometrics are the completely regular ones. (On the other hand, it is clear that any topology can be generated by some family of quasi-pseudometrics (see e.g. [5]).)

In this paper we define and study the class of those topological spaces (X, τ) which have the property that for any quasi-pseudometric q on X such that $\tau_q \subseteq \tau$ there is a pseudometric p on X such that $\tau_q \subseteq \tau_p \subseteq \tau$.

Let us compare this definition with a similar definition given in [5] : A topological space (X, τ) is called *transitive* provided that for any quasi-pseudometric q on X such that $\tau_q \subseteq \tau$ there is a non-archimedean quasi-pseudometric n on X such that $\tau_n \subseteq \tau$ and $\mathcal{U}_q \subseteq \mathcal{U}_n$. (Here $\mathcal{U}_q$ (resp. $\mathcal{U}_n$) denote the quasi-uniformities determined by q (resp. n) on X.)

If we had just modelled our definition according to the definition of transitivity we were led to the following concept: A topological space (X, τ) is, say, strongly upholstered, provided that for each quasi-pseudometric q on X such that $\tau_q \subseteq \tau$ there is a pseudometric p on X such that $\mathcal{U}_q \subseteq \mathcal{U}_p$ and $\tau_p \subseteq \tau$.

However, this condition is easily seen to be equivalent to the condition that in X each open set is closed (or, equivalently, that for any point x in X, $\overline{\{x\}}$ is the smallest neighborhood of x). Hence it is too restrictive to be of any interest. This explains the choice above.

Let us also note that we can obtain an interesting notion of weak transitivity that is completely analogous to our concept. We shall say that a topological space (X, τ) is called *weakly transitive* provided that for each quasi-pseudometric q on X such that $\tau_q \subseteq \tau$ there is a non-archimedean quasi-pseudometric n on X such that $\tau_q \subseteq \tau_n \subseteq \tau$.

Clearly any quasi-pseudometric space that is not non-archimedeanly quasi-pseudometrizable (e.g. the Kofner plane, see [5]) is not weakly transitive. On the other hand, a non-archimedeanly quasi-pseudometrizable space that is not transitive would yield an example of a weakly transitive space that is not transitive. Unfortunately, it is unknown whether such a space exists. (In fact, it is straightforward to verify that if there exists a weakly transitive nontransitive space, then there exists a non-archimedean quasi-pseudometric space that is not transitive.)

Let us finally observe that the basic idea behind the definition of the concept of upholsteredness can be used to define numerous other properties that may be of interest elsewhere. In general, we could say that a topological space (X, τ) has property $[P/Q]$ provided that for any topology τ_1 satisfying $\tau_1 \subseteq \tau$ and having property P, there is a topology τ_2 satisfying $\tau_1 \subseteq \tau_2 \subseteq \tau$ and possessing property Q. Obviously, in this terminology the upholstered spaces are just the [quasi-pseudometrizable/pseudometrizable] ones.

We refer the reader to [5] for concepts of the theory of quasi-uniform spaces that are not explained in this paper. We just mention here that we shall be dealing mainly with three compatible quasi–uniformities of a topological space X, the *fine quasi–uniformity* $\mathcal{FINE}$, the *fine transitive quasi–uniformity* $\mathcal{FT}$ and the *point–finite quasi–uniformity* $\mathcal{PF}$. The first consists of all normal neighbornets of X, the second is (the filter) generated by all transitive neighbornets of X while the third one is generated by all neighbornets of the form $D\mathcal{U}$, where $\mathcal{U}$ is a point–finite open family of X (here the neighbornet $D\mathcal{U}$ is defined by the condition $D\mathcal{U}(x) = \bigcap(\mathcal{U})_x$, where $(\mathcal{U})_x = \{U \in \mathcal{U} : x \in U\})$. Note that the quasi–uniformity $\mathcal{FT}$ can also be generated by open families, since transitive neighbornets of X are exactly the relations of the form $V = D\mathcal{U}$, where $\mathcal{U}$ is an interior–preserving open family of X.

2. Main Results

A *quasi-pseudometric* q on a set X is a map $q : X \times X \to [0, \to[$ such that

(i) $q(x,x) = 0$ whenever $x \in X$,

(ii) $q(x,y) \leq q(x,z) + q(z,y)$ whenever $x, y, z \in X$.

If q also satisfies

(iii) $q(x,y) = q(y,x)$ whenever $x, y \in X$,

then we shall say, as usual, that q is a *pseudometric*. For any $\epsilon > 0$ and any $x \in X$ set $B^q_\epsilon(x) = \{y \in X : q(x,y) < \epsilon\}$. The collection $\{B^q_\epsilon(x) : x \in X, \epsilon > 0\}$ is a base for a topology τ_q, said to be *induced by* q on X.

Definition 1. Let (X, τ) be a topological space. We shall say that X (or τ) is *upholstered* provided that whenever q is a quasi-pseudometric on X such that $\tau_q \subseteq \tau$, then there is a pseudometric p on X such that $\tau_q \subseteq \tau_p \subseteq \tau$.

Remark 1. (a) Each pseudometrizable space is upholstered and each quasi-pseudometrizable upholstered space is pseudometrizable.

(b) An upholstered space that can be condensed onto a T_0-quasi-pseudometrizable space is submetrizable.

Proof. The assertions are obvious. □

Lemma 1. *Each upholstered space* (X, τ) *is a perfect and paracompact regular space.*

Proof. Let G be open in X and let q be a quasi-pseudometric on X that induces the quasi-uniformity fil$\{(G\times G)\cup[(X\setminus G)\times X]\}$ on X. Because X is upholstered, there is some pseudometric p on X such that $G\in\tau_q\subseteq\tau_p\subseteq\tau$. Since τ_p is perfect, we conclude that τ is perfect. Furthermore it follows from the argument that τ is the supremum of pseudometrizable topologies. Hence it is completely regular.

Let $\mathcal{C}=\{C_\alpha:\alpha<\beta\}$ be a well-monotone open increasing cover in (X,τ). Let q be a quasi-pseudometric on X that induces the quasi-uniformity fil$\{D\mathcal{C}\}$. Then there is a pseudometric p on X such that $\mathcal{C}\subseteq\tau_q\subseteq\tau_p\subseteq\tau$. Since τ_p is paracompact, we conclude that $\mathcal{C}$ has a τ_p-locally finite τ_p-open refinement. Hence $\mathcal{C}$ has a locally finite open refinement in (X,τ). It follows that X is paracompact [13, Corollary 6]. $\square$

Corollary 1. *Each quasi-developable upholstered T_0-space X is metrizable.*

Proof. Each quasi-developable perfect space is developable [6, p. 480]. Hence X is metrizable by Bing's Theorem [6, p. 426], since it is paracompact. $\square$

The following result shows that upholstered spaces are transitive in a very strong sense.

Lemma 2. *The fine quasi-uniformity $\mathcal{FINE}$ of each upholstered space is equal to its point-finite quasi-uniformity $\mathcal{PF}$.*

Proof. Let $\langle V_n\rangle_{n\in\omega}$ be a normal sequence of neighbornets of an upholstered space (X,τ), and let q be a quasi-pseudometric on X that is compatible with the quasi-uniformity generated by $\{V_n : n\in\omega\}$. Let p be a pseudometric on X such that $\tau_q\subseteq\tau_p\subseteq\tau$. Hence $\langle V_n\rangle_{n\in\omega}$ is a normal sequence of neighbornets on (X,τ_p). Since the fine quasi-uniformity of a pseudometrizable space is equal to its point-finite quasi-uniformity (by [9, Corollary 4.13]), we conclude that the fine quasi-uniformity of (X,τ) is equal to its point-finite quasi-uniformity. $\square$

In many situations, we do not need the full strength of the condition $\mathcal{FINE} = \mathcal{PF}$, since the weaker condition $\mathcal{FT} = \mathcal{PF}$ often suffices. The last condition has a natural formulation which does not involve quasi–uniformities. In [11], a closure–preserving closed family $\mathcal{K}$ of a topological space X is called *special* provided that X has a point–finite open family $\mathcal{U}$ such that we have $X \setminus K = \bigcup\{U \in \mathcal{U} : U \cap K = \emptyset\}$ for every $K \in \mathcal{K}$. Since a family $\mathcal{K}$ of subsets of X is closure–preserving if, and only if, the family $\{X \setminus K : K \in \mathcal{K}\}$ is interior–preserving, we see that the equality $\mathcal{FT} = \mathcal{PF}$ holds for a space X if, and only if, every closure–preserving closed family of X is special. In particular, we have the following consequence of Lemma 2.

Corollary 2. *Every closure–preserving family of closed subsets of an upholstered space is special.*

Remark 2. Note that a result of Sconyers [19] (for a proof, see [10]) and an argument similar to that used in the proof of Lemma 1 show that any space having $\mathcal{FINE} = \mathcal{PF}$ (or just $\mathcal{FT} = \mathcal{PF}$) is hereditarily metacompact.

Next we intend to show that the properties of upholstered spaces mentioned in Lemmas 1 and 2 actually yield a characterization of upholsteredness. We shall need the following auxiliary result.

Lemma 3. *Let $\mathcal{U}$ be a σ–point–finite family of open subsets of a perfect paracompact regular space X. Then $\mathcal{U}$ is an open family in some coarser pseudometrizable topology.*

Proof. The proof depends on A.H. Stone's "Coincidence Theorem" [22] according to which paracompact regular spaces are "fully normal" in the sense of J.W. Tukey [24]. Note that the result of the lemma follows easily once we prove the corresponding result for "point–finite" instead of "σ–point–finite". Hence we may assume that $\mathcal{U}$ is point–finite.

For every $n \in \mathbf{N}$, let $\mathcal{V}_n = \{\bigcap \mathcal{U}' : \mathcal{U}' \subseteq \mathcal{U} \text{ and } |\mathcal{U}'| = n\}$; further, let $\langle F_{n,k} \rangle_{k=1}^{\infty}$ be a sequence of closed subsets of the perfect space X such that $\bigcup_{k=1}^{\infty} F_{n,k} = \bigcup \mathcal{V}_n$. Note that, for all $n, k \in \mathbf{N}$, the family $\mathcal{W}_{n,k} = \mathcal{V}_n \cup \{X \setminus F_{n,k}\}$ is an open cover of X and therefore, by full normality, there exists a continuous pseudometric $d_{n,k}$ on X such that, for every $x \in X$, we have that $\{y \in X : d_{n,k}(x, y) < 1\} \subseteq \mathrm{St}(x, \mathcal{W}_{n,k})$; we may assume that $d_{n,k}$ is bounded by 1.

Let $d = \Sigma_{n,k \in \mathbf{N}} 2^{-n-k} d_{n,k}$ and observe that d is a continuous pseudometric on X. Also note that, for every $x \in X$, there exist $i, j \in \mathbf{N}$ such that we have $\mathrm{St}(x, \mathcal{W}_{i,j}) = \bigcap (\mathcal{U})_x$; then the set $\bigcap (\mathcal{U})_x$ is a neighborhood of x in the topology $\tau_{d_{i,j}}$ and hence also in the topology τ_d. It follows that $\mathcal{U} \subseteq \tau_d$. $\square$

Remark 3. The above result remains true with "collectionwise normal" replacing "paracompact", but then the proof is a slightly more complicated adaption of the Michael–Nagami technique (see [16] and [17]).

Proposition 1. *A topological space is upholstered if, and only if, the space is perfect, paracompact and regular and the equality $\mathcal{PF} = \mathcal{FINE}$ holds.*

Proof. We have already seen that the stated conditions are necessary for a space to be upholstered. To show that they are sufficient, assume that a space (X, τ) satisfies them. Let d be a quasi–pseudometric on X such that $\tau_d \subseteq \tau$. For every $n \in \mathbf{N}$, the neighbornet V_n of (X, τ), where $V_n(x) = \{y \in X : d(x, y) < \frac{1}{n}\}$ for each $x \in X$, is normal; it follows, since $\mathcal{PF} = \mathcal{FINE}$, that there exists a point–finite open family $\mathcal{U}_n$ of the space (X, τ) such that $D\mathcal{U}_n \subseteq V_n$. By the previous lemma, there exists a continuous pseudometric ρ in the space (X, τ) such that the family $\bigcup_{n \in \mathbf{N}} \mathcal{U}_n$ is open in τ_ρ. It is easily seen that we have $\tau_d \subseteq \tau_\rho$. $\square$

Remark 4. Remarks 2 and 3 together with the result that every metacompact and collectionwise normal regular space is paracompact ([16] and [17]) indicate that the result of Proposition 1 remains true if "paracompact" is weakened to "collectionwise normal".

Now we can easily exhibit one quite large class of upholstered spaces.

Proposition 2. *Every paracompact semi–stratifiable regular space is upholstered.*

Proof. Every semi–stratifiable space is perfect and hence the result follows from [9, Corollary 4.13] and Proposition 1. □

Note that it has been earlier shown by S. Oka [18] and Ju. Bregman [1] that every paracompact (Hausdorff) σ–space is "weakly upholstered" in the sense that every closure–preserving and closed family of such a space is closure–preserving and closed in some coarser metrizable topology (the latter condition formulated for "pseudometrizable" instead of "metrizable" is equivalent to "upholsteredness for non–archimedean quasi–pseudometrics").

We shall next derive another characterization for upholsteredness using the concept of mosaical families, due to K. Tamano [23]. Recall that a family $\mathcal{L}$ of subsets of a topological space X is said to be *mosaical* if the partition of X generated by $\mathcal{L}$ (in other words, the family $\{\bigcap(\mathcal{L})_x \setminus \bigcup(\mathcal{L} \setminus (\mathcal{L})_x) : x \in X\}$) has a σ–discrete closed refinement; such a refinement is called a *mosaic* for the family $\mathcal{L}$. Let us say that a space X is *cp–mosaical* if every closure–preserving family of closed subsets of X is mosaical. It is obvious that a family $\mathcal{L}$ of subsets of X is mosaical if, and only if, the family $\{X \setminus L : L \in \mathcal{L}\}$ is mosaical; as a consequence, X is cp–mosaical if, and only if, every interior–preserving family of open subsets of X is mosaical.

Cp–mosaical spaces first appeared (implicitly) in [20], where the proof of the main result showed that every σ-space is cp–mosaical; later, Theorem 4.8 of [9] established that every semi-stratifiable space is cp–mosaical.

Lemma 4. *Every upholstered space is cp–mosaical.*

Proof. Let X be an upholstered space and let $\mathcal{K}$ be a closure–preserving family of closed subsets of X. By Corollary 2, there exists a point–finite open family $\mathcal{U}$ of X such that we have $X \setminus K = \bigcup\{U \in \mathcal{U} : U \cap K = \emptyset\}$ for every $K \in \mathcal{K}$. For every $n \in \omega$, let $H_n = \{x \in X : |(\mathcal{U})_x| = n\}$ and note that H_n is the difference of two closed subsets of X. By Lemma 1, X is perfect; consequently, for every $n \in \omega$, we can write $H_n = \bigcup_{k\in\omega} F_{n,k}$, where the sets $F_{n,k}$ are closed in X. For every $n \in \omega$, the family $\mathcal{H}_n = \{H_n \cap \bigcap \mathcal{V} : \mathcal{V} \subseteq \mathcal{U} \text{ and } |\mathcal{V}| = n\}$ is discrete and closed in the subspace H_n of X; it follows that, for each $k \in \omega$, the family $\mathcal{F}_{n,k} = \{F_{n,k} \cap H : H \in \mathcal{H}_n\}$ is discrete and closed in X. It is easily seen that the family $\mathcal{F} = \bigcup\{\mathcal{F}_{n,k} : n, k \in \omega\}$ is a mosaic for $\mathcal{U}$. Since we have that $X \setminus K = \bigcup\{U \in \mathcal{U} : U \cap K = \emptyset\}$ for every $K \in \mathcal{K}$, we see that $\mathcal{U}$ generates a finer partition of X than $\mathcal{K}$. It follows that $\mathcal{F}$ is also a mosaic for $\mathcal{K}$. $\square$

Remark 5. Note that the above proof actually establishes a stronger result: a space is cp–mosaical provided that the space is perfect and every closure–preserving family of closed subsets of the space is special.

We shall next indicate some properties of cp–mosaical spaces. To start with, we give weak versions of Lemmas 1 and 2 for cp–mosaical spaces.

Lemma 5. *(a) Every cp–mosaical space is perfect and subparacompact.*

(b) We have $\mathcal{FT} = \mathcal{PF}$ in every metacompact cp–mosaical space.

Proof. The argument used in the last paragraph of the proof of Lemma 1 and Theorem 2 of [2] prove part (a) of the lemma. A straightforward modification of the proof of [9, Theorem 4.12] proves part (b). $\square$

Recall that a space Y is *right–separated* (*left–separated*) if there exists a well–order $\prec$ on Y such that the set $\{z \in Y : z \prec y\}$ is open (closed) for every $y \in Y$. A space is right–separated if, and only if, it is scattered. The following result shows that, in a cp–mosaical space, all right–separated subspaces and all closed left–separated subspaces are σ–discrete.

Lemma 6. *(a) Every right–separated subspace of a perfect subparacompact space is σ–discrete.*

(b) Every closed left–separated subspace of a cp–mosaical space is σ–discrete.

Proof. Since every subspace of a perfect subparacompact space is perfect and subparacompact, part (a) follows directly from [2, Theorem 2].

(b) Let S be a closed left–separated subspace of a cp–mosaical space X. Let $\prec$ be a well–order on S such that the set $F_y = \{z \in S : z \prec y\}$ is closed in S for every $y \in S$. Note that, for every $A \subseteq S$, we have either $\bigcup_{y\in A} F_y = S$ or $\bigcup_{y\in A} F_y = F_v$ for some $v \in S$; in both cases, the set $\bigcup_{y\in A} F_y$ is closed in S and hence also in X. By the foregoing, the family $\mathcal{F} = \{F_y : y \in S\}$ is closure–preserving and closed in X. Since X is cp–mosaical, the family $\mathcal{F}$ is mosaical. It is easily seen that the partition of X generated by $\mathcal{F}$ consists of the set $X \setminus S$ together with the singletons $\{y\}$, for $y \in S$. Since the partition has a σ–discrete refinement, the subspace S is σ–discrete. $\square$

Corollary 3. *Every hereditarily cp–mosaical space has a σ–discrete dense subset.*

Proof. This follows from the previous lemma together with the result of I. Juhász [8] that every space has a left–separated dense subset. $\square$

Note that it follows from the corollary that every hereditarily cp–mosaical Lindelöf–space is hereditarily separable.

Since collectionwise normal subparacompact regular spaces are paracompact ([15]), Lemmas 4 and 5 together with Proposition 1 establish the following characterization of upholstered spaces.

Proposition 3. *A space is upholstered if, and only if, the space is collectionwise normal, cp–mosaical and transitive.*

Since every suborderable space (i.e., every GO–space) is both collectionwise normal ([14] and [21]) and transitive ([12]), we have the following consequence of Proposition 3.

Corollary 4. *Every cp–mosaical suborderable space is upholstered.*

Next we shall show that, for compact regular spaces, perfectness alone is sufficient for upholsteredness. We shall need the following auxiliary result, which establishes the equality $\mathcal{FINE} = \mathcal{PF}$ for every hereditarily metacompact compact regular space. In fact, each hereditarily metacompact locally compact regular space satisfies the latter equality. This follows either by modifying the proof given below or by combining Proposition 4 with known results (see e.g. [5, Theorem 6.19], [11, Theorem 1.2 and Theorem 7.4]).

Proposition 4. *Let O be a neighbornet of a hereditarily metacompact compact regular space X. Then there exists a point–finite open family $\mathcal{V}$ of X such that $D\mathcal{V} \subseteq O^3$.*

Proof. Without loss of generality we suppose that $O(x)$ is open whenever $x \in X$. We shall modify the proof of Theorem 6.7 of [11]. Here we are not able to consider a game $G(\mathcal{J}, X)$ for some ideal $\mathcal{J}$ of closed subsets of X. Instead, we shall use as our collection of "small" sets the family

$$\{S \subseteq X : S \text{ is closed and there exists } x \in X \text{ such that } S \subseteq O(x) \cap \overline{O^{-1}(x)}\},$$

which is not, in general, an ideal. Moreover, we are not able to define the sets $s(F)$ witnessing the existence of a "winning strategy for Player I" for all closed sets $F \subseteq X$, but only for the sets belonging to the following family:

$$\mathcal{F} = \{F \subseteq X : F \text{ is closed and there exists } x \in X \text{ such that } x \in F \subseteq O(x)\}.$$

For every $F \in \mathcal{F}$, we define a point p_F and closed subset $s(F)$ of F as follows: for p_F we pick any point x satisfying $x \in F \subseteq O(x)$ and then we set $s(F) = F \cap \overline{O^{-1}(p_F)}$. These choices give us something corresponding to a "winning strategy":

Claim 1: There is no infinite sequence $\langle F_n \rangle$ of members of $\mathcal{F}$ such that we have $F_{n+1} \subseteq F_n \setminus s(F_n)$ for every $n \in \omega$.

Proof of Claim 1. Assume that $\langle F_n \rangle$ is such a sequence. Let $K = \bigcap_{n\in\omega} F_n$. The set $O(K)$ is a neighborhood of the set K and it follows by compactness of X, since $\langle F_n \rangle$ is a decreasing sequence of closed sets, that there exists $k \in \omega$ such that $F_k \subseteq O(K)$. We now have that $p_{F_k} \in O(K)$, in other words, that $O^{-1}(p_{F_k}) \cap K \neq \emptyset$. Since $K \subseteq F_k$, it follows from the foregoing that $s(F_k) \cap K \neq \emptyset$; however, from this it further follows, since $F_{k+1} \cap s(F_k) = \emptyset$, that $K \nsubseteq F_{k+1}$, and this is a contradiction.

We set $\mathcal{G} = \{V \subseteq X : V \text{ is open and } \overline{V} \in \mathcal{F}\}$.

Claim 2: To every open subset U of X there corresponds a finite family $\mathcal{G}(U) \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}(U) = U$.

Proof of Claim 2. For every $x \in \overline{U}$, let V_x be an open neighborhood of x such that $\overline{V_x} \subseteq O(x)$. Let A be a finite subset of $\overline{U}$ such that $\overline{U} \subseteq \bigcup_{x\in A} V_x$ and let $\mathcal{G}(U) = \{V_x \cap U : x \in A\}$.

We shall now modify the proof of Theorem 6.7 of [11] to get the following result:

Claim 3: There exists a point–finite family $\mathcal{U} \subseteq \mathcal{G}$ such that the family $\mathcal{H} = \{U \cap s(\overline{U}) : U \in \mathcal{U}\}$ covers X.

Proof of Claim 3. Let $U \in \mathcal{G}$. By regularity and hereditary metacompactness of X there exists a point–finite family $\mathcal{V}(U)$ of open subsets of X such that we have

$$\bigcup\{V : V \in \mathcal{V}(U)\} = \bigcup\{\overline{V} : V \in \mathcal{V}(U)\} = U \setminus s(\overline{U}) \ .$$

Note that, by Claim 2, we may assume that $\mathcal{V}(U) \subseteq \mathcal{G}$: if this were not already the case, we could replace $\mathcal{V}(U)$ by $\bigcup\{\mathcal{G}(V) : V \in \mathcal{V}(U)\}$.

We define inductively point–finite families $\mathcal{U}_n \subseteq \mathcal{G}$ by setting $\mathcal{U}_0 = \mathcal{G}(X)$ and $\mathcal{U}_{n+1} = \bigcup\{\mathcal{V}(U) : U \in \mathcal{U}_n\}$. Similarly as in the proof of Theorem 6.7 of [11], we shall show that Claim 3 holds true for the family $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$:

Suppose that $\mathcal{U}$ is not point-finite at some point $x \in X$. By König's Lemma there is a sequence $\langle U_n \rangle_{n\in\omega}$ such that $x \in \cap_{n\in\omega} U_n$ and for each $n \in \omega$, $U_{n+1} \in \mathcal{V}(U_n)$. Thus $\overline{U_{n+1}} \subseteq U_n \setminus s(\overline{U_n})$ for every $n \in \omega$ — a contradiction to Claim 1, because $\overline{U_n} \in \mathcal{F}$ for every n. We conclude that $\mathcal{U}$ is point-finite.

Suppose that some point $x \in X$ is not covered by $\mathcal{H}$. Then x is not in any set $U \cap s(\overline{U})$ where $U \in \mathcal{U}$. We construct inductively a sequence $\langle V_n \rangle_{n\in\omega}$ such that we have $x \in V_n \in \mathcal{U}_n$ for every $n \in \omega$ and $V_n \in \mathcal{V}(V_{n-1})$ whenever $n > 0$. Since $\bigcup \mathcal{U}_0 = \bigcup \mathcal{G}(X) = X$, there exists $V_0 \in \mathcal{U}_0$ such that $x \in V_0$. Suppose that $n > 0$ and that $V_{n-1} \in (\mathcal{U}_{n-1})_x$ has already been chosen. By our assumption, we have that $x \notin s(\overline{V_{n-1}})$ and it follows that there exists $V_n \in \mathcal{V}(V_{n-1})$ such that $x \in V_n$. This concludes the induction. As above, we deduce that $\overline{V_{n+1}} \subseteq V_n \setminus s(\overline{V_n})$ whenever $n \in \omega$ —a contradiction to Claim 1.

To complete the proof of Proposition 4, let $\mathcal{V} = \{U \cap O(p_{\bar{U}}) : U \in \mathcal{U}\}$. Note that $\mathcal{V}$ is a point–finite family of open subsets of X. We show that $D\mathcal{V} \subseteq O^3$. Let $x \in X$. There exists $U \in \mathcal{U}$ such that $x \in U \cap s(\overline{U})$. Note that, since $\overline{U} \in \mathcal{F}$, we have that $\overline{U} \subseteq O(p_{\bar{U}})$ and hence that $x \in s(\overline{U}) \subseteq O(p_{\bar{U}}) \cap \overline{O^{-1}(p_{\bar{U}})}$.

Furthermore, we deduce that $x \in U \cap O(p_{\bar{U}}) \in \mathcal{V}$ and hence that $D\mathcal{V}(x) = \bigcap\{V \in \mathcal{V} : x \in V\} \subseteq O(p_{\bar{U}})$. Since we have that $x \in \overline{O^{-1}(p_{\bar{U}})}$, there exists a point $y \in O(x) \cap O^{-1}(p_{\bar{U}})$. We now conclude that $y \in O(x)$ and $p_{\bar{U}} \in O(y)$ and it follows that $p_{\bar{U}} \in O^2(x)$ and, further, that $O(p_{\bar{U}}) \subseteq O^3(x)$. As a consequence, we see that $D\mathcal{V}(x) \subseteq O(p_{\bar{U}}) \subseteq O^3(x)$. □

Remark 6. According to Corollary 4.13 of [9], the corresponding result holds also for all metacompact semi–stratifiable spaces.

Combining Propositions 1 and 4, we get the following result:

Proposition 5. *Every perfect compact regular space is upholstered.*

Since a compact regular space is perfect if, and only if, the space is hereditarily Lindelöf, the above result gives many "exotic" consistent examples, such as a Souslin line, of upholstered spaces. A well-known absolute example, the "two arrows space" (see [4, p. 212]), can be used to show that the property of being upholstered is neither productive nor hereditary:

Remark 7. Since the two arrows space X does not have a G_δ-diagonal, the product $X \times X$ cannot be upholstered.

Because the two arrows space contains quasi-metrizable subspaces that are not metrizable, it is not hereditarily upholstered.

We close the paper by showing that F_σ–subspaces and closed images of upholstered spaces are upholstered.

Proposition 6. *An F_σ–subspace of an upholstered space is upholstered.*

Proof. Let X be an upholstered space, let $\langle F_n \rangle_{n\in\omega}$ be a sequence of closed subsets of X and let $A = \bigcup_{n\in\omega} F_n$. We shall use Proposition 1 to show that A is upholstered. Since X is perfect, paracompact and regular, so is the subspace A. As an F_σ–subspace of the transitive space X, the space A is transitive (see [5]), i.e., the fine transitive quasi–uniformity $\mathcal{FT}$ of A is the fine quasi–uniformity. Hence to show that the conditions of Proposition 1 are satisfied, it suffices to show that $\mathcal{PF} = \mathcal{FT}$ in the space A. Let U be a transitive neighbornet of A. For every $n \in \omega$, define a transitive neighbornet U_n of X by setting $U_n(x) = X \setminus F_n$ for $x \notin F_n$ and $U_n(x) = U(x) \cup (X \setminus F_n)$ for $x \in F_n$. Since $\mathcal{PF} = \mathcal{FINE}$ in X, we can find, for each $n \in \omega$, a point–finite open family $\mathcal{V}_n$ in X such that $D\mathcal{V}_n \subseteq U_n$; we may assume that $\mathcal{V}_n$ is closed under finite intersections. For every $n \in \omega$ and for each $V \in \mathcal{V}_n$, denote by $\hat{V}$ the open subset $\bigcap\{U(x) : x \in A \text{ and } V \cap F_n \subseteq U(x)\}$ of A; further, denote by $\mathcal{W}_n$ the point–finite open family $\{(V \cap \hat{V}) \setminus \bigcup_{k<n} F_k : V \in \mathcal{V}_n\}$ of A. Let $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$ and note that $\mathcal{W}$ is a point–finite family of open subsets of A. To see that $D\mathcal{W} \subseteq U$, let $x \in A$. Denote by l the least $n \in \omega$ satisfying $x \in F_n$. Let $V = D\mathcal{V}_l(x)$ and note that $V \in \mathcal{V}_l$. We have that $x \in V \subseteq U_l(x)$ and it follows, since $x \in F_l$, that $V \subseteq U(x) \cup (X \setminus F_l)$, in other words, that $V \cap F_l \subseteq U(x)$; as a consequence, we have that $\hat{V} \subseteq U(x)$. Therefore $x \in (\hat{V} \cap V) \setminus \bigcup_{k<l} F_k \in \mathcal{W}_l \subseteq \mathcal{W}$. By the foregoing, we conclude that $D\mathcal{W}(x) \subseteq \hat{V} \subseteq U(x)$. $\square$

The following lemma is quite useful to show that some spaces are upholstered. Recall that a quasi-pseudometric q is called *strong* provided that $\tau_q \subseteq \tau_{q^{-1}}$.

Lemma 7. *Let (X, τ) be a perfect, collectionwise normal space and q a strong quasi-pseudometric on X such that $\tau_q \subseteq \tau$. Then there is a pseudometric p on X such that $\tau_q \subseteq \tau_p \subseteq \tau$.*

Proof. The topology τ_q is developable and thus has, by subparacompactness [6, p. 428], a τ_q-closed network $\mathcal{N} = \bigcup_{n\in\omega} \mathcal{N}_n$ such that each collection $\mathcal{N}_n$ is τ_q-discrete. Fix $n \in \omega$. Since X is collectionwise normal, for each $m \in \omega$ in (X, τ) there exists a discrete open collection $\mathcal{G}_{nm}$ such that $\mathcal{G}_{nm} = \{G(m, F) : F \in \mathcal{N}_n\}$

where $F \subseteq G(m,F) \subseteq B^q_m(F)$. Since (X,τ) is perfectly normal, each $\mathcal{G}_{nm}$ consists of co-zero sets. For each $G \in \mathcal{G}_{nm}$ choose a continuous function $f_G : (X,\tau) \to [0,1]$ such that $X \setminus G = f_G^{-1}\{0\}$. Set $\rho_{n,m}(x,y) = \sum_{G\in\mathcal{G}_{n,m}} |f_G(x) - f_G(y)|$ whenever $x,y \in X$. We now define by $\rho = \sum_{n,m\in\omega} 2^{-n-m}\rho_{n,m}$ a pseudometric ρ on X such that $\tau_q \subseteq \tau_\rho \subseteq \tau$. □

Proposition 7. *Let X be upholstered and let $f : X \to Y$ be a closed continuous surjection. Then Y is upholstered.*

Proof. Denote by τ and π the topologies of X and Y, respectively.

Let q_0 be a quasi-pseudometric on Y such that $\tau_{q_0} \subseteq \pi$. Without loss of generality assume that $q_0 \le 1$. By induction we shall define a sequence $\langle q_n\rangle_{n\in\omega}$ of quasi-pseudometrics on Y such that for each $n \in \omega$, $\tau_{q_n} \subseteq \tau_{q_{n+1}} \subseteq \pi$. Suppose that for some $n \in \omega$, q_k is defined whenever $k \le n$. Let q'_n be the quasi-pseudometric on X defined by $q'_n(x,y) = q_n(f(x), f(y))$ whenever $x, y \in X$. Since f is continuous, we see that $\tau_{q'_n} \subseteq \tau$. Because X is upholstered, there is a pseudometric p'_n on X such that $\tau_{q'_n} \subseteq \tau_{p'_n} \subseteq \tau$.

Let $\mathcal{N}_n = \bigcup_{m\in\omega}\mathcal{N}_{nm}$ be a $\tau_{p'_n}$-closed network such that each collection $\mathcal{N}_{nm}$ is $\tau_{p'_n}$-locally finite. Since f is closed, $f(\mathcal{N}_{nm})$ is closure-preserving and closed in π. For each $m \in \omega$ set $T_{nm} = (\bigcup_{x\in Y}(\{x\} \times T_{nm}(x))$ where $T_{nm}(x) = Y \setminus \bigcup\{f(F) \in \mathcal{N}_{nk} : k \le m \text{ and } x \notin f(F)\}$ and $x \in Y$. Let q_{n+1} be a quasi-pseudometric bounded by 1 and inducing the quasi-uniformity fil$\{S_m : m \in \omega\}$ on Y where $S_m = \{(x,y) \in Y \times Y : q_n(x,y) < 2^{-m}$ and $y \in T_{nm}(x)\}$ whenever $m \in \omega$. Clearly $\tau_{q_n} \subseteq \tau_{q_{n+1}} \subseteq \pi$. Let us note that $\tau_{q_n} \subseteq \tau_{q^{-1}_{n+1}}$:

Indeed, consider $y \in Y, s \in \omega$ and let $y' \in X$ be such that $f(y') = y$. Then there are $m \in \omega$ and $F \in \mathcal{N}_{nm}$ such that $y' \in F \subseteq \{x' \in X : q'_n(y',x') < 2^{-s}\}$. Hence $y \in f(F) \subseteq \{z \in Y : q_n(y,z) < 2^{-s}\}$. Furthermore $T^{-1}_{nm}(y) \subseteq f(F)$. We conclude that $\tau_{q_n} \subseteq \tau_{q^{-1}_{n+1}}$.

Set $q = \sum_{n\in\omega} 2^{-n} q_n$. Clearly $\tau_{q_0} \subseteq \tau_q \subseteq \pi$. Moreover q is strong by construction, i.e. $\tau_q \subseteq \tau_{q^{-1}}$. We conclude by Lemma 7 that there is a pseudometric p on Y such that $\tau_{q_0} \subseteq \tau_p \subseteq \pi$, since collectionwise normality and perfectness are preserved by closed continuous surjections [4, p. 510]. Hence Y is upholstered. □

Problem 1. Characterize the hereditarily upholstered spaces. Are they semi-stratifiable? Is every compact hereditarily upholstered Hausdorff space metrizable (see e.g. [3])?

Problem 2. Characterize the upholstered suborderable spaces by a condition weaker than "cp-mosaical" (compare e.g. [7]).

Problem 3. Is the product of a compact metric space and an upholstered space upholstered?

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