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# SOME NOTIONS OF SIZE IN PARTIAL SEMIGROUPS

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**Abstract**

In a semigroup, the combinatorial definitions of *syndetic, piecewise syndetic*, and *IP* are equivalent to their algebraic characterizations in terms of $\beta S$. We introduce the analogous definitions and characterizations of *syndetic, piecewise syndetic*, and *IP* for an *adequate partial semigroup* and show that equivalence between the combinatorial definition and algebraic characterization is lost once we move from a semigroup to a partial semigroup. Where they exist, we show some of the interrelationships between the notions; and in the case of *IP*, we give some conditions for when the algebraic characterization and combinatorial definition are in fact equivalent.

## 1. Introduction

Given a set $S$, and a natural binary operation, it is often convenient to define the operation for only a subset of $S \times S$. Consider for instance the semigroup $(\mathcal{P}_f(\mathbb{N}), \cup)$, where $\mathcal{P}_f(\mathbb{N}) = \{F : F \text{ is a finite nonempty subset of } \mathbb{N}\}$. If we define $\varphi : (\mathcal{P}_f(\mathbb{N}), \cup) \to (\mathbb{N}, +)$ by $\varphi(F) = |F|$, then $\varphi$ is not a homomorphism.

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However, if we let

$$A \uplus B = \begin{cases} A \cup B & \text{if } A \cap B = \emptyset \\ \text{undefined} & \text{if } A \cap B \neq \emptyset \end{cases}$$

then $\varphi$ is a homomorphism on $(\mathcal{P}_f(\mathbb{N}), \uplus)$, in the sense that $\varphi(A \uplus B) = \varphi(A) + \varphi(B)$ whenever $(A \uplus B)$ is defined.

Another case in which we may need to restrict the domain of the operation occurs when the natural operation does not satisfy the closure property. For example, given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in the semigroup $(S, \cdot)$, let

$$T = FP(\langle x_n \rangle_{n=1}^{\infty}) = \left\{ \prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \right\},$$

where the products are taken in increasing order of indices. Then $(x_1 \cdot x_3) \cdot (x_2 \cdot x_4)$ is not likely to be in $T$ unless $x_2$ and $x_3$ commute, and $(x_1 \cdot x_3) \cdot (x_3 \cdot x_4)$ is not likely to be in $T$ at all. On the other hand, if we let $(\prod_{n \in F} x_n) * (\prod_{n \in G} x_n)$ be

$$\begin{cases} \prod_{n \in F \cup G} x_n & \text{if } \max F < \min G \\ \text{undefined} & \text{if } \max F \geq \min G \end{cases}$$

Then $T$ is closed under $*$.

$(\mathcal{P}_f(\mathbb{N}), \uplus)$ and $(T, *)$ above, are examples of adequate partial semigroups [1], which are defined next.

**Definition 1.1.** A *partial semigroup* is a pair $(S, *)$ where $*$ maps a subset of $S \times S$ to $S$ and for all $a, b, c, \in S$, $(a * b) * c = a * (b * c)$ in the sense that if either side is defined, then so is the other and they are equal.

There are several notions of size in an arbitrary semigroup, all of which have simple characterizations in terms of $\beta S$, the Stone-Čech compactification of $S$. Partial semigroups lead to a natural and interesting subsemigroup of $\beta S$, which will be the focus of much of this paper. Therefore, we remind the reader of the algebraic structure of $\beta S$. For a discrete semigroup $S$, we take $\beta S$ to be the set of all ultrafilters on $S$; and we identify the principal ultrafilters with the points of $S$. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets , as well as a basis for the closed sets of $\beta S$. We denote by $\cdot$ the natural extension of the operation on $S$ which makes $\beta S$ a compact right topological semigroup with $S$ contained in its topological center. So that, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$, is continuous; and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$, is continuous. The reader is referred to [4] for an elementary introduction to the algebra of $\beta S$.

**Definition 1.2.** Let $(S, *)$ be a partial semigroup.

(a) For $s \in S$, $\varphi(s) = \{t \in S : s * t \text{ is defined}\}$.

(b) For $H \in \mathcal{P}_f(S)$, $\sigma(H) = \bigcap_{s \in H} \varphi(s)$.

(c) $(S, *)$ is *adequate* iff $\sigma(H) \neq \emptyset$ for all $H \in \mathcal{P}_f(S)$.

(d) $\delta S = \bigcap_{x \in S} c\ell_{\beta S}(\varphi(x)) = \bigcap_{H \in \mathcal{P}_f(S)} c\ell_{\beta S}(\sigma(H))$.

Notice that adequacy of $S$ is exactly what is required to guarantee that $\delta S \neq \emptyset$. Also, if $S$ is in fact a semigroup then $\delta S = \beta S$. For an adequate partial semigroup $S$, $\delta S$ is in a natural way a compact right topological semigroup (see [3, Theorem 2.10]). This fact provides a natural context for the notions of size we wish to consider in an semigroup.

In an arbitrary semigroup the notions of *syndetic*, *piecewise syndetic*, and *IP*, all have simple algebraic characterizations in terms of $\beta S$. In turn these characterizations lead to simple definitions for partial semigroups in terms of $\delta S$. However, as the main results of this paper show that these equivalences are lost, for the most part, for partial semigroups.

In general, notions preceded by *"č-"*, will refer to the partial semigroup analog of the combinatorial definition of that notion. After a short section describing the algebra in $\delta S$, the paper is organized by the notions of size we consider. In each section we begin by giving the definition and characterizations as they are known for a semigroup. We then give the analogous definitions for an adequate partial semigroup. We will write all partial semigroups multiplicatively as $(S,*)$, and we assume that $S$ is discrete.

## 2. Algebra in $\delta S$

In this section we introduce some of the basic properties of the operation $*$ in $\delta S$. We will make repeated use of these properties throughout this paper. This section overlaps with much of [3, Section 2], and the results we use are stated here, without proof, for the reader's convenience.

**Definition 2.1.** Let $(S,*)$ be a partial semigroup.
For $s \in S$ and $A \subseteq S$, $s^{-1}A = \{t \in \varphi(s) : s * t \in A\}$.

Note that, as in a semigroup, and even more strongly here, the notation $s^{-1}A$ does not imply that the element $s$ has an inverse in $S$. However, we do see that in some sense the behavior does in fact resemble the case in which $s$ has an inverse.

**Lemma 2.2.** *Let $(S,*)$ be a partial semigroup, let $A \subseteq S$ and let $a,b,c \in S$. Then*

$$c \in b^{-1}(a^{-1}A) \Leftrightarrow b \in \varphi(a) \text{ and } c \in (a*b)^{-1}A.$$

*In particular, if $b \in \varphi(a)$, then $b^{-1}(a^{-1}A) = (a*b)^{-1}A$.*

*Proof.* [3, Lemma 2.3]. □

As a subsemigroup of $\beta S$, the members of $\delta S$ are ultrafilters. The following definition and results show how $*$ behaves on members of $\delta S$ and how $*$ is extended to $\beta S$.

First recall from [4, Theorem 4.12] that if $(S,\cdot)$ is a semigroup, $A \subseteq S$, $a \in S$, and $p$, $q \in \beta S$, then

$$A \in a \cdot q \Leftrightarrow a^{-1}A \in q$$

and

$$A \in p \cdot q \Leftrightarrow \{a \in S : a^{-1}A \in q\} \in p.$$

We have the following analog in the case of an adequate partial semigroup.

**Definition 2.3.** Let $(S, *)$ be an adequate partial semigroup.

(a) For $a \in S$ and $q \in \overline{\varphi(a)}$, $a * q = \{A \subseteq S : a^{-1}A \in q\}$.

(b) For $p \in \beta S$ and $q \in \delta S$, $p * q = \{A \subseteq S : \{a^{-1}A \in q\} \in p\}$.

**Lemma 2.4.** *Let $(S, *)$ be an adequate partial semigroup.*

*(a) If $a \in S$ and $q \in \overline{\varphi(a)}$, then $a * q \in \beta S$.*

*(b) If $p \in \beta S$ and $q \in \delta S$, then $p * q \in \beta S$.*

*(c) Let $p \in \beta S, q \in \delta S$, and $a \in S$. Then $\varphi(a) \in p * q$ if and only if $\varphi(a) \in p$.*

*(d) If $p, q \in \delta S$, then $p * q \in \delta S$.*

*Proof.* [3, Lemma 2.7]. □

**Lemma 2.5.** *Let $(S, *)$ be an adequate partial semigroup and let $q \in \delta S$. Then the function $\rho_q : \beta S \to \beta S$ defined by $\rho_q(p) = p*q$ is continuous.*

*Proof.* [3, Lemma 2.8]. □

**Lemma 2.6.** *Let $p \in \beta S$ and let $q, r \in \delta S$. Then $p * (q * r) = (p * q) * r$.*

*Proof.* [3, Lemma 2.9]. □

As a consequence of the above results, we have that if $(S, *)$ is an adequate partial semigroup, then $(\delta S, *)$ is a compact right topological semigroup.

## 3. Syndetic Sets

In this section we present some results about syndetic sets in an arbitrary partial semigroup. The terminology is, as mentioned in the introduction, borrowed from topological dynamics. The notion of a *syndetic* set originated in the context of $(\mathbb{N}, +)$ where a set $A$ is syndetic if and only if it has bounded gaps.

**Definition 3.1.** Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set $A$ is *syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $S \subseteq \bigcup_{t \in H} t^{-1}A$.

**Theorem 3.2.** *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set $A$ is syndetic if and only if for every left ideal $L$ of $\beta S$, $\overline{A} \cap L \neq \emptyset$.*

*Proof.* [2, Theorem 2.9(d)]. □

The combinatorial definition and the algebraic characterization of syndetic in a semigroup can both be extended to partial semigroups in a natural way. Since $*$ is defined for only a subset of $S$, we are unlikely to find a finite subset $H$ of $S$ such that $S \subseteq \bigcup_{t \in H} \varphi(t)$. Thus we cannot transfer, verbatim, the definition for syndetic to partial semigroups. However, a minor adjustment is sufficient.

**Definition 3.3.** Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$.

(a) The set $A$ is *č-syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}A$.

(b) The set $A$ is *syndetic* if and only if for every left ideal $L$ of $\delta S$, $\overline{A} \cap L \neq \emptyset$.

Note that the combinatorial definition of syndetic in a partial semigroup, (č-syndetic), guarantees that $S$ itself is syndetic.

Notice also that if $S$ is a semigroup, Definition 3.3 agrees with our semigroup definition and characterization of "syndetic".

The notions "syndetic" and and "$\check{c}$-syndetic" are not equivalent, though we shall see that every syndetic subset of an adequate partial semigroup is also $\check{c}$-syndetic. As an example of a set which is $\check{c}$-syndetic but not syndetic, we have the following.

**Theorem 3.4.** *There exists an adequate partial semigroup* $(T, *)$ *and a* $\check{c}$*-syndetic subset* $A$ *of* $T$ *which is not syndetic.*

*Proof.* Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in a semigroup $(S, \cdot)$ which satisfies uniqueness of finite products (meaning $\prod_{n\in F} x_n = \prod_{n\in G} x_n$ only when $F = G$), and $(T, *)$ is the partial semigroup introduced earlier, where

$$T = FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\textstyle\prod_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\},$$

and products are taken in increasing order of indices, with $(\prod_{n\in F} x_n) * (\prod_{n\in G} x_n)$ defined as

$$\begin{cases} \prod_{n\in F\cup G} x_n & \text{if } \max F < \min G \\ \text{undefined} & \text{if } \max F \geq \min G \end{cases}$$

Then the set $A = \{\prod_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } 1 \in F\}$ is $\check{c}$-syndetic but not syndetic.

To see that $A$ is $\check{c}$-syndetic, let $H = \{x_1\}$, so that $\sigma(H) = \varphi(x_1) = \{\prod_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F > 1\}$. Then $\sigma(H) \subseteq {x_1}^{-1}A$.

To see that $A$ is not syndetic, we show in fact that for *any* $p \in \delta S$, $\overline{A} \cap (\delta S * p) = \emptyset$. Suppose instead that we have $q \in \delta S$ such that $A \in q * p$. Then $\{x \in S : x^{-1}A \in p\} \in q$ and $\varphi(x_1) \in q$ so pick $y \in \varphi(x_1)$ such that $y^{-1}A \in p$. But $y^{-1}A = \emptyset$, a contradiction. Thus $A$ is not syndetic. □

As mentioned earlier, every syndetic set is a $\check{c}$-syndetic set. The proof of this fact leads to an algebraic characterization of $\check{c}$-syndetic sets in terms of $\beta S$. For emphasis, we state these as two separate results.

**Lemma 3.5.** *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. Then $A$ is č-syndetic if and only if for all $p \in \delta S$, $\overline{A} \cap (\beta S * p) \neq \emptyset$.*

*Proof.* Assume $A$ is č-syndetic. Let $p \in \delta S = \bigcap_{H \in \mathcal{P}_f(S)} \overline{\sigma(H)}$. Pick $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}A$. Since $\sigma(H) \in p$, $\bigcup_{t \in H} t^{-1}A \in p$. So there exists $t \in H$ such that $t^{-1}A \in p$. Then $t * p \in \overline{A}$. Therefore $\overline{A} \cap (\beta S * p) \neq \emptyset$.

Assume that for all $p \in \delta S$, $\overline{A} \cap (\beta S * p) \neq \emptyset$. Suppose that for all $H \in \mathcal{P}_f(S)$, $\sigma(H) \not\subseteq \bigcup_{t \in H} t^{-1}A$. Then for all $H \in \mathcal{P}_f(S)$, $\sigma(H) \setminus \bigcup_{t \in H} t^{-1}A \neq \emptyset$. Therefore $\{\sigma(H) \setminus \bigcup_{t \in H} t^{-1}A : H \in \mathcal{P}_f(S)\}$ has the finite intersection property. So pick $p \in \beta S$ such that $\{\sigma(H) \setminus \bigcup_{t \in H} t^{-1}A : H \in \mathcal{P}_f(S)\} \subseteq p$. Since $\{\sigma(H) : H \in \mathcal{P}_f(S)\} \subseteq p$ we have that $p \in \delta S$. So pick $q \in \beta S$ such that $A \in q * p$. Then $\{x \in S : x^{-1}A \in p\} \in q \neq \emptyset$. So pick $x$ such that $x^{-1}A \in p$. Since $\{x\} \in \mathcal{P}_f(S)$, $\sigma(\{x\}) \setminus x^{-1}A \in p$. But $x^{-1}A \in p$. This is a contradiction. Therefore $A$ must be č-syndetic. □

**Theorem 3.6.** *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. If $A$ is syndetic then $A$ is č-syndetic.*

*Proof.* Assume that $A \subseteq S$ is syndetic. Let $p \in \delta S$, then $\overline{A} \cap (\delta S * p) \neq \emptyset$. And $\delta S \subseteq \beta S$, so $\overline{A} \cap (\beta S * p) \neq \emptyset$. So by Lemma 3.5, $A$ is č-syndetic. □

Even though the notions of "syndetic" and "č-syndetic" are not equivalent, we see that they play an identical role in the characterization of members of the smallest ideal.

**Theorem 3.7.** *Let $(S,*)$ be an adequate partial semigroup and let $p \in \delta S$. The following statements are equivalent:*

*(a) $p \in K(\delta S)$.*

*(b) For all $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is syndetic.*

*(c) For all $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is č-syndetic.*

*(d) For all $q \in \delta S$, $p \in \delta S * q * p$.*

*Proof.* (b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (d). [3, Theorem 2.15].

(d) $\Rightarrow$ (a). Trivial.

(a) $\Rightarrow$ (b). Let $A \in p$ and let $B = \{x \in S : x^{-1}A \in p\}$. Let $L$ be a minimal left ideal of $\delta S$ with $p \in L$. We show that for every left ideal $L'$ of $\delta S$, $\overline{B} \cap L' \neq \emptyset$. Let $L'$ be a left ideal of $\delta S$. Then $L' * p$ is a left ideal of $\delta S$ and $L' * p \subseteq L$ because $L$ is a left ideal. So $L' * p = L$ (by the minimality of $L$). Pick $q \in L'$ such that $p = q * p$. Since $A \in p = q * p$, $B = \{x \in S : x^{-1}A \in p\} \in q$ and so $q \in \overline{B}$. □

## 4. Piecewise Syndetic Sets

In the semigroup $(\mathbb{N}, +)$, a set $A$ is piecewise syndetic if and only if there exist a fixed bound $b$ and arbitrary long intervals in which the gaps of $A$ are bounded by $b$. The term *piecewise syndetic* originated in this context. With respect to $\beta S$, piecewise syndetic sets are of particular importance because they characterize the smallest ideal $K(\beta S)$.

We begin, again, by reminding the reader of the definition and equivalent characterization of piecewise syndetic for a semigroup.

**Definition 4.1.** Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set $A$ is *piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that for all $T \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $T \cdot x \subseteq \bigcup_{t \in H} t^{-1}A$.

**Theorem 4.2.** *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set $A$ is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.*

*Proof.* [4, Theorem 4.40]. □

As was the case for the notion of syndetic, the definition and characterization of piecewise syndeticity in a semigroup extend naturally to partial semigroups.

**Definition 4.3.** Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$.

(a) The set $A$ is *č-piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that for all $T \in \mathcal{P}_f(S)$ there exists $x \in \sigma(T)$ such that $(T \cap \sigma(H)) * x \subseteq \bigcup_{t\in H} t^{-1}A$.

(b) The set $A$ is *piecewise syndetic* if and only if $\overline{A} \cap K(\delta S) \neq \emptyset$.

Notice that the references to $\sigma(T)$ and $\sigma(H)$ in the definition of "č-piecewise syndetic" are needed to guarantee that the operations occurring therein are defined.

As was the case with "syndetic", there is an algebraic characterization of "č-piecewise syndetic" which allows us to establish that it is implied by "piecewise syndetic".

**Theorem 4.4.** *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. $A$ is č-piecewise syndetic if and only if there exists $p \in K(\delta S)$ such that $\overline{A} \cap (\beta S * p) \neq \emptyset$.*

*Proof.* Assume that $A$ is č-piecewise syndetic and pick $H \in \mathcal{P}_f(S)$ as guaranteed. For $T \in \mathcal{P}_f(S)$ let $B(T) = \{x \in \sigma(T) : (T \cap \sigma(H)) * x \subseteq \bigcup_{t\in H} t^{-1}A\}$. Note that $B(T_1 \cup T_2) \subseteq B(T_1) \cap B(T_2)$ and by assumption each $B(T) \neq \emptyset$. So $\{B(T) : T \in \mathcal{P}_f(S)\}$ has the finite intersection property. So pick $p \in \beta S$ such that $\{B(T) : T \in \mathcal{P}_f(S)\} \subseteq p$. Since for all $T \in \mathcal{P}_f(S), B(T) \subseteq \sigma(T)$, we have $p \in \delta S$. Then $\delta S * p$ is a left ideal of $\delta S$ and so we can pick $q \in K(\delta S)$ such that $q \in \delta S * p$. We claim that $\overline{A} \cap (\beta S * q * p) \neq \emptyset$. It suffices to show that there exists $t \in S$ such that $A \in t * q * p$. Suppose not. Then $\bigcup_{t\in H} t^{-1}A \notin q * p$. So $\{s \in S : s^{-1}(\bigcup_{t\in H} t^{-1}A) \notin p\} \in q$. Also, $\sigma(H) \in q$ so pick $s \in \sigma(H)$ such that $s^{-1}(\bigcup_{t\in H} t^{-1}A) \notin p$. Let $T = \{s\}$. Then $B(T) \in p$. Pick $x \in B(T) \setminus (s^{-1}(\bigcup_{t\in H} t^{-1}A))$. Then $x \in B(T)$ so $s * x \in \bigcup_{t\in H} t^{-1}A$. This is a contradiction. So $\bigcup_{t\in H} t^{-1}A \in q*p$. Thus $\overline{A} \cap (\beta S * q * p) \neq \emptyset$. Since $q*p \in K(\delta S)$, the result follows.

Now pick $p \in K(\delta S)$ such that $\overline{A} \cap (\beta S * p) \neq \emptyset$. So pick $t \in S$ such that $A \in t * p$. Let $B = \{a \in S : a^{-1}(t^{-1}A) \in p\}$. By Theorem 3.6, $B$ is $\check{c}$ -syndetic, so pick $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{s \in H} s^{-1}B$. Let $G = (t * H) \cup H$. Then $G \in \mathcal{P}_f(S)$. For $T \in \mathcal{P}_f(S)$, we show that there exists $x \in \sigma(T)$ such that $(T \cap \sigma(G)) * x \subseteq \bigcup_{t \in G} t^{-1}A$. Given $y \in (T \cap \sigma(G))$, choose $s_y \in H$ such that $s_y * y \in B$. So $(s_y * y)^{-1}(t^{-1}A) \in p$. Pick $x \in \bigcap_{y \in (T \cap \sigma(G))} (s_y * y)^{-1}(t^{-1}A)$. Then $s_y * y * x \in t^{-1}A$ and thus $t * s_y * y * x \in A$ and so $y * x \in (t * s_y)^{-1}A$. Thus $(T \cap \sigma(G)) * x \subseteq \bigcup_{t \in G} t^{-1}A$ and so $A$ is $\check{c}$-piecewise syndetic. □

**Theorem 4.5.** *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$ be piecewise syndetic. Then $A$ is $\check{c}$-piecewise syndetic.*

*Proof.* Pick $p \in K(\delta S)$ such that $A \in p$. Let $L$ be a minimal left ideal of $\delta S$ containing $p$. Then $L * p$ is a left ideal and $L * p \subseteq L$. So $L * p = L$. Since $p \in L * p$ we have $\overline{A} \cap (\beta S * p) \neq \emptyset$. Thus, by Theorem 4.4 $A$ is $\check{c}$-piecewise syndetic. □

The notions of "$\check{c}$-piecewise syndetic" and "piecewise syndetic" are not equivalent for an adequate partial semigroup. The following example shows this using the adequate partial semigroup $(T, *)$ introduced earlier.

**Theorem 4.6.** *There exists a partial semigroup $(S, *)$ and a subset $A$ of $S$ such that $A$ is $\check{c}$-piecewise syndetic but not piecewise syndetic.*

*Proof.* Let $T$ and $A$ be as in the proof of Theorem 3.4. Then $\delta S \subseteq \varphi(x_1) = \overline{FP(\langle x_n \rangle_{n=2}^{\infty})}$ and $A \cap FP(\langle x_n \rangle_{n=2}^{\infty}) = \emptyset$. In particular $\overline{A} \cap K(\delta S) = \emptyset$, so $A$ is not piecewise syndetic.

To see that $A$ is $\check{c}$-piecewise syndetic, let $H = \{x_1\}$. Let $G \in \mathcal{P}_f(S)$ be given. For each $w \in G$, pick $F_w \in \mathcal{P}_f(\mathbb{N})$ such that $w = \prod_{n \in F_w} x_n$. Let $m = \max \bigcup_{w \in G} F_w$. Then $x_{m+1} \in \sigma(G)$. And, since $\sigma(H) = \varphi(x_1) = FP(\langle x_n \rangle_{n=2}^{\infty})$, we have that $(G \cap \sigma(H)) * x_{m+1} \subseteq \bigcup_{t \in H} t^{-1}A$. □

The following theorem shows some of the interrelationships between syndetic sets and piecewise syndetic sets for an adequate partial semigroup.

**Theorem 4.7.** *Let $S$ be an adequate partial semigroup and suppose $A \subseteq S$. The following statements are equivalent:*

*(a) $A$ is piecewise syndetic.*

*(b) There exists $p \in K(\delta S)$ such that $\{x \in S : x^{-1}A \in p\}$ is syndetic.*

*(c) There exists $p \in \delta S$ such that $\{x \in S : x^{-1}A \in p\}$ is syndetic.*

*(d) There exists $p \in \delta S$ such that $\{x \in S : x^{-1}A \in p\}$ is piecewise syndetic.*

*Proof.* (a) $\Rightarrow$ (b). Pick $p \in K(\delta S) \cap \overline{A}$. Then by by Theorem 3.6, $\{x \in S : x^{-1}A \in p\}$ is syndetic.

(b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (d). Pick $p \in \delta S$ such that $B = \{x \in S : x^{-1}A \in p\}$ is syndetic. Then $\overline{B}$ intersects every left ideal of $\delta S$, so in particular $\overline{B} \cap K(\delta S) \neq \emptyset$. Thus $B$ is piecewise syndetic.

(d) $\Rightarrow$ (a). Pick $p$ as guaranteed. Let $B = \{x \in S : x^{-1}A \in p\}$. Since $B$ is piecewise syndetic, pick $q \in K(\delta S)$ such that $B \in q$. So $\{x \in S : x^{-1}A \in p\} \in q$ so $A \in q * p$. Therefore $\overline{A} \cap K(\delta S) \neq \emptyset$. □

## 5. *IP* Sets

The terminology of this section is due to Furstenberg and is commonly used in Topological Dynamics. $IP$ sets are of particular interest because of their intimate relationship with idempotents.

**Definition 5.1.** Let $(S, \cdot)$ be a semigroup. A subset $A$ of $S$ is an *IP set* if and only if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $S$, such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

**Theorem 5.2.** *Let $(S,\cdot)$ be a semigroup and let $A$ be a subset of $S$. Then $A$ is an IP set if and only if there is some idempotent $p \in \beta S$ such that $A \in p$.*

*Proof.* [4, Theorem 16.4]. □

For an adequate partial semigroup we have the following natural extensions of the definition and algebraic characterization of an $IP$ set.

**Definition 5.3.** Let $(S,*)$ be an adequate partial semigroup and suppose $A \subseteq S$.

(a) $A$ is $IP$ if and only if there exists an idempotent $p \in \delta S$ such that $A \in p$.

(b) $A$ is *č-IP* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $S$ such that for all $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n\in F} x_n$ is defined and $\prod_{n\in F} x_n \in A$.

The following is an example of a subset of an adequate partial semigroup which is č-$IP$ but not $IP$.

**Theorem 5.4.** *There exists a partial semigroup $(S,*)$ and a subset $A$ of $S$ such that $A$ is č-$IP$ but not $IP$.*

*Proof.* Let $S = \{A \subseteq \mathbb{N} : |A \backslash 2\mathbb{N}| < \omega\}$. So $S$ is the collection of subsets of $\mathbb{N}$ with finitely many odd numbers. Define $\uplus$ on $S$ such that

$$A \uplus B = \begin{cases} A \cup B & \text{if } A \cap B = \emptyset \\ \text{undefined} & \text{if } A \cap B \neq \emptyset \end{cases}$$

Then $(S,\uplus)$ is an adequate partial semigroup. To see this, let $\mathcal{H} = \{A_1, A_2, \ldots, A_n\} \subseteq S$. Then $|\bigcup_{i=1}^{n} A_i \backslash 2\mathbb{N}| < \omega$. So pick $x \in \mathbb{N} \backslash \bigcup_{i=1}^{n} A_i$. Then $\{x\} \in \varphi(A_i)$ for $i \in \{1,2,\ldots,n\}$. So $\{x\} \in \bigcap_{i=1}^{n} \varphi(A_i) = \sigma(\mathcal{H})$. Therefore $\sigma(\mathcal{H}) \neq \emptyset$ so $(S,\uplus)$ is adequate.

Let $A = \mathcal{P}_f(2\mathbb{N})$. We claim that $A$ is č-$IP$ but not $IP$.

$A$ is $\check{c}$-$IP$ since $A = FP(\langle\{2n\}\rangle_{n=1}^{\infty})$. To see $A$ is not $IP$, suppose there exists $p \in \delta S$ such that $A = \mathcal{P}_f(2\mathbb{N}) \in p$. Notice that $2\mathbb{N} \in S$, and $\varphi(2\mathbb{N}) = \mathcal{P}_f(2\mathbb{N} - 1) \cup \{\emptyset\} \in p$. However, $\mathcal{P}_f(2\mathbb{N}) \cap \varphi(2\mathbb{N}) = \emptyset$. This is a contradiction. Thus $A$ is not $IP$. □

Though the notions of "IP" and "č-IP" are not equivalent, we have the following implication. The proof follows exactly the proof for an ordinary semigroup [4, Theorem 5.8].

**Theorem 5.5.** *Let $(S, *)$ be an adequate partial semigroup and suppose $A \subseteq S$. If $A$ is $IP$, then $A$ is $\check{c}$-$IP$.*

*Proof.* Pick $p \in \delta S$ with $p * p = p$, such that $A \in p$. Let $A_1 = A$ and let $B_1 = \{x \in S : x^{-1}A_1 \in p\}$. $A_1 \in p * p$ (since $p = p * p$), so $\{x \in S : x^{-1}A_1\} \in p$. So $B_1 \in p$. Pick $x_1 \in B_1 \cap A_1$ and let $A_2 = A_1 \cap (x^{-1}A_1)$. So $A_2 \in p$. Inductively, given $A_n \in p$, let $B_n = \{x \in S : x^{-1}A_n \in p\}$. Then $B_n \in p$, so pick $x_n \in B_n \cap A_n$, and let $A_{n+1} = A_n \cap (x_n^{-1}A_n)$. We have produced a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $S$.

We show that if $F \in \mathcal{P}_f(\mathbb{N})$ and $m = \min F$ then $\prod_{n\in F} x_n \in A_m$. To see this, if $|F| = 1$, then $\prod_{n\in F} x_n = x_m \in A_m$. If $|F| > 1$, let $G = F\backslash\{m\}$, and let $k = \min G$. Since $k > m$, $A_k \subseteq A_{m+1}$. Then by the induction hypothesis, $\prod_{n\in G} x_n \in A_k \subseteq A_{m+1} \subseteq x_m^{-1}A_m$. So $\prod_{n\in F} x_n = x_m * \prod_{n\in G} x_n \in A_m$. Therefore $A$ is $\check{c}$-$IP$. □

The following results provide some conditions that guarantee equivalence of the notions of $\check{c}$-$IP$ and $IP$.

**Lemma 5.6.** *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence such that $\prod_{n\in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$. The following are equivalent:*

*(a) $\{FP(\langle x_n\rangle_{n=m}^{\infty}) \cap \sigma(H) : m \in \mathbb{N}, H \in \mathcal{P}_f(S)\}$ has the finite intersection property.*

*(b) $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n\rangle_{n=m}^{\infty})} \cap \delta S$ is a semigroup.*

*Proof.* (a)⇒(b). Let $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_m \rangle_{n=m}^{\infty})} \cap \delta S$

$$= \bigcap_{m=1}^{\infty} \overline{FP(\langle x_m \rangle_{n=m}^{\infty})} \cap \bigcap_{H \in \mathcal{P}_f(S)} \overline{\sigma(H)}.$$

$T \neq \emptyset$ by assumption. Let $p, q \in T$. To see that $p * q \in T$, let $m \in \mathbb{N}$, $H \in \mathcal{P}_f(S)$, and let $A = FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \sigma(H)$. We show that $A \subseteq \{s \in S : s^{-1}A \in q\}$ so that $A \in p * q$. To see this, let $s \in A$, and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $s = \prod_{n \in F} x_n$. Let $k = \max F + 1$, and let $L = H * s$. (Notice that since $s \in \sigma(H)$, $y * s$ is defined for all $y \in H$.) We claim that $FP(\langle x_n \rangle_{n=k}^{\infty}) \cap \sigma(L) \subseteq s^{-1}A$, so that $s^{-1}A \in q$. To see this, let $t \in FP(\langle x_n \rangle_{n=k}^{\infty}) \cap \sigma(L)$. One has immediately that $s * t \in FP(\langle x_n \rangle_{n=m}^{\infty})$. To see that $s * t \in \sigma(H)$, let $h \in H$. Then $h * s \in L$, so $(h * s) * t$ is defined. So $h * (s * t)$ is also defined. Therefore $(s * t) \in \sigma(H)$. Thus, $t \in s^{-1}A$.

(b) ⇒ (a). Since $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_m \rangle_{n=m}^{\infty})} \cap \delta S$ is a semigroup, $T \neq \emptyset$. Given $p \in T$, $\{FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \sigma(H) : m \in \mathbb{N}, H \in \mathcal{P}_f(S)\} \subseteq p$. □

The following theorem answers the question: "When is a č-$IP$ set $IP$?".

**Theorem 5.7.** *Let $(S, *)$ be an adequate partial semigroup. The following are equivalent:*

*(a) For all $A \subseteq S$, $A$ is č-$IP$ if and only if $A$ is $IP$.*

*(b) Whenever $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $S$ such that $\prod_{n \in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$ and $H \in \mathcal{P}_f(S)$, $FP(\langle x_n \rangle_{n=1}^{\infty}) \cap \sigma(H) \neq \emptyset$.*

*(c) Whenever $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $S$ such that $\prod_{n \in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$, $\{FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \varphi(y) : m \in \mathbb{N}$ and $y \in S\}$ has the finite intersection property.*

*Proof.* (a) ⇒ (b). Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $S$ such that for all $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n \in F} x_n$ is defined and let $A = FP(\langle x_n \rangle_{n=1}^{\infty})$.

Let $H \in \mathcal{P}_f(S)$. Pick $p \in \delta S$ such that $p = p*p$ and $A \in p$. Then $\sigma(H) \in p$ since $p \in \delta S$. Therefore $FP(\langle x_n\rangle_{n=1}^\infty) \cap \sigma(H) \neq \emptyset$.

(b) $\Rightarrow$ (c). Let $F \in \mathcal{P}_f(\mathbb{N})$ and let $H \in \mathcal{P}_f(S)$. Let $k = \max F$. Then (b) applied to the sequence $\langle x_n\rangle_{n=k}^\infty$ says that $\emptyset \neq FP(\langle x_n\rangle_{n=k}^\infty) \cap \sigma(H) \subseteq \bigcap_{m\in F} FP(\langle x_n\rangle_{n=m}^\infty) \cap \bigcap_{y\in H} \varphi(y)$.

(c) $\Rightarrow$ (a). Let $A$ be $\check{c}$-$IP$ and let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $S$ such that $\prod_{n\in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$ and $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq A$. Then by Lemma 5.6, $T = \bigcap_{m=1}^\infty \overline{FP(\langle x_n\rangle_{n=m}^\infty)} \cap \delta S$ is a semigroup. So pick $p$, an idempotent in $T$. Then $FP(\langle x_n\rangle_{n=1}^\infty) \in p$ and $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq A$. So $A \in p$. Therefore $A$ is $IP$. By Theorem 5.5 we know that $IP$ implies $\check{c}$-$IP$. $\square$

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