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SEPARATION PSEUDOCHARACTER AND THE CARDINALITY OF TOPOLOGICAL SPACES

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Abstract

We investigate the relation between the cardinality of a topological space and various types of pseudocharacters. In this paper, using Hausdorff pseudocharacter $H\psi(X)$ we strengthen a result of Gryzlov and Stavrova [6] concerning Lindelöf degree with respect to a selected subset and obtain $|X \setminus X_o| \leq expL(X, X_o) \cdot H\psi(X)$ for a Hausdorff space X and its subset X_o. We also introduce a cardinal invariant, Urysohn pseudocharacter $U\psi(X)$ for a Uryshon space X, and prove that $|X| \leq exp(aL(X) \cdot U\psi(X))$ for a Uryshon space X, which strengthens a result of Bella and Cammaroto [3].

1. Introduction

Standard notations from [5] and [8] will be used. Let us remind that $\mathcal{O}(X)$ will stand for the set of all open subsets of a given topological space X.

It is well known that in T_1 - topological space X a cardinal invariant called pseudocharacter can be introduced in a natural way - we say that $\psi(X) \leq \tau$ iff for every $x \in X$ there is a neighborhood system $\mathcal{H}(x) = \{U_\alpha(x) : \alpha \in \tau\}$ such that $\{x\} = \cap\mathcal{H}(x)$ for every $x \in X$.

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In [7] Richard Hodel introduced a similar invariant for Hausdorff space X - we say that the Hausdorff pseudocharacter $H\psi(X) \leq \tau$ iff for every $x \in X$ there is a neighborhood system $\mathcal{H}(x) = \{U_\alpha(x) : \alpha \in \tau\}$ such that for every two points $x \neq y$ there are $U_\alpha(x) \in \mathcal{H}(x)$ and $U_\beta(y) \in \mathcal{H}(y)$ such that $U_\alpha(x) \cap U_\beta(y) = \emptyset$. He also proved that $\psi(X) \leq H\psi(X) \leq \chi(X)$ and gave examples that these inequalities can be strict. He strenghtens the main theorems in the theory of cardinal invariants for topological spaces by replacing $\chi(X)$ with $H\psi(X)$ in Hausdorff cases.

In the present paper we use this invariant to strenghten a result of A.Gryzlov and D.Stavrova [6] concerning Lindelöf degree with respect to a selected subset (sometimes called relative Lindelöf number of the space) (Theorem 2). In [6] it was proved that for a given subset X_o of a Hausdorff space we have that $|X \setminus X_o| \leq expL(X, X_o) \cdot \psi_c(X) \cdot t(X)$ where $L(X, X_o) = \omega \cdot min\{\tau :$ for every open cover γ of X there is a $\gamma_1 \subseteq \gamma$ of cardinality at most τ such that $X \setminus X_o \subseteq \cup\gamma_1\}$. Theorem 2 replaces local invariants in this inequality by $H\psi(X)$.

Also in this paper we introduce a similar invariant in an Urysohn space X depending on this separation axiom - we say that the Urysohn pseudocharachter of X - $U\psi(X) \leq \tau$ iff for every $x \in X$ there is a neighborhood system $\mathcal{H}(x) = \{U_\alpha(x) : \alpha \in \tau\}$ such that for every two points $x \neq y$ there are $U_\alpha(x) \in \mathcal{H}(x)$ and $U_\beta(y) \in \mathcal{H}(y)$ such that $Cl(U_\alpha(x)) \cap Cl(U_\beta(y)) = \emptyset$.

In [4] U.N.B.Dissanayeke and S.Willard introduced the almost Lindelöf number for a topological space X as $aL(X) = \omega \cdot min\{\tau :$ for every open cover γ of X there is a $\gamma_1 \subseteq \gamma$ with cardinality at most τ such that $\cup\{Cl(U) : U \in \gamma_1\} = X\}$. In [3] A.Bella and F.Cammaroto proved that for Urysohn spaces we have that $|X| \leq exp(aL(X) \cdot \chi(X))$. We strenghten this result by replacing $\chi(X)$ with $U\psi(X)$.

Let us note that :

Lemma 1. *If X is Urysohn then $U\psi(X) \leq \chi(X)$.*

2. Cardinality of Urysohn spaces

Theorem 1. *If X is an Urysohn space then*

$$|X| \leq exp(aL(X) \cdot U\psi(X)).$$

Before presenting the proof of Theorem 1 we need some more notations and results:

Definition 1. For every $A \subseteq X$ we define the so called θ - closure of A , namely $Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \emptyset$ for every open $U \ni x\}$.

We obviously have $A \subseteq Cl(A) \subseteq Cl_\theta(A)$.

Definition 2. Let for every $x \in X$ (which is an Urysohn space) we have a family of neighborhoods $\mathcal{H}(x)$ that is closed under finite intersection, $\{x\} = \cap\mathcal{H}(x)$ and if $x \neq y$ then there exist $U_x \in \mathcal{H}(x)$ and $U_y \in \mathcal{H}(y)$ such that $Cl(U_x) \cap Cl(U_y) = \emptyset$. In that case we say that the family $\{\mathcal{H}(x) : x \in X\}$ realizes the Urysohn property on X.

Definition 3. Let $\mathcal{H} = \{\mathcal{H}(x) : x \in X\}$ realizes the Urysohn separation property on X and for every $x \in X$ let $\mathcal{H}(x) = \{U_\alpha(x) : \alpha \in \tau\}$. Then we say that $\mathcal{H}$ realizes $U\psi(X) \leq \tau$.

Definition 4. Define $Cl_\theta^{\mathcal{H}}(A) = \{x \in X : Cl(U_\alpha(x)) \cap A \neq \emptyset$ for every $\alpha \in \tau\}$.

We have that $A \subseteq Cl(A) \subseteq Cl_\theta(A) \subseteq Cl_\theta^{\mathcal{H}}(A)$.

Lemma 2. *If X is an Urysohn space, $U\psi(X) \leq \tau$ and $\mathcal{H}$ realizes it then $|Cl_\theta^{\mathcal{H}}(A)| \leq |A|^\tau$.*

Proof. Let $x \in Cl_\theta^{\mathcal{H}}(A)$.

Let $A_x = \{a(\alpha, x) \in Cl(U_\alpha(x)) \cap A : \alpha \in \tau\}$. We have that $x \in Cl_\theta^{\mathcal{H}}(A_x)$. Moreover $x \in Cl_\theta^{\mathcal{H}}(Cl(U_\alpha(x) \cap A_x))$ for every $\alpha \in \tau$ - indeed let $U_\beta(x) \in \mathcal{H}(x)$. Since $\mathcal{H}(x)$ is closed under finite intersections there is $\gamma \in \tau$ such that $U_\gamma(x) = U_\beta(x) \cap U_\alpha(x)$. From $x \in Cl_\theta^{\mathcal{H}}(A_x)$ it follows that $Cl(U_\gamma(x)) \cap A_x \neq \emptyset$ i.e., $\emptyset \neq Cl(U_\gamma(x)) \cap A_x = Cl(U_\beta(x) \cap U_\alpha(x)) \cap A_x \subseteq Cl(U_\beta(x)) \cap (Cl(U_\alpha(x)) \cap A_x)$. Moreover $Cl_\theta^{\mathcal{H}}(Cl(U_\alpha(x)) \cap A_x) \subseteq Cl_\theta^{\mathcal{H}}(Cl(U_\alpha(x))$ and $\{x\} = \cap\{Cl_\theta^{\mathcal{H}}(Cl(U_\alpha(x))) : \alpha \in \tau\}$ - indeed, we saw that $x \in Cl_\theta^{\mathcal{H}}(Cl(U_\alpha(x)))$ for every $\alpha \in \tau$. Let now $x \neq y$. By $U\psi(x) \leq \tau$ and the fact that $\mathcal{H}$ realizes it we have that there exist α_1 and $\alpha_2 \in \tau$ such that $Cl(U_{\alpha_1}(x)) \cap Cl(U_{\alpha_2}(y)) = \emptyset$ i.e., $y \notin Cl_\theta^{\mathcal{H}}(Cl(U_{\alpha_1}(x)))$. Hence $\{x\} = \cap\{Cl_\theta^{\mathcal{H}}(Cl(U_\alpha(x))) : \alpha \in \tau\}$ and from here it follows that $\{x\} = \cap\{Cl_\theta^{\mathcal{H}}(Cl(U_\alpha(x)) \cap A_x) : \alpha \in \tau\}$. We have that $A_x \in [A]^{\leq\tau}$, so $Cl(U_\alpha(x)) \cap A_x \subseteq A_x \in [A]^{\leq\tau}$ for every $\alpha \in \tau$. Let $\Gamma_x = \{Cl(U_\alpha(x)) \cap A_x : \alpha \in \tau\}$. Then $\Gamma_x \in [[A]^{\leq\tau}]^{\leq\tau}$. Hence $x \longrightarrow \Gamma_x$ is one -to -one mapping, hence $|Cl_\theta^{\mathcal{H}}(A)| \leq |A|^\tau$. ◻

Lemma 3. *If X is an Urysohn space, $\mathcal{H}$ realizes $U\psi(X) \leq \tau$ and $\mathcal{H}(x)$ is closed under finite intersections for every $x \in X$ then there is a special family $\mathcal{L} \subseteq expX$ each element of which has cardinality at most 2^τ such that whenever $M \subseteq X$ is of cardinality at most 2^τ then there is an element $L \in \mathcal{L}$ such that $M \subseteq L$. We shall call such family a 2^τ - dominating family in X.*

Proof. Let $|A| \leq 2^\tau$ and $A \subseteq X$. We shall construct $\tilde{A} \subseteq X, A \subseteq \tilde{A}$ and $|\tilde{A}| \leq 2^\tau$.

Let $A_o = A$. Let $A_1 = Cl_\theta^{\mathcal{H}}(A_o)$. Let $\alpha \in \tau^+$ and $\{A_\beta : \beta \in \alpha\}$ be already defined such that - (1) for $\beta \in \beta' \in \alpha$ we have that $A_\beta \subseteq Cl_\theta^{\mathcal{H}}(A_\beta) \subseteq A_{\beta'}$. Define $A_\alpha = Cl_\theta^{\mathcal{H}}(\cup\{A_\beta : \beta \in \alpha\}) \supseteq \cup\{Cl_\theta^{\mathcal{H}}(A_\beta) : \beta \in \alpha\} \supseteq \cup\{A_\beta : \beta \in \alpha\} \supseteq A_o = A$. For $\alpha \in \alpha' \in \tau^+$ we obviously have that $A_\alpha \subseteq Cl_\theta^{\mathcal{H}}(A_\alpha) \subseteq A_{\alpha'} \subseteq Cl_\theta^{\mathcal{H}}(A_{\alpha'})$ and from here we obtain - (2) $Cl_\theta^{\mathcal{H}}(Cl_\theta^{\mathcal{H}}(A_\alpha)) \subseteq Cl_\theta^{\mathcal{H}}(A_{\alpha'})$.

In that way we have defined $\{A_\alpha : \alpha \in \tau^+\}$ beginning with A. Let then $\tilde{A} = A_\theta^{\mathcal{H}} = Cl_\theta^{\mathcal{H}}(\cup\{A_\alpha : \alpha \in \tau^+\})$. From Lemma 2 and the construction we have that $|A_\theta^{\mathcal{H}}| \leq 2^\tau$. We also have that $A_\theta^{\mathcal{H}} = \cup\{Cl_\theta^{\mathcal{H}}(A_\alpha) : \alpha \in \tau^+\}$. Indeed let $x \in Cl_\theta^{\mathcal{H}}(\cup\{A_\alpha : \alpha \in \tau^+\})$. For every $\gamma \in \tau$ choose $a(\gamma, x) \in Cl(U_\gamma(x)) \cap (\cup\{A_\alpha : \alpha \in \tau^+\})$. Let $A_x = \{a(\gamma, x) : \gamma \in \tau\}$. Then A_x is a subset of cardinality $\leq \tau$ of the increasing union - $\cup\{A_\alpha : \alpha \in \tau^+\}$ and by the regularity of τ^+ we have that there is an $\alpha \in \tau^+$ such that $A_x \subseteq A_\alpha$. Then $x \in Cl_\theta^{\mathcal{H}}(A_x) \subseteq Cl_\theta^{\mathcal{H}}(A_\alpha) \subseteq \cup\{Cl_\theta^{\mathcal{H}}(A_\alpha) : \alpha \in \tau^+\}$. i.e., $A_\theta^{\mathcal{H}} = \cup\{Cl_\theta^{\mathcal{H}}(A_\alpha) : \alpha \in \tau^+\}$ i.e. $A_\theta^{\mathcal{H}} = Cl_\theta^{\mathcal{H}}(\cup\{A_\alpha : \alpha \in \tau^+\}) = \cup\{Cl_\theta^{\mathcal{H}}(A_\alpha) : \alpha \in \tau^+\}$ and the latter two unions are increasing unions. Let $\mathcal{L} = \{A_\theta^{\mathcal{H}} : A \subseteq X, |A| \leq 2^\tau\}$. Then $\mathcal{L}$ is a 2^τ-dominating family in X. □

Lemma 4. *Let $\mathcal{L} = \{A_\theta^{\mathcal{H}} : A \subseteq X, |A| \leq 2^\tau\}$ and Y is an increasing union of τ^+ elements of $\mathcal{L}$ (in that case we shall call Y - $\mathcal{L}(\tau^+)$ - inductive. Then $Y = Cl_\theta^{\mathcal{H}}(Y)$ and $Y_\theta^{\mathcal{H}} = Y$. (The notations are from the proof of the previous Lemma).*

Proof. Let $Y = \cup\{Y_\alpha : \alpha \in \tau^+\}$ be $\mathcal{L}(\tau^+)$ - inductive and $Y_\alpha = (A_\alpha)_\theta^{\mathcal{H}}$ for every $\alpha \in \tau^+$. Let $x \in Cl_\theta^{\mathcal{H}}(Y)$. Then for every $\gamma \in \tau$ let us choose $a(\gamma, x) \in Cl(U_\gamma(x)) \cap Y$ and let $Y' = \{a(\gamma, x) : \gamma \in \tau\} \in [Y]^{\leq \tau}$. By the regularity of τ^+ there exists $\alpha \in \tau^+$ such that $Y' \subseteq Y_\alpha$. Then $Y' \subseteq Y_\alpha = (A_\alpha)_\theta^{\mathcal{H}} = \cup\{Cl_\theta^{\mathcal{H}}(A_\alpha^\beta) : \beta \in \tau^+\}$ i.e., $Y' \in [\cup\{Cl_\theta^{\mathcal{H}}(A_\alpha^\beta) : \beta \in \tau^+\}]^{\leq \tau}$. Hence there is $\beta \in \tau^+$ such that $Y' \subseteq Cl_\theta^{\mathcal{H}}(Y') \subseteq Cl_\theta^{\mathcal{H}}(Cl_\theta^{\mathcal{H}}(A_\alpha^\beta)) \subseteq Cl_\theta^{\mathcal{H}}(A_\alpha^{\beta+1}) \subseteq Y_\alpha \subseteq Y$ i.e., $x \in Y$. Hence $Cl_\theta^{\mathcal{H}}(Y) = Y$. Moreover we have that $Y_\theta^{\mathcal{H}} = Y$. Remind from Lemma 3 that $Y_\theta^{\mathcal{H}}$ was constructed in the way that $Y_\theta^{\mathcal{H}} = Cl_\theta^{\mathcal{H}}(\cup\{Y_\alpha : \alpha \in \tau^+\}) = \cup\{Cl_\theta^{\mathcal{H}}(Y_\alpha) : \alpha \in \tau^+\}$ where $Y_1 = Cl_\theta^{\mathcal{H}}(Y) = Y$; so by induction if for some $\alpha \in \tau^+$ we suppose that $Y_\beta = Y$ for $\beta \in \alpha$ then $\cup\{Y_\beta : \beta \in \alpha\} = Y$ and $Y_\alpha = Cl_\theta^{\mathcal{H}}(Y) = Y$. □

Proof of Theorem 1. We shall use Theorem 1 from [12]. Let $\tau = \lambda = aL(X) \cdot U\psi(X)$; for a family $\gamma \subseteq expX$ let $p(\gamma) = \cup\{Cl(U) : U \in \gamma\}$ Let $\mathcal{H} = \cup\{\mathcal{H}(x) : x \in X\}$ realizes $U\psi(X) \leq \tau$ (in particular that means that $\mathcal{H}(x)$ is closed under finite intersections). Let $\mathcal{L} = \{A_\theta^{\mathcal{H}} : A \subseteq X, |A| \leq 2^\tau\}$. By Lemma 3, $\mathcal{L}$ is 2^τ-dominating in X. By Lemma 4 for each Y which is $\mathcal{L}(\tau^+)$-inductive we have that $Cl_\theta^{\mathcal{H}}(Y) = Y$ and $Y_\theta^{\mathcal{H}} = Y$. Let us check the requirements of Theorem 1, [12]: Let Y be $\mathcal{L}(\tau^+)$-inductive and $q \in X \setminus Y$. For each $x \in X \setminus Y$ there is an $\alpha(x) \in \tau$ such that $Cl(U_{\alpha(x)}(x)) \cap Y = \emptyset$ (because $x \notin Cl_\theta^{\mathcal{H}}(Y) = Y$). For each $y \in Y$ there is $\alpha(q) \in \tau$ such that $q \notin Cl(U_{\alpha(q)}(y))$. Let $\mathcal{W} = \{U_{\alpha(x)}(x), U_{\alpha(q)}(y) : x \in X \setminus Y, y \in Y\}$. By $aL(X) \leq \tau$ there are $\mathcal{W}^{'}, \mathcal{W}^{''} \in [\mathcal{W}]^{\leq\tau}$ such that $X = \cup\{Cl(U) : U \in \mathcal{W}^{'}\} \cup \cup\{Cl(U); U \in \mathcal{W}^{''}\}$ and $\mathcal{W}^{'} \in [\{U_{\alpha(x)}(x) : x \in X \setminus Y\}]^{\leq\tau}, \mathcal{W}^{''} \in [\{U_{\alpha(q)}(y) : y \in Y\}]^{\leq\tau}$. But $(\cup\{Cl(U) : U \in \mathcal{W}^{'}\}) \cap Y = \emptyset$. Hence $Y \subseteq \cup\{Cl(U) : U \in \mathcal{W}^{''}\} = p(\mathcal{W}^{''})$ and $q \notin p(\mathcal{W}'')$. By Theorem 1,[12] we have that $|X| \leq 2^\tau$.

3. Cardinality of Hausdorff Spaces with a Selected Subset

Theorem 2. *If X is a Hausdorff space, $X_o \subseteq X$ then*

$$|X \setminus X_o| \leq expL(X, X_o) \cdot H\psi(X).$$

Before proving Theorem 2 let us make some preliminary notes and remarks. The definition of $L(X, X_o)$ came as a common view point to the notion of relative Lindelöfness (introduced in [2] - we say that $X_o \subseteq X$ is Lindelöf in X if from every open cover γ of X a countable $\gamma^{'} \subseteq \gamma$ can be found that covers X_o). From the other side in [10] the following invariant is introduced - $kL(X) = \omega \cdot min\{\tau$:there is an $A \subseteq X, |A| \leq 2^\tau$ such that - (*) for each open cover γ of X there are $\gamma^{'} \in [\gamma]^{\leq\tau}$ and $B \in [A]^{\leq\tau}$ such that $X = \cup\gamma^{'} \cup Cl(B)\}$ and it was observed that

$kL(X) \leq min\{d(X), L(X), s(X)\}$. It was also shown that for Hausdorff spaces $|X| \leq expkL(X) \cdot \psi_c(X) \cdot t(X)$. Figuratively speaking this notion shows that we can have some "bad" part of a certain space but if the cardinality of this part is not "too big" we still can get a result about the cardinality of the main space. In [6] it is shown that similar results can be obtained by using $L(X, X_o)$ and we can looked at it as a generalization of $kL(X)$ in the sense that if $kL(X) \leq \tau$ then we can find $X_o \subseteq X$ such that $L(X, X_o) \leq \tau$. It can also be easily seen that $L(X, \emptyset) = L(X), L(X, X_o) \leq L(X), L(X, X_o) \leq L(X \setminus X_o) \leq hL(X), L(X) \leq L(X_o) \cdot L(X, X_o)$ and if $X \setminus X_o$ is Lindelöf in X then $L(X, X_o) \leq \omega$. As in section 2 we shall need some more technical definitions :

Definition 5. Let $H\psi(X) \leq \tau$. The family $\mathcal{U} = \{\mathcal{U}_x : x \in X\} \subseteq \mathcal{O}(X)$ such that - for every $p \in X$ we have that $p \in \cap\mathcal{U}_p, |\mathcal{U}_p| \leq \tau$ and if $p \neq q$ there exist $A \in \mathcal{U}_p$ and $B \in \mathcal{U}_q$ such that $A \cap B = \emptyset$ then the family $\mathcal{U}$ will said to be a realizing family for $H\psi(X) \leq \tau$.

It is easy to be noticed that if $\mathcal{U}$ is a realizing family for $H\psi(X) \leq \tau$ then the family $\mathcal{V} = \{\mathcal{V}_x : x \in X\}$ where $\mathcal{V}_x$ consists of all finite intersections of members of $\mathcal{U}_x$ will also be a realizing family for $H\psi(X) \leq \tau$.

Definition 6. The family $\mathcal{U} = \{\mathcal{U}_x : x \in X\}$ will be called canonically realizing $H\psi(X) \leq \tau$ if the following holds : 1) $\mathcal{U}_x$ is closed under finite intersections. 2) For every $p \in X$ we have $|\mathcal{U}_p| \leq \tau$. 3)$p \in \cap\mathcal{U}_p$ for every $p \in X$. 4) $\mathcal{U}_p \subseteq \mathcal{O}(X)$ for every $p \in X$. 5) Whenever $p \neq q$ there exist $A \in \mathcal{U}_p$ and $B \in \mathcal{U}_q$ such that $A \cap B = \emptyset$.

Definition 7. Having a family $\mathcal{U} = \{\mathcal{U}_x : x \in X\}$ satisfying 3) and 4) from Definition 6 we define an operator $(\quad)_{\mathcal{U}} : expX \longrightarrow expX$ as follows : if $A \subseteq X$ then $A_{\mathcal{U}} = \{x \in X : U \cap A \neq \emptyset$ for every $U \in \mathcal{U}_x\}$. We shall say that the family $\mathcal{U} = \{\mathcal{U}_x : x \in X\}$ generates the operator $(\quad)_{\mathcal{U}}$.

This operator is defined by Hodel,[7] and for every $A \subseteq X$ we have that $A \subseteq Cl(A) \subseteq (A)_{\mathcal{U}}$. We shall use the following two results proved in [7] :

Lemma 5. *If X is a Hausdorff topological space and $\mathcal{U} = \{\mathcal{U}_x : x \in X\}$ is canonically realizing $H\psi(X) \leq \tau$ and $A \in [X]^{\leq\tau}$ then $(A)_{\mathcal{U}} \in [X]^{\leq\tau}$.*

Lemma 6. *If $\mathcal{U} = \{\mathcal{U}_x : x \in X\}$ generates the operator $(\quad)_{\mathcal{U}}$ and $A = \cup\{(A_\alpha)_{\mathcal{U}} : \alpha \in \tau^+\} \subseteq expX$ is such that for every $\alpha \in \tau^+$ we have that $\cup\{(A_\beta)_{\mathcal{U}} : \beta \in \alpha\} \subseteq A_\alpha$ then $A_{\mathcal{U}} = A$.*

Proof of Theorem 2:

Let $L(X, X_o) \cdot H\psi(X) \leq \tau$ and let $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ is canonically realizing $H\psi(X) \leq \tau$. By transfinite induction we shall define two families - $\{H_\alpha : \alpha \in \tau^+\} \subseteq X \setminus X_o$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\} \subseteq \mathcal{O}(X)$ such that :

1) $H_\alpha \subseteq X \setminus X_o$ for every $\alpha \in \tau^+$.

2) $|H_\alpha| \leq 2^\tau$ for every $\alpha \in \tau^+$.

3) $(\cup\{(H_\beta)_{\mathcal{W}} : \beta \in \alpha\}) \cap (X \setminus X_o) \subseteq H_\alpha$ for every $\alpha \in \tau^+$.

4) If $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ are already defined then $\mathcal{B}_\alpha = \cup\{\mathcal{W}(x) : x \in \cup\{(H_\beta)_{\mathcal{W}} : \beta \in \alpha\}\}$.

5) If $\mathcal{V} \in [\mathcal{B}_\alpha]^{\leq\tau}$ and $X \setminus (\cup\mathcal{V} \cup X_o) \neq \emptyset$ then $H_\alpha \setminus (\cup\mathcal{V} \cup X_o) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ and $\{\mathcal{B}_\beta : \beta \in \alpha\}$ be already defined with properties 1) - 5).

Let $\mathcal{E}_\alpha = \{\mathcal{V} : \mathcal{V} \in [\mathcal{B}_\alpha]^{\leq\tau}$ and $X \setminus (\cup\mathcal{V} \cup X_o) \neq \emptyset$. For every $\mathcal{V} \in \mathcal{E}_\alpha$ we choose a point $\phi(\mathcal{V}) \in X \setminus (\cup\mathcal{V} \cup X_o) \neq \emptyset$ and let $C_\alpha = \{\phi(\mathcal{V}) : \mathcal{V} \in \mathcal{E}_\alpha\}$. Since $|\mathcal{E}_\alpha| \leq 2^\tau$ we have that $|C_\alpha| \leq 2^\tau$. Finally we put $H_\alpha = (X \setminus X_o) \cap ((C_\alpha)_{\mathcal{W}} \cup \cup\{(H_\beta)_{\mathcal{W}} : \beta \in \alpha\}$. We obviously have that $\cup\{H_\beta : \beta \in \alpha\} \subseteq H_\alpha$. Using Lemma 6 we obtain that $|H_\alpha| \leq 2^\tau$. It can be easily seen that the conditions 1) - 5) are satisfied.

Let $H = \cup\{H_\alpha : \alpha \in \tau^+\}$. We have that $(H)_{\mathcal{W}} = \cup \{(H_\alpha)_{\mathcal{W}} : \alpha \in \tau^+\}$ - in order to prove this let $X \in (H)_{\mathcal{W}}$. Then $W_x \cap H \neq \emptyset$ for every $W_x \in \mathcal{W}(x)$. Let us choose a point $a(W_x) \in W_x \cap H$ for every $W_x \in \mathcal{W}(x)$ and let $B_x = \{a(W_x) : x \in W_x \in \mathcal{W}(x)\} \in [H]^\tau$. Therefore there is an $\alpha \in \tau^+$ such that $B_x \subseteq H_\alpha$. Then $x \in (B_x)_\mathcal{W} \subseteq (H_\alpha)_\mathcal{W}$. The converse inclusion is obvious.

We also have that $(H)_\mathcal{W} \cap (X \setminus X_o) = H$.

Let us show that $X \setminus X_o = H$. Suppose there is a $q \in X \setminus H \setminus X_o$. Then $q \notin (H)_\mathcal{W}$ (because if we suppose that $q \in (H)_\mathcal{W}$ we shall have that $q \in (X \setminus X_o) \cap (H)_\mathcal{W} = H$ - a contradiction). Hence for every $p \in (H)_\mathcal{W}$ we can choose $V_p \in \mathcal{W}(p)$ such that $q \notin V_p$. From the other side for every $z \in X \setminus (H)_\mathcal{W}$ because of $z \notin (H)_\mathcal{W}$ we have $U_z \in \mathcal{W}(z)$ such that $U_z \cap H = \emptyset$. Let $\mu = \{V_p, U_z : p \in (H)_\mathcal{W}, z \in X \setminus (H)_\mathcal{W}\}$. We have that $\cup\mu \supseteq X$ and from $L(X, X_o) \leq \tau$ we can choose $\mu_o \in [\mu]^\tau$ such that $H \subseteq X \setminus X_o \subseteq \cup\mu_o$. Let us consider $\mu' = \{U \in \mu_o : U \cap H \neq \emptyset\}$. Therefore $H \subseteq \cup\mu'$ and $\mu^{'} = \{V_p : p \in H^{'} \in [(H)\mathcal{W}]^\tau\}$. By the regularity of τ^+ it follows that there is $\alpha_o \in \tau^+$ such that $H^{'} \subseteq (H_\alpha)_\mathcal{W}$ i.e., $|\mu^{'}| \leq \tau, \mu^{'} \subseteq B_{\alpha_o}$ and $q \notin \cup\mu' \cup X_o$. Then we have already chosen a point $\phi(\mu') \supseteq H$ - a contradiction.

From Theorem 2 (as well as from Theorem 1) we can obtain several known inequalities :

Corollary 1. *For every Hausdorff topological space X we have that*

$$|X| \leq expL(X) \cdot \chi(X).$$

This is a result first obtained in [1] for countable case.

Corollary 2. *For every regular topological space X we have that*

$$|X| \leq expkL(X) \cdot \chi(X).$$

This is a result first obtained in [9].

Corollary 3. *For every Hausdorff topological space X we have that*

$$|X \setminus X_o| \leq expkL(X, X_o) \cdot \chi(X).$$

This is a result first obtained in [6] for $X_o \neq \emptyset$ and for $X_o = \emptyset$ first obtained in [10].

Corollary 4. *For every Hausdorff topological space X we have that*

$$|X \setminus X_o| \leq expL(X, X_o) \cdot \chi(X).$$

This is a result first obtained in [6].

Corollary 5. *For every Hausdorff topological space X we have that*

$$|X| \leq expL(X) \cdot H\psi(X).$$

This is a result first obtained in [7].

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