# Topology Proceedings

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Summer 2000

Pages 361–379

# TOPOLOGY 2000 CONTRIBUTED PROBLEMS

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**Alexander Arhangel'skiĭ**
Ohio University, Athens, Ohio 45701, USA
arhangel@bing.math.ohiou.edu

## Locally Compact Linearly Lindelöf Spaces

**Formulation.** Let $X$ be a compact Hausdorff space and $a \in X$ such that for every uncountable subset $A$ of $X$ of regular cardinality there exists an open neighbourhood $U$ of $a$ such that the cardinality of $X \setminus A$ is the same as the cardinality of $A$. Is then $X$ first countable at $a$?

**Comment.** The above question is obviously equivalent to the following one: Is every linearly Lindelöf locally compact Hausdorff space Lindelöf? Certain results in the direction of the above problem were obtained in [1] (where the question was formulated for the first time). Recall that a space $Y$ is said to be *linearly Lindelöf* if every uncountable set of regular cardinality has a point of complete accumulation in $Y$. There are linearly Lindelöf Tychonoff spaces that are not Lindelöf. However, it is still unknown if there exists a normal linearly Lindelöf space that is not Lindelöf.

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**Alexander Arhangel'skiĭ**
Ohio University, Athens, Ohio 45701, USA
arhangel@bing.math.ohiou.edu

## First Countable Linearly Lindelöf Spaces

**Formulation.** Let $X$ be a first countable linearly Lindelöf Tychonoff (regular) space. Is then $X$ Lindelöf?

**Comment.** The answer to the above question is "yes" under $CH$ (and even under some weaker assumptions). This follows from the following result of Arhangel'skiĭ and Buzyakova, proved in $ZFC$ (see [1]): the cardinality of every first countable linearly Lindelöf Tychonoff space does not exceed $2^\omega$. One can find other results, related to the problem, in [1], where the question was formulated for the first time. Recall that a space $Y$ is said to be *linearly Lindelöf* if every uncountable set of regular cardinality has a point of complete accumulation in $Y$. There are linearly Lindelöf Tychonoff spaces that are not Lindelöf. However, it is still unknown if there exists a normal linearly Lindelöf space that is not Lindelöf.

### References

[1] A. V. Arhangel'skiĭ and R. Z. Buzyakova, *On linearly Lindelöf and strongly discretely Lindelöf spaces*, Proc. Amer. Math. Soc. **127:8** (1999), 2449–2458.

**Alexander Arhangel'skiĭ**
Ohio University, Athens, Ohio 45701, USA
arhangel@bing.math.ohiou.edu

## Discretely Lindelöf Spaces

**Definition.** A space $X$ is called *discretely Lindelöf* if for every discrete subspace $A$ of $X$ the closure of $A$ is Lindelöf.

**Formulation.** Is every Lindelöf discretely Lindelöf Tychonoff (regular) space Lindelöf?

**Comments.** Discretely Lindelöf spaces were called *strongly discretely Lindelöf* in [1]. Every discretely Lindelöf regular space is linearly Lindelöf [1]. However, it is not even known if every discretely Lindelöf locally compact Hausdorff space is Lindelöf. I believe, the answer to the last question should be in positive, at least, consistently. Recall that a space $Y$ is said to be *linearly Lindelöf* if every uncountable set of regular cardinality has a point of complete accumulation in $Y$. There are linearly Lindelöf Tychonoff spaces that are not Lindelöf. However, it is still unknown if there exists a normal linearly Lindelöf space that is not Lindelöf.

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**Sergey Antonyan**

Fac. de Ciencias, Universidad Nacional Autonoma de Mexico
antonyan@servidor.unam.mx

### Paracompactness of Open Images

**Problem.** Let $X$ be a paracompact (metrizable, if necessary) space, $Y$ be a completely regular, Hausdorff space and $f : X \to Y$ be a continuous, open, surjective map with connected, second countable fibers. Furthermore, assume that $Y$ has an open cover $\{U_\alpha\}$ satisfying the following conditions:

(1) each $f^{-1}(U_\alpha)$ is dense in $X$,

(2) there exists a subset $S_\alpha \subset f^{-1}(U_\alpha)$, closed in $f^{-1}(U_\alpha)$, such that $f(S_\alpha) = U_\alpha$, and the restriction $f|_{S_\alpha}$ is a perfect and open map.

Is then $Y$ paracompact (normal)? What if $f|_{S_\alpha}$ is a homeomorphism ?

Some important problems in the theory of topological transformation groups can be reduced to this purely general-topological problem. Namely, let $G$ be a separable Lie group and $Z$ be a proper (in the sense of R. Palais [Pa]) $G$-space. By making use the results of Palais [Pa], it can be shown that if $G$ is connected then the orbit map $f : Z \to Z/G$ with $Z$ paracompact (metrizable), satisfies to the conditions of Problem 1 with $X = Z$ and $Y = Z/G$.

A positive answer to Problem 1 will provide a solution of the Hájek-Abels conjecture on paracompactness of the orbit space of a paracompact proper $G$-space (see [Ha] and [Ab]). As it was shown by H. Abels, this conjecture will imply, for instance, the parallelizability of dispersive dynamical systems on arbitrary paracompact phase spaces (see [Ha] and [Ab]), a generalization of a classical Antosiewicz-Dugundji-Nemytski theorem.

On the other hand, as it is shown in the preprint [AN], the paracompactness of the orbit space will imply the existence of a consistent $G$-invariant metric on each metrizable proper $G$-space $X$ (for $X$ second countable the result was established first by Palais [Pa]). This fact can have important applications in equivariant theory of retracts.

Finally, we recall that a $G$-space $X$ is called *proper* (see R. Palais [Pa]) if: (1) $G$ is a locally compact Hausdorff topological group, (2) $X$ is completely regular Hausdorff space and (3) every point of $X$ has a neighborhood $V$ such that for every point of $X$ there is a neighborhood $U$ with the property that the set $\langle U, V\rangle = \{g \in G : gU \cap V \neq \emptyset\}$ has compact closure in $G$.

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**K. P. Hart**

TU Delft

k.p.hart@its.tudelft.nl

**Problem 1.** This problem is due to Alan Dow: is it consistent that all extremally disconnected continuous images of $\omega^*$ are separable?

Indeed, most of the ZFC results on continuous images of $\omega^*$ are quite general: *all* separable compact spaces, *all* compact spaces of weight $\aleph_1$ (or less) and *all* perfectly normal compact spaces are continuous images of $\omega^*$. What distinguishes the separable spaces from the rest is that for these spaces the proof is nearly trivial: enumerate a dense subset of $X$ with infinite repetitions and take the Čech-Stone extension of this enumeration; the repetitions ensure that $\omega^*$ gets mapped onto $X$. The other types of spaces are, in some sense, small so that a recursive construction of an onto map is possible; but note that none is extremally disconnected.

It seems likely that OCA or PFA will settle this problem positively because both axioms tend to dictate that many maps must be nearly trivial, in the sense that the map has a lifting to all of $\beta\omega$ and so the map on $\omega^*$ decides where $\omega$ must go.

In our case one would expect the following to hold under OCA or PFA: if $f : \omega^* \to 2^{\mathfrak{c}}$ is continuous and $X = f[\omega^*]$ is extremally disconnected then any extension of $f$ to all of $\beta\omega$ will have to map all but finitely many elements of $\omega$ into $X$.

**Problem 2.** Give a topological characterization of $\mathbb{H}^*$ under CH. Here $\mathbb{H} = [0, \infty)$ the half-line.

This topological characterization should be in the spirit of Parovičenko's characterization of $\omega^*$. Remember that $\omega^*$ is, under CH, the only compact space of weight $\mathfrak{c}$ that is zero-dimensional and without isolated points, and which is also an $F$-space in which nonempty $G_\delta$-sets have non-empty interior.

The characterization of $\mathbb{H}^*$ should replace 'zero-dimensional and without isolated points' by some, preferably finite, list of topological properties.

**Problem 3.** This problem is due to Stevo Todorčević: does OCA imply that there is no Borel lifting for the measure algebra?

The measure algebra is defined as $M = \mathrm{Bor}/\mathcal{N}$, where Bor denotes the $\sigma$-algebra of Borel sets and $\mathcal{N}$ is the ideal of measure-zero sets. A lifting is a homomorphism $l : M \to \mathrm{Bor}$ such that $q \circ l$ is the identity on $M$, where $q : \mathrm{Bor} \to M$ is the quotient homomorphism. The Continuum Hypothesis implies such a lifting exists and Shelah has shown, by his oracle-cc method, that it is consistent that no lifting exists.

A recent metatheorem of Ilijas Farah states that if there is no reason (in ZFC) for two quotients of $\mathcal{P}(\omega)$ to be isomorphic then OCA + MA implies that they are not isomorphic. A positive answer to the present question, possibly using MA, would reinforce the idea that OCA and MA together generate a quite complete theory of the reals — with the right theorems.

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**Lew Ludwig**

Denison University

### A ZFC Example of a $\beta$-normal Non-normal Space

A space $X$ is called $\beta$-normal if for any two disjoint closed subsets $A$ and $B$ of $X$, there exists open subsets $U$ and $V$ of $X$ such that $A \cap U$ is dense in $A$, $B \cap V$ is dense in $B$, and the closure of $U$ in $X$ and the closure of $V$ in $X$ have empty intersection [AL].

In [LS], we demonstrated the existence of a $\beta$-normal non-normal space assuming the existence of a normal, right-separated in type-$\omega_1$, S-space.

**Question 1.** Does there exist in ZFC a $\beta$-normal, non-normal space?

**Question 2.** Does there exist a space $X$ in ZFC which is normal, right separated of type $\kappa$ with $hd(X) < \kappa$?

**Question 3.** Does there exist a space $Y$ in ZFC which is not normal, has scattered height 2, and the set of non-isolated points is of cardinality $\lambda$, such that any two disjoint closed sets, one of which has size less than $\lambda$, can be separated?

**Question 4.** Does there exist an uncountable $\lambda$ that gives a positive answer to Questions 2 and 3 simultaneously?

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**Mikhail Matveev**
Irvine CA
misha_matveev@hotmail.com

### Spaces Without Minimal Pseudocompact Extensions

A space $pX$ is called a *pseudocompact extension of a space* $X$ if $pX$ is pseudocompact and contains $X$ as a dense subspace; $pX$ is a *minimal pseudocompact extension* of $X$ if no proper subspace of $pX$ is a pseudocompact extension of $X$.

**Problem.** Is there a Tychonoff space without a minimal (Tychonoff) pseudocompact extension?

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**Mikhail Matveev**
Irvine CA
misha_matveev@hotmail.com

### Variations of Homogeneity

A space is called *basically homogeneous* (B-homogeneous for short) if it has a base every element of which can be mapped onto every other by an autohomeomorphism of the entire space.

**Problem 1.** Is every topological vector space B-homogeneous?

A separable space is called CDH (abbreviation for Countably Dense Homogeneous) if for every two dense countable subspaces there is an autohomeomorphism of the entire space mapping one onto the other.

**Problem 2.** Is it consistent that $2^\tau$ is CDH for some uncountable $\tau$?

**Reference**

Stanislav Shkarin has given a partial positive answer for Problem 1 in [Q and A in Gen. Topol. **18** (2000) 1-16]: "yes" for locally convex TVS.

For a survey on CDH see the book Open Problems in Topology.

Recently the author has proven the consistensy of the following statement:

For some uncountable $\tau$, all compactifications of $\omega$ with remainders homeomorphic to $2^\tau$ are homeomorphic to each other.

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**Mikhail Matveev**
Irvine CA
`misha_matveev@hotmail.com`

### Monotonic Compactness and Lindelöfness

A space $X$ is *monotonically compact (monotonically Lindelöf)* if there is a mapping that assigns to every open cover $U$ of $X$ a finite (resp. countable) open refinement $R(U)$ so that $R(U_1)$ refines $R(U_2)$ as soon as $U_1$ refines $U_2$.

**Problem 1.** Is every Hausdorff monotonically compact space metrizable?

**Problem 2.** Is every countable monotonically Lindelöf space metrizable?

**Reference**

The author and Jerry Vaughan have shown (unpublished) that some well-known examples of non metrizable compacta are not monotonically compact.

**Mikhail Matveev**
Irvine CA
misha_matveev@hotmail.com

### Inverse Properties

Let $F$ be a family of subsets of a set $X$. A *partial inversement* of $F$ is a family $\{p(A) : A \in F\}$ such that for every $A \in F$, $p(A)$ is either $A$ or $X \setminus A$. A space $X$ is called inversely compact (inversely Lindelöf) if every open cover of $X$ has a partial inversement which contains a finite (resp. countable) subcover of $X$. In other words, a space is inversely compact if every independent family of closed sets has nonempty intersection.

**Problem 1.** Is there a Hausdorff (regular, Tychonoff, normal) inversely compact space which is not compact?

**Problem 2.** Is the discrete sum of two inversely Lindelöf spaces inversely Lindelöf?

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**Mikhail Matveev**
Irvine CA
misha_matveev@hotmail.com

### Can Jones' Lemma be Improved?

For a topological space $X$, $\mathrm{St}-l(X)$ denotes the minimum of such cardinals $\tau$ that for every open cover $U$ of $X$ there is a subset $A$ of $X$ with $|A| \leq \tau$ and $\mathrm{St}(A, U) = X$.

**Problem.** Is the inequality $2^{|K|} \leq (\chi(X))^{\mathrm{St}-l(X)}$ true for every closed discrete subset $K$ in a normal space $X$?

**Reference**

See the authors preprint "Some Questions" in Topology Atlas.

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**Justin Tatch Moore**

Boise State University, Boise, ID 83725

justin@diamond.boisestate.edu

**Problem 1.** (J. T. Moore). If $2^{\aleph_0} < 2^{\aleph_1}$ is there a nontrivial automorphism of $\mathbb{N}^*$?

**Problem 2.** Does OCA imply that all automorphisms of $\mathbb{N}^*$ are trivial?

**Remark.** If OCA is supplemented with MA then "yes." It is also known that OCA implies that every automorphism of $\mathbb{N}^*$ is somewhere trivial. I'm not sure if this question has been explicitly asked elsewhere, though it is certainly natural in light of the results of [V2].

**Problem 3.** (S. Todorčević). Assume $\mathrm{MA}_{\aleph_1}$. After forcing with an arbitrary measure algebra, must every nonmetrizable compact space contain an uncountable discrete set in its square?

**Remark.** If the measure algebra is trivial then "yes" since any counterexample would have to have a square which is a compact S space. See also [T2] and [T4].

**Problem 4.** (J. T. Moore). Assume $\mathrm{MA}_{\aleph_1}$. After adding one random real, does $\mathrm{MA}_{\aleph_1}$ hold for all partial orders which have a c.c.c. product with every c.c.c. partial order?

**Problem 5.** (S. Todorčević). If the countable chain condition is productive, must $\mathrm{MA}_{\aleph_1}$ hold?

**Remark.** While the answer seems to be "no", one can still prove partial positive results. For instance $\mathfrak{b}$ and $\mathrm{cf}(\mathfrak{c})$ are both greater than $\aleph_1$ if the countable chain condition is productive. The most interesting open question of this sort is probably: If the countable chain condition is productive must $2^{\aleph_0} = 2^{\aleph_1}$? See, e.g., [T3], [T5], and [T6].

**Problem 6.** (M. E. Rudin). After adding $\aleph_2$ Laver reals, is there a Lindelöf space which has a non-Lindelöf product with the irrationals?

**Remark.** A solution in either direction would be interesting. A negative solution would solve Michael's problem. A positive solution would give a fundamentally new construction of a Michael space since the argument could not involve Baire category (or the assumption $\mathfrak{b} = \aleph_1$). See [M] for a survey of this problem. M. E. Rudin also has an unpublished note surveying Michael's problem (which I regrettably was unaware of when I wrote [M]).

**Problem 7.** Is it consistent that for every $c : [\omega_1]^2 \to 2$ there is a pair of uncountable sets $A, B \subseteq \omega_1$ such that $c$ is constant on $A \otimes B = \{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha < \beta\}$?

**Remark.** I believe this problem is due to F. Galvin. If the answer is "yes" then it is consistent that every regular space is hereditarily separable iff it is hereditarily Lindelöf iff it does not contain an uncountable discrete set (by results of Woodin, if this statement is consistent at all, then it is consistent with $\mathrm{MA}_{\aleph_1}$). See [T1].

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**Peter Nyikos**
University of South Carolina
nyikos@math.sc.edu

### Separable Normal Manifolds

Is every separable normal manifold $\omega_1$-compact? [A space is said to be $\omega_1$-*compact* if every closed discrete subspace is countable.]

An equivalent problem is: In a separable normal manifold, is every closed discrete subspace a $G_\delta$ – a countable intersection of open sets? This problem is unsolved even for hereditarily normal manifolds.

Jones' Lemma shows that the answer to this problem is Yes if $2^{\aleph_0} < 2^{\aleph_1}$. Some theorems on manifolds would be improved upon if we could even show that MA+¬CH or even PFA is consistent with an affirmative answer. A Yes answer would follow from one to Mary Ellen Rudin's favorite problem about manifolds: is every normal manifold collectionwise Hausdorff?

There are plenty of separable normal manifolds from ZFC alone, but all the ones I know of are $\omega_1$-compact.

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**Stanislav Peregudov**

State University of Management

`matem@acman.msk.su`


### Weakly Uniform Bases

**Problem 1.** Is every Lindelof regular space that has a weakly uniform base first countable? (The answer is No in the class of Lindelöf Hausdorff spaces).

**Problem 2.** Is there an L-space with a weakly uniform base? (The answer is No under MA+¬CH).

**Problem 3.** Is every pseudocompact space that has a weakly uniform base compact? (The space is Cech complete first countable).

**Roman Pol**
Warsaw University
`pol@mimuw.edu.pl`

## Light Maps on Continua

Let $f : X \to Y$ be a light map (i.e., with 0-dimensional fibers), where $X$ is a metrizable continuum with $\dim X > 2$. Does there exist a non-trivial continuum $C$ in $X$ such that $f$ restricted to $C$ is injective ?

**Comment.** For some related information cf. Pol, Tsukuba J. Math. **20** (1996), 11 - 19. In particular, the answer is positive for $Y$ finite-dimensional.

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**James T. Rogers, Jr.**
Tulane University
`jim@math.tulane.edu`

Let $M$ be a hereditarily indecomposable continuum. Assume $\dim M = n > 1$. Let $H(M)$ be the homeomorphism group of $M$.

**Question 1.** Can $H(M)$ contain a nontrivial continuum? A nontrivial connected set?

For each integer $n > 1$, Rogers has exhibited an $M$ such that $H(M)$ contains no nontrivial connected set.

**Question 2.** Can $M$ be rigid? i.e., the identity map is the only element of $H(M)$?

**Mary Ellen Rudin**
University of Wisconsin–Madison
mrudin@math.wisc.edu

## The Linearly Lindelöf Problem

**Problem.** Is there a Hausdorff, normal, Lindelöf not Linearly Lindelöf space? A space is Linearly Lindelöf if every increasing open cover has a countable subcover.

**Comment.** There are several regular but not normal examples known. It is normality that is hard to achieve.

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**Mary Ellen Rudin**
University of Wisconsin–Madison
mrudin@math.wisc.edu

## The Point Countable Base Problem

**Problem.** We are given a $T_1$ space $X$ such that each point $x$ of $X$ has a countable open basis $\mathcal{B}(x)$ having the property that for all open $U$ with $x \in U$ there is an open $V$ with $x \in V \subset U$ such that $y \in V$ implies there is $B \in \mathcal{B}(y)$ with $x \in B \subset U$. Does this imply that the space must have a point countable base (i.e., that each point is in only countably many members of the base.)?

**Observation.** It is true if the density of the space is $\leq \omega_1$.

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**Mary Ellen Rudin**
University of Wisconsin–Madison
`mrudin@math.wisc.edu`

## Max Burke's Spaces

**Problem.** Max Burke would like a nice characterization of the class of spaces which are the continuous image of an arbitrary product of compact, linearly ordered spaces.

**Comment.** Burke has recently proved that if $X$ is in this class then every separately continuous function $f : X \times Y \to \mathbb{R}$ (where $Y$ is any space) is Borel. Also, if $I : C(X) \to \mathbb{R}$ is integration with respect to finite Borel measure on $X$, then $I$ is Borel measurable when $C(X)$ has the topology of pointwise convergence.

## Reference

Max Burke is at Prince Edward Island University in Canada and his e-mail address is `burke@upei.ca`.

**Krishnan Shankar**
University of Michigan, Ann Arbor, MI
shankar@umich.edu

### Is the Berger Space an $S^3$ Bundle Over $S^4$?

$S^3$ bundles over $S^4$ have been important ever since J. Milnor showed that the total spaces of such bundles with Euler class $\pm 1$ are homeomorphic to the standard sphere $S^7$ but not always diffeomorphic to it. In 1974, D. Gromoll and W. Meyer constructed a metric of nonnegative curvature on one of these exotic spheres (a generator in the group of homotopy 7-spheres) and it remained the only exotic sphere known to admit nonnegative curvature. More recently K. Grove and W. Ziller constructed metrics of nonnegative curvature on all the total spaces of $S^3$ bundles over $S^4$ thus showing that all the Milnor 7-spheres admit nonnegative curvature. They also asked whether the homogeneous Berger space admits the structure of such a bundle. They showed that it cannot be a principal $S^3$ bundle over $S^4$.

The Berger space is described as the quotient $Sp(2)/Sp(1) = SO(5)/SO(3)$ where the embedding of $SO(3)$ is nonstandard. If we think of the space of $\mathbf{R}^5$ as the space of symmetric, $3\times 3$, traceless matrices, then $SO(3)$ acts on this space by conjugation which gives a representation of $SO(3)$ into $SO(5)$. The Berger space is described as the resulting quotient space.

In a recent paper, Kitchloo and Shankar showed that the Berger space $SO(5)/SO(3)$ is $PL$-homeomorphic to an $S^3$ bundle over $S^4$. They were unable to settle the diffeomorphism question since this requires computing the Eells–Kuiper invariant. To do this one needs to exhibit the Berger space as a spin coboundary. It remains open whether the Berger space is diffeomorphic to an $S^3$ bundle over $S^4$.

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