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MOSCOW SPACES AND TOPOLOGICAL GROUPS

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Abstract

The main goal of the article is to demonstrate how large is the class of Moscow topological spaces, and especially, to show how extremely large is the class of Moscow topological groups. A series of new results in this direction is presented. Special attention is given to the question when the product of a family of Moscow spaces is a Moscow space. In particular, it turns out that the product of any family of regular spaces with a countable network is a Moscow space. We also review the main recent applications of Moscow spaces in the theory of topological groups and of homogeneous spaces in general, providing the first systematic survey of these applications. The applications are based on a non-trivial interaction between Moscow spaces and C-embeddings. New challenging open problems are formulated.

0. Introduction

The main goal of this article is to consider in detail the class of Moscow spaces, introduced in [2], to show how large it is, and, in particular, to investigate which classes of topological

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spaces and groups are contained in the class of Moscow spaces. Moscow spaces work especially well in combination with homogeneity; this explains why this notion turned out to be useful in the theory of topological groups. Recent applications of Moscow spaces involve Dieudonné completions of topological groups and C-embeddings (see for more details [3], [5], [7]). In part, the article is the first systematic survey on Moscow spaces, but it also contains a series of new results and open questions (see, in particular, Corollary 2.5, Theorem 2.7, Corollary 2.8, Theorem 2.9, Example 2.13, Example 2.16, Lemma 2.22, Proposition 2.24, Theorem 2.27, Corollary 2.29, Theorem 3.8, Corollary 3.14, Theorem 3.19, Problems 3.21, 3.22, 3.23, 3.26, and 3.29).

All spaces considered are assumed to be Tychonoff, unless the restrictions on separation axioms are explicitly formulated. Notation and terminology are as in [11]. In particular, a space X is called *homogeneous* if for any two points x and y in X there exists a homeomorphism h of X onto itself such that $h(x) = y$. If A is a subset of a space X, then the G_δ*-closure* of A in X is defined as the set of all points $x \in X$ such that every G_δ-subset of X containing x intersects A. If X is the G_δ-closure of A, we say that A is G_δ*-dense in* X. If the G_δ-closure of A coincides with A, we say that A is G_δ*-closed*.

If X is a space, then $C(X)$ is the set of all continuous real-valued functions on X. By R we denote the usual space of real numbers, $[a, b]$ stands for the closed interval in R with endpoints a and b, D is the two-point discrete space $\{0, 1\}$.

In the first section of this article we mostly present some known results on Moscow spaces, sometimes with proofs for the sake of completeness.

1. Some Basic Known Facts on Moscow Spaces

A Hausdorff space X is called *Moscow* [2], [3], if, for each open subset U of X, the closure of U in X is the union of a family of G_δ-subsets of X, that is, for each $x \in \overline{U}$ there exists a G_δ-subset P of X such that $x \in P \subset \overline{U}$.

Clearly, the notion of a Moscow space generalizes the notion of a perfectly κ-normal space introduced independently by R. Blair [9], E.V. Ščepin [17], and T. Terada [19] under different names. A space X is called *perfectly κ-normal* if the closure of any open set (that is, every canonical closed set) is a zero-set. The class of Moscow spaces is much wider than the class of perfectly κ-normal spaces, since every first countable space, and even every space of countable pseudocharacter, is a Moscow space, while not every first countable compact space is perfectly κ-normal. Indeed, the square of the "two arrows" space X of Alexandroff and Urysohn is a first countable compact space which is not perfectly κ-normal [16]. Indeed, assume the contrary. Then the diagonal Δ_X in $X \times X$ can be represented as the intersection of two canonical closed sets, since X is linearly ordered. Therefore, Δ_X is a G_δ in $X \times X$. It remains to refer to the fact that every compact space with the diagonal G_δ is metrizable [11]. This contradiction shows that $X \times X$ is not perfectly κ-normal.

The notion of Moscow space can be also considered as a straightforward generalization of extremal disconnectedness. Recall, that a space X is *extremally disconnected* if the closure of each open subset of X is open.

It is also clear that if every point in a space X is a G_δ, then X is Moscow.

We sum up the above observations in the following statement:

Proposition 1.1. *a) Every perfectly κ-normal space is Moscow;*

b) Every extremally disconnected space is Moscow;

c) Every space of countable pseudocharacter is Moscow.

Since the product of arbitrary family of metrizable spaces is perfectly κ-normal (see [9], [18]), the next statement is a corollary to Proposition 1.1. Note that in the next section we prove a much more general theorem (Theorem 2.18).

Corollary 1.2. [23] *The product of arbitrary family of metrizable spaces is Moscow.*

It follows from Corollary 1.2, that every Tychonoff cube I^τ, as well as every generalized Cantor discontinuum D^τ, is Moscow.

Notice, that the next statement is obviously true:

Proposition 1.3. *Every dense subspace of a Moscow space is Moscow.*

However, we cannot claim that every closed subspace of a Moscow space is Moscow.

Example 1.4. [5] Let $D(\tau)$ be an uncountable discrete space and $\alpha D(\tau)$ the one point compactification of $D(\tau)$. Then $D(\tau)$ is a Moscow space, and $D(\tau)$ is G_δ-dense in $\alpha D(\tau)$, while $\alpha D(\tau)$ is not a Moscow space. Indeed, let U be any infinite countable subset of $D(\tau)$. Then U is open in $\alpha D(\tau)$, and $\overline{U} = U \cup \{\alpha\}$, where α is the only non-isolated point in $\alpha D(\tau)$. Every G_δ-subset of $\alpha D(\tau)$ containing the point α is easily seen to be uncountable; therefore, $\overline{U}$ is not the union of any family of G_δ-subsets of $\alpha D(\tau)$. Since $\alpha D(\tau)$ is a closed subspace of a Tychonoff cube, we conclude that the class of Moscow spaces is not closed hereditary.

The notion of Moscow space plays a major role in the theory of C-embeddings. Recall that a subspace Y of a space X is said to be *C-embedded in X* if every continuous real-valued function f on Y can be extended to a continuous real-valued function on X. It is well known [12] that if a dense subspace Y of a space X is C-embedded in X, then Y is G_δ-dense in X. The converse to this statement is not true. Indeed, in Example 1.4 the subspace $D(\tau)$ is G_δ-dense in $\alpha D(\tau)$, while, obviously, $D(\tau)$ is not C-embedded in $\alpha D(\tau)$.

V.V. Uspenskij observed in [23] (making use of an argument of M.G. Tkačenko [20]) that Moscow spaces have the following important property.

Theorem 1.5. *For every Moscow space X, every G_δ-dense subset Y of X is C-embedded in X.*

Proof. Assume that Y is not C-embedded in X. Then there are open subsets V_1 and V_2 of Y such that their closures in Y are disjoint, while the intersection of the closures of V_1 and V_2 in X is not empty. Fix a point x in $\overline{V_1} \cap \overline{V_2}$, and let U_i be the interior of the closure of V_i in X, $i = 1, 2$. Obviously, $V_i \subset U_i$; therefore, U_i is not empty.

Since X is a Moscow space, we can find G_δ-sets P_i in X such that $x \in P_i \subset \overline{U_i}$, $i = 1, 2$. Then $P = P_1 \cap P_2$ is a G_δ-subset of X and $x \in P$; therefore, $P \cap Y$ is not empty. Clearly, every point of $P \cap Y$ belongs to the intersection of the closures of the sets V_1 and V_2 in Y, which is impossible, since this intersection is empty, by the choice of V_1 and V_2. □

Example 1.4 shows that it is not enough to assume Y to be a Moscow space in Theorem 1.5. However, the following fact was established in [5].

Theorem 1.6. *If a Moscow space Y is a G_δ-dense subspace of a homogeneous space X, then X is also a Moscow space and Y is C-embedded in X.*

Proof. Let U be an open subset of X and x a point in the closure of U. We have to show that there exists a G_δ-subset P in X such that $x \in P \subset \overline{U}$.

Since X is homogeneous, we may assume that $x \in Y$. Then $x \in \overline{U \cap Y}$ and, since Y is a Moscow space, there exists a G_δ-set Q in the space Y such that $x \in Q \subset \overline{U \cap Y}$. Thus, there exists a countable family $\{U_n : n \in \omega\}$ of open subsets of X such that their intersection $P = \cap\{U_n : n \in \omega\}$ satisfies the condition:

$$x \in P \cap Y \subset \overline{U}.$$

We claim that $P \subset \overline{U}$. Indeed, assume the contrary. Then $P \setminus \overline{U}$ is a non-empty G_δ-subset in X, and since Y is G_δ-dense in X, it follows that $(P \setminus \overline{U}) \cap Y$ is not empty. On the other hand,

$(P \setminus \overline{U}) \cap Y = (P \cap Y) \setminus \overline{U} = \emptyset$, by the above formula. This contradiction shows that $x \in P \subset \overline{U}$. Thus, X is a Moscow space. It remains to apply Theorem 1.5. □

From Theorem 1.6 and Proposition 1.3 we immediately obtain the following result:

Corollary 1.7. [5] *Let X be a homogeneous space and Y a G_δ-dense subspace of X. Then X is a Moscow space if and only if Y is a Moscow space.*

We can sum up the information obtained above as follows:

Corollary 1.8. [5] *Let Y be a dense subspace of a homogeneous space X. Then the next conditions are equivalent:*

1) X is a Moscow space and Y is G_δ-dense in X;

2) X is a Moscow space and Y is C-embedded in X;

3) Y is a Moscow space and Y is G_δ-dense in X;

4) Y is a Moscow space and Y is C-embedded in X.

The next characterization of Moscow spaces shows that the relationship of this class of spaces to C-embeddings is even deeper than one might presume.

Theorem 1.9. [5] *A space X is Moscow if and only if every dense subspace Y of X is C-embedded in the G_δ-closure of Y in X.*

Example 1.10. [5] In connection with Theorem 1.9 we should observe that there exists a non-Moscow space X such that every G_δ-dense subspace Y of X is C-embedded in X. Indeed, the space $\omega_1 + 1$ of ordinals is such a space, since every G_δ-dense subspace Y of ω_1+1 is either ω_1 or ω_1+1. To see that ω_1+1 is not Moscow, take two disjoint uncountable sets U and V consisting of isolated ordinals. Then the point ω_1 is in the intersection of their closures. Now let us assume that $\omega_1 + 1$ is Moscow. It follows that there exists a G_δ-subset P of $\omega_1 + 1$ such that $\omega_1 \in P$ and no point of P is isolated in $\omega_1 + 1$. However, such P, obviously, does not exist, a contradiction. Therefore, $\omega_1 + 1$ is not Moscow.

We now present a generalization of Theorem 1.6. It is a relative version of this theorem.

Let us say that a subset Y of a space X is *residually Moscow in* X if for each open subset U of X and each $z \in \overline{U} \setminus Y$ there exists a G_δ-subset P in X such that $z \in P$ and $P \cap Y \subset \overline{U}$.

The next result can be proved in the same way as Theorem 1.6.

Theorem 1.11. *If Y is a subspace of X, Y is G_δ-dense in X, and Y is residually Moscow in X, then Y is C-embedded in X.*

Notice, that if X is any space and $Y = X$, then Y is residually Moscow in X.

Here are some curious applications of Theorems 1.5 and 1.6 (see [6]).

Corollary 1.12. *If X is a pseudocompact Moscow space, and bX is a homogeneous compactification of X, then bX is the Stone-Čech compactification of X.*

Proof. Since X is pseudocompact, X is G_δ-dense in bX. By Theorem 1.6, X is C-embedded in bX. Therefore, $bX = \beta X$. □

Corollary 1.13. *Every G_δ-dense subspace Y of a compact Moscow space X is pseudocompact, and X is the Stone-Čech compactification of Y.*

Proof. By Theorem 1.5, Y is C-embedded in X. Since X is compact, it follows that Y is pseudocompact, and that X is the Stone-Čech compactification of Y. □

There is another generalization of Theorem 1.6 (see [7]).

Let Y be a subspace of a space X. We say that Y is *h-dense* in X, if Y is dense in X and for each $x \in X$ there exists a homeomorphism h of X onto itself such that $h(x) \in Y$. We also say in this case that X is *Y-homogeneous.*

Theorem 1.14. [7] *If a Moscow space Y is a G_δ-dense subspace of a Y-homogeneous space X, then X is also a Moscow space.*

The assumption that X is Y-homogeneous cannot be dropped. To see this, we can take the same spaces $D(\tau)$ and $\alpha D(\tau)$ as in Example 1.4.

Problem 1.15. When the Stone-Čech compactification of a space X is Moscow?

Problem 1.16. Given a space X, when there exists a compactification of X which is Moscow?

In connection with Problems 1.15 and 1.16 see [10].

2. Further Properties of Moscow Spaces

In this section, we consider the behaviour of the class of Moscow spaces with respect to products and mappings. We also discuss how to construct non-Moscow spaces.

Recall that *a P-space* is a space, in which every G_δ-set is open. The next assertion obviously follows from the definitions.

Proposition 2.1. [3] *If a topological space X is a P-space and a Moscow space, then X is extremally disconnected.*

J.R. Isbell has shown [13] that every extremally disconnected P-space of Ulam non-measurable cardinality is discrete. Using this fact and Proposition 2.1, we obtain the following result:

Theorem 2.2. *If a topological space X of Ulam non-measurable cardinality is a P-space and a Moscow space, then X is discrete.*

Theorem 2.2 suggests a certain strategy in looking for examples of non-Moscow spaces and of non-Moscow topological groups. We may start with any topological group G of uncountable pseudocharacter and of Ulam non-measurable cardinality, and introduce on G a new topology: the G_δ-modification of the original topology on G (G_δ-sets form a base of it). The topological group G^* so obtained is a P-space. Then Theorem 2.2 guarantees that G^* is not a Moscow space. See in this connection [4].

An interesting property of Moscow spaces involves the next generalization of the notion of pseudocharacter.

The κ-pseudocharacter $\kappa\psi(x, X)$ of a space X at a point $x \in X$ is the smallest infinite cardinal number τ such that there exists a family γ of canonical closed sets in X such that $\{x\} = \cap\gamma$. We define *the κ-pseudocharacter* $\kappa\psi(X)$ of a space X as the supremum of the numbers $\kappa\psi(x, X)$ when x runs over X. Clearly, the κ-pseudocharacter at any point never exceeds the pseudocharacter at the same point. On the other hand, we have the following result:

Proposition 2.3. *If X is a Moscow space, then the pseudocharacter and the κ-pseudocharacter coincide at every point of X.*

Proof. Let γ be a family of canonical closed sets such that $\{x\} = \cap\gamma$. Since X is Moscow, for each $V \in \gamma$ we can fix a G_δ-subset P_V in X such that $x \in P_V \subset V$. Then, clearly, $\{x\} = \cap\{P_V : V \in \gamma\}$. Therefore, $\psi(x, X) \leq |\gamma|$. □

Corollary 2.4. *If X is a Moscow space, $x \in X$, and V and W are canonical closed sets in X such that $\{x\} = V \cap W$, then x is a G_δ-point in X.*

Corollary 2.5. *For every bisequential Moscow space, the pseudocharacter of X is countable.*

Proof. In view of Proposition 2.3, it is enough to refer to the next statement which is a reformulation of a result in [1]: □

Proposition 2.6. *For every bisequential space X, the κ-pseudocharacter of X is countable.*

Using the simple results presented above, we can easily construct non-Moscow spaces by means of natural mappings. Recall, that a space X is said to be *perfect* if every closed subset of it is a G_δ-set in X. The next result clarifies the behaviour of the class of Moscow spaces with respect to closed mappings. Recall that the cardinality of a set X is said to be *Ulam non-measurable* if every countably centered ultrafilter on X is fixed. It is consistent to assume that every cardinal number is Ulam non-measurable [12], [11].

Theorem 2.7. *A regular space X of Ulam non-measurable cardinality is perfect if and only if, for each closed continuous mapping of X onto a Hausdorff space Y, the space Y is Moscow.*

Proof. Clearly, if Y is an image of a perfect space X under a closed continuous mapping, then every point in Y is a G_δ-set. Thus, it remains to prove sufficiency.

Take any closed subset F of X. Obviously, we may assume that F is not open. Then $F \neq X$. Now we have to distinguish two cases.

Case 1. Not all points of $X \setminus F$ are isolated. Then we can fix a non-isolated point a in $X \setminus F$. Now, let Y be the quotient space obtained, when we collapse all points of the set $A = F \cup \{a\}$ into a point $b \in Y$. We denote by p the natural quotient mapping of X onto Y. It is well known that the mapping p is closed and continuous. Since X is regular, the space Y is Hausdorff.

By the assumption, the space Y is Moscow. Let us show that the κ-pseudocharacter of Y at the point b is countable.

Fix open neighbourhoods OF and Oa of F and a, respectively, such that the closures of OF and Oa are disjoint. Put $U = OF \setminus F$ and $V = Oa \setminus \{a\}$. Clearly, the sets U and V are open in X, and the sets $f(U)$ and $f(V)$ are open in Y. We also have $\overline{f(U)} \cap \overline{f(V)} = \{b\}$, since a is not isolated, F is not open, and the closures (in X) of OF and Oa are disjoint. It follows that the κ-pseudocharacter of Y at the point b is countable, which implies, by Proposition 2.3, that b is a G_δ-point in Y. Therefore, by continuity of p, the set $p^{-1}(b) = F \cup \{a\}$ is a G_δ in X. Hence, F is a G_δ-set in X, and the space X is perfect.

Case 2. Every point of the set $X \setminus F$ is isolated in X. Let Z be the quotient space obtained when we collapse F into a point $c \in Z$. It is enough to show that c is a G_δ-point in the space Z. Clearly, the quotient mapping is again closed, the space Z is Hausdorff, and all points of $Z \setminus \{c\}$ are isolated. By the assumption, Z is Moscow.

Let us consider the family η of all subsets A of $Z \setminus \{c\}$ such that $c \in \overline{A}$. Obviously, $\cap\eta = \emptyset$.

Claim: There exists a countable subfamily ξ of η such that $\cap\xi = \emptyset$.

Indeed, otherwise η is a countably centered ultrafilter on X, and η is not fixed. Therefore, the cardinality of $Z \setminus \{c\}$ is Ulam measurable, which implies that the cardinalities of Z and X are Ulam measurable, a contradiction. The Claim is proved.

Hence, we can fix ξ such as in the Claim. For every A in ξ, there exists a G_δ-subset P_A of Z such that $c \in P_A \subset \overline{A}$, since Z is Moscow, and every $A \in \xi$ is open in Z. Now we have $\{c\} = \cap\{P_A : A \in \xi\}$, since all points of $Z \setminus \{c\}$ are isolated. Since ξ is countable, we conclude that $\{c\}$ is a G_δ-subset of Z. This completes the proof of Theorem 2.7.

Note the next two cases in which, to complete the proof of Theorem 2.7, we do not need the assumption on the cardinality of X. □

First, if all points of X are non-isolated in X, the second case in the proof is impossible. Second, assume that X is Lindelöf, and consider the second case. If Z is countable, then the Claim is obviously true. If Z is uncountable, then take any two disjoint uncountable subsets A_1 and A_2 of $Z\setminus\{c\}$. Clearly, $\{c\} = \overline{A_1} \cap \overline{A_2}$, since Z is Lindelöf and c is the only non-isolated point of Z. The Claim is proved, and the argument can be completed as above. Taking this observation into account, we have

Corollary 2.8. *A compact space X is perfect if and only if every continuous image of X is a Moscow space.*

Note also that if there is an Ulam measurable cardinal, then there exists a non-perfect space X such that every closed continuous image of it is Moscow: it is enough to take the ultrafilter space corresponding to a countably centered non-fixed ultrafilter. Since there are nonperfect first countable compacta, it follows from Corollary 2.8 that perfect mappings do not preserve the class of Moscow spaces. Therefore, the next result is, probably, the best possible.

Theorem 2.9. *If f is an open continuous compact mapping of a Moscow space X onto a space Y, then Y is also a Moscow space.*

Proof. Let V be an open subset of Y and $U = f^{-1}(V)$. Put $F = \overline{U}$. Since f is open and continuous, we have $f(F) = \overline{V}$. Take any $y \in \overline{V}$, and fix $x \in F$ such that $f(x) = y$. Since F is a canonical closed set in X and X is Moscow, there exist open sets W_n in X, for each $n \in \omega$, such that $x \in W_n$ and $\cap\{W_n : n \in \omega\} \subset F$. Since the space X is regular, we may also assume that $x \in \overline{W_{n+1}} \subset W_n$. Put $H_n = f(W_n)$. Then, obviously, $y = f(x) \in H_n$ and H_n is open in Y, for each $n \in \omega$. Therefore, $y \in P$, where $P = \cap\{H_n : n \in \omega\}$ is a G_δ-subset of Y.

Let us show that $P \subset \overline{V}$. Take any $y_0 \in Y \setminus \overline{V}$, and put $F_0 = f^{-1}(y_0)$. Since $f(F) = \overline{V}$, it follows that $F \cap F_0 = \emptyset$. Put $M_n = \overline{W_n} \cap F_0$ and consider the family $\eta = \{M_n : n \in \omega\}$. Since F_0 is compact, η is a decreasing sequence of compact subsets of X. Clearly, the intersection of η is empty, since it is contained in $F_0 \cap F = \emptyset$. Therefore, some element M_k of η is empty. Then $W_k \cap F_0 = \emptyset$, which implies that y_0 is not in $f(W_k) = H_k$. It follows that $y \in P \subset \overline{V}$. Hence, Y is a Moscow space. □

Here is another preservation result for Moscow spaces.

Theorem 2.10. *Every retract of a Moscow space is a Moscow space.*

Theorem 2.10 follows from a slightly more general result below (Proposition 2.11). Let us say that a subspace Y of a space X is *canonically embedded* in X, if for every open subset V of Y there exists an open subset U of X such that the closure of V in Y is the set $\overline{U} \cap Y$.

The next statement is obvious:

Proposition 2.11. *If a subspace Y of a Moscow space X is canonically embedded in X, then Y is also a Moscow space.*

Corollary 2.12. *If a Moscow space Z is the topological product of some family $\mathcal{F}$ of spaces, then every element of $\mathcal{F}$ is a Moscow space.*

Proof. Clearly, it is enough to consider the case, when $\mathcal{F}$ consists of two spaces X and Y, that is, $Z = X \times Y$. Now it is easy to see that the conclusion follows from Proposition 2.11, since every factor embeds canonically into the product space.

Unfortunately, it is not true in general that the product of two Moscow spaces is a Moscow space. □

Example 2.13. There exists a separable compact Hausdorff Moscow space B such that $B \times B$ is not Moscow. Indeed, let B be βN, the Stone-Čech compactification of the discrete space N of integers. Then B is extremally disconnected (see [11]). Therefore, B is a Moscow space. Besides, B is compact and separable.

Let us show that $B \times B$ is not a Moscow space. Fix any point $p \in \beta N \setminus N$, and put $X = N \cup \{p\}$ and $Y = B \setminus \{p\}$. Since X is countable and Tychonoff, it is realcompact.

The set Y is G_δ-dense in B, since B is not first countable at p (see [11]). Therefore, $X \times Y$ is G_δ-dense in $X \times B$. Put $Z = (X \times Y) \cup \{(p, p)\}$. Since $(p, p) \in X \times B$, it follows that $X \times Y$ is G_δ-dense in the space Z.

Assume now that $B \times B$ is Moscow. Since Z is dense in $B \times B$, it follows that Z is also a Moscow space.

Put $U = \{(n, n) : n \in N\}$. Then U is a closed subset of $X \times Y$, since $N = X \cap Y$. Therefore, the set $V = (X \times Y) \setminus U$ is open in $X \times Y$. The set U is also open in $X \times Y$, since each $n \in N$ is isolated in X and in Y. Since $X \times Y$ is obviously open in Z, it follows that the sets U and V are open in Z. Clearly, U and V are disjoint, and $(p, p) \in \overline{U} \cap \overline{V}$. Since $U \cup V = X \times Y$, and U and V are disjoint open sets, it follows that $\{(p, p)\} = \overline{U} \cap \overline{V}$ (in the space Z). Therefore, (p, p) is a G_δ-point in Z. This contradicts the earlier established fact that $X \times Y$ is G_δ-dense in the space Z.

Though the class of Moscow spaces is not productive, as it is witnessed by Example 2.13, there are quite a few large classes of Moscow spaces each of which is closed under the product operation (*is productive* as we say below).

First, we remind some definitions from [18]. Suppose that for every point x in a space X and every canonical closed subset C of X a non-negative real number $\rho(x, C)$ is defined in such a way that the next conditions are satisfied:

1) $\rho(x, C) = 0$ if and only if $x \in C$;

2) If C_1 and C_2 are canonical closed sets such that $C_1 \subset C_2$, then $\rho(x, C_2) \leq \rho(x, C_1)$, for each $x \in X$;

3) whenever a canonical closed set C is fixed, $\rho(x, C)$ is a continuous function of x on X;

4) if $\mathcal{C}$ is any chain of canonical closed sets in X, then $\rho(x, \overline{\cup \mathcal{C}}) = inf\{\rho(x, C) : C \in \mathcal{C}\}$, for each $x \in X$.

Then we say that ρ is a *κ-metric* on X, and $\rho(x, C)$ is called *the distance* from x to C (with respect to the κ-metric ρ).

Every metrizable space is, obviously, κ-metrizable. An important theorem of Ščepin says [18] that the topological product of any family of κ-metrizable spaces is κ-metrizable. In particular, it follows that the product of any family of metrizable spaces is κ-metrizable.

Ščepin also proved [18] that every dense subspace of any κ-metrizable space is κ-metrizable. Together the two results show that the class of κ-metrizable spaces is very large, much larger, than the class of metrizable spaces.

A well known property of metrizable spaces is that every closed subset of such a space is a G_δ-set. A trace of this property we find in κ-metrizable spaces: every canonical closed set in any κ-metrizable space X is a zero-set of some continuous real-valued function on X, that is, every κ-metrizable space is perfectly κ-normal. As we noticed in Section 1, the class of perfectly

κ-normal spaces is not (even, finitely) productive. However, we have the next corollary to Ščepin's results:

Theorem 2.14. *The class of κ-metrizable spaces is a productive class of Moscow spaces.*

Recall that a space X is said to be *a σ-space* if it has a σ-discrete network. It is easy to see (and well known) that the product of any countable family of σ-spaces is again a σ-space. It is also clear that every point in a σ-space is a G_δ. Therefore, every σ-space is a Moscow space. Clearly, every metrizable space is a σ-space. We have already observed that the product of any family of metrizable spaces is κ-metrizable and, therefore, perfectly κ-normal and Moscow. This naturally brings to mind the question whether the product of any family of σ-spaces is a Moscow space. The answer is "no". This can be seen from the next theorem which is a slightly modified version of Theorem 3 in [14], and the proof of which practically coincides with the proof of Theorem 3 in [14] and because of that is omitted.

First, we recall that $S(\omega_1)$ is the space obtained from the free topological sum of ω_1 copies of the usual convergent sequence by identifying the limits of these sequences (the resulting set is given the quotient topology). Further, let us call a space X *κ-perfect* if every canonical closed set in X is a G_δ.

Theorem 2.15. [14] *Let Y be a space such that the weight of Y does not exceed ω_1 and the product space $S(\omega_1) \times S(\omega_1) \times Y$ is κ-perfect. Then Y is perfect.*

Theorem 2.15 is instrumental in establishing that not every product of σ-spaces is a Moscow space.

Example 2.16. The space $X = S(\omega_1) \times S(\omega_1) \times D^{\omega_1}$ is not Moscow, though obviously X is the product of σ-spaces. To prove this, we need the following result of Y. Yajima (see [24], Theorem 6):

Theorem 2.17. [24] *Let X be a product of σ-spaces. Then a closed set in X is a G_δ-set if and only if it is the union of some family of G_δ-sets in X.*

Assume now that the space $X = S(\omega_1) \times S(\omega_1) \times D^{\omega_1}$ is Moscow, and let P be any canonical closed set in X. Then P is the union of G_δ-sets in X and therefore, by Theorem 2.17, P is a G_δ-set in X. Thus, the space X is κ-perfect. Now Theorem 2.15 implies that D^{ω_1} is perfect, which is not the case. This contradiction means that X is not Moscow.

However, under some not very strong additional restrictions, the product of σ-spaces has to be a Moscow space. This conclusion can be derived from another remarkable result of Yukinobu Yajima. The next assertion is an immediate corollary to his Theorem 7 in [24]:

Theorem 2.18. *Let X be the product of a family $\mathcal{F}$ of spaces of countable pseudocharacter such that the tightness of the product of every finite subfamily of $\mathcal{F}$ is countable. Then X is a Moscow space.*

Corollary 2.19. *Let X be the product of a family of spaces with a countable network. Then X is a Moscow space.*

Corollary 2.20. *Let X be the product of some family of bisequential spaces of the countable pseudocharacter. Then X is a Moscow space.*

Corollary 2.21. *The product of any family of first countable spaces is a Moscow space.*

Of course, Corollary 2.21 also implies that every dense subspace of the product of any family of metrizable spaces is a Moscow space.

We will now establish a general theorem, from which Theorem 2.18 follows. First, we introduce some notation.

Let $\mathcal{F} = \{X_a : a \in A\}$ be a family of topological spaces, and let $X = \Pi\{X_a : a \in A\}$ be their topological product. Then *ω-cube in X* is any subset B of X that can be represented as follows: $B = \Pi\{B_a : a \in A\}$, where B_a is a subset of X_a, for each $a \in A$, and the set $A_B = \{a \in A : B_a \neq X_a\}$ is countable. The set A_B in this case will be called *the core of the ω-cube B*.

We also put $X_K = \Pi\{X_a : a \in K\}$, for every subset K of A, and denote by p_K the natural projection mapping of X onto X_K.

An interesting tightness type invariant was introduced by M.G. Tkačenko [20]. The *o-tightness* of a space X is said to be countable (notation: $ot(X) \leq \omega$) if whenever a point a belongs to the closure of $\cup\gamma$, where γ is any family of open sets in X, then there exists a countable subfamily η of γ such that a is in the closure of $\cup\eta$.

Note, that if the tightness of X is countable, or the Souslin number $c(X)$ is countable, then the o-tightness of X is countable.

Lemma 2.22. *Let $\{X_n : n \in \omega\}$ be a countable family of spaces such that the space $X_K = \Pi\{X_n : n \in K\}$ is Moscow, for every finite subset K of ω. Then the product space $X = \Pi\{X_a : n \in \omega\}$ is also Moscow.*

Proof. Let U be any open set in X, and x a point in the closure of U. Take any finite subset K of ω. Since $X_K = \Pi\{X_n : n \in K\}$ is Moscow, and the set $p_K(U)$ is open in X_K, there exists a G_δ-subset P_K of X_K such that $p_K(x) \in P_K$ and P_K is contained in the closure of the set $p_K(U)$ in the space X_K.

Put $P = \cap\{p_K^{-1} P_K : K \in \mathcal{K}\}$, where $\mathcal{K}$ is the family of all finite subsets of ω. Then, clearly, $x \in P$, and P is a G_δ-subset of X.

Let us show that $P \subset \overline{U}$, which will complete the proof.

Take any $y \in P$ and any standard open neighbourhood $O(y)$ of y in X. Then $O(y) = p_{K^*}^{-1} p_{K^*}(O(y))$, for some finite $K^* \subset \omega$. We have $p_{K^*}(y) \in P_{K^*}$, since $y \in P$. Therefore, $p_{K^*}(y) \in \overline{p_{K^*}(U)}$. Since, obviously, $p_{K^*}(O(y))$ is an open neighbourhood of $p_{K^*}(y)$ in the space X_{K^*}, the set $p_{K^*}(O(y)) \cap p_{K^*}(U)$ is not empty. Since $O(y) = p_{K^*}^{-1} p_{K^*}(O(y))$, it follows that

$$O(y) \cap U \neq \emptyset.$$

Hence, $y \in \overline{U}$ and $P \subset \overline{U}$. □

The next lemma was proved by Tkačenko. We present its proof, for the sake of completeness.

Lemma 2.23. [20] *Let $\{X_n : n \in \omega\}$ be a countable family of spaces such that the o-tightness of $X_K = \Pi\{X_n : n \in K\}$ is countable, for every finite subset K of ω. Then the o-tightness of the product $X = \Pi\{X_n : n \in \omega\}$ is countable.*

Proof. Let γ be any family of open sets in X and x a point in the closure of the set $U = \cup\gamma$. Take any finite subset K of ω. Since $ot(X_K) \leq \omega$, there exist a countable subfamily γ_K of γ such that $p_K(x)$ is in the closure of the set $\cup\{p_K(V) : V \in \gamma_K\}$ in the space X_K.

Put $\eta = \cup\{\gamma_K : K \in \mathcal{K}\}$, where $\mathcal{K}$ is the family of all finite subsets of A. Then, clearly, η is a countable subfamily of γ, and $x \in \overline{\cup\eta}$. □

The next statement is a modification of a result of Tkačenko [20].

Proposition 2.24. *Let $\mathcal{F} = \{X_a : a \in A\}$ be a family of topological spaces such that the o-tightness of X_K is countable, for every finite subset K of A, and let $X = \Pi\{X_a : a \in A\}$ be their topological product. Then, for any family γ of open sets in X and for any point x in the closure of the set $U = \cup\gamma$, there exist a countable subfamily η of γ and an ω-cube B such that $x \in B \subset \overline{\cup\eta}$. In particular, we then have $ot(X) \leq \omega$.*

Proof. Without loss in generality, we may assume that γ consists of open ω-cubes in X with the finite core. We are going to define, by induction, an increasing sequence of countable subsets A_n of A and a sequence of countable subfamilies γ_n of γ in the following way.

Take A_0 to be any nonempty countable subset of A. Suppose that a countable subset A_n is already defined, and put $K = A_n$. By Lemma 2.23, $ot(X_K) \leq \omega$. Therefore, there exists a countable subfamily γ_n of γ such that $p_K(x)$ is in the closure of the set $\cup\{p_K(V) : V \in \gamma_n\}$ in the space X_K. Now put

$A_{n+1} = A_n \cup (\cup\{A_B : B \in \gamma_n\})$ (recall that A_B is the core of the ω-cube B). The inductive step is complete.

Put $M = \cup\{A_n : n \in \omega\}$ and $\eta = \cup\{\gamma_n : n \in \omega\}$. Clearly, η is a countable subfamily of γ. Let H be the closure of $\cup\eta$, and let y be any point in H. Since η consists of ω-cubes, there exists an ω cube B in X such that $y \in B \subset H$. Indeed, let B be the set of all $z \in X$ such that $z_a = y_a$, for each $a \in M$. Since $y \in H$, we clearly have $B \subset H$.

It remains to show that $x \in H$. Take any standard open neighbourhood $O(x)$ of x in X. Then $O(x) = p_{K^*}^{-1} p_{K^*}(O(x))$, for some finite $K^* \subset A$. Since $p_M^{-1} p_M(H) = H$, it suffices to consider the case when $K^* \subset M$.

Since the sequence $\{A_n : n \in \omega\}$ is increasing, there exists $n \in \omega$ such that $K^* \subset A_n$. By the choice of γ_n, then $p_{K^*}(x)$ is in the closure of the set $\cup\{p_{K^*}(V) : V \in \gamma_n\}$ in the space X_{K^*}. Therefore, there exists a point $z \in \cup\eta$ such that $p_{K^*}(z) \in p_{K^*}(O(x))$. It follows that z is in $O(x) \cap (\cup\eta)$. Hence, $x \in H$. □

We also need the next obvious lemma.

Lemma 2.25. *Let $\{X_a : a \in A\}$ be a family of topological spaces such that the pseudocharacter of X_a is countable, for each $a \in A$. Then every ω-cube B in the product space $X = \Pi\{X_a : a \in A\}$ is the union of a family of G_δ-subsets of X.*

Now we can easily prove the main assertion:

Theorem 2.26. *Let $\mathcal{F} = \{X_a : a \in A\}$ be a family of topological spaces such that the o-tightness of X_K is countable, for every finite subset K of A, and the pseudocharacter of X_a is countable, for each $a \in A$. Then the product space $X = \Pi\{X_a : a \in A\}$ is Moscow.*

Proof. Take any open set U in X, and let x be any point in the closure of U. Put $\gamma = \{U\}$. Now from Proposition 2.24 it follows that there exists an ω-cube B in X such that $x \in B \subset \overline{U}$. By Lemma 2.25, there exists a G_δ-subset P in X such that $x \in P \subset B$. Now we have $x \in P \subset B \subset \overline{U}$. Hence, X is Moscow. □

The last theorem can be generalized as follows:

Theorem 2.27. *Let $\mathcal{F} = \{X_a : a \in A\}$ be a family of topological spaces such that X_K is a Moscow space of the countable o-tightness, for every finite subset K of A. Then the product space $X = \Pi\{X_a : a \in A\}$ is Moscow.*

Proof. Take any open set U in X, and let x be any point in the closure of U. Put $\gamma = \{U\}$. Now it suffices to repeat the proof of Proposition 2.24, with a few minor changes in it. We use notation from this proof. By Lemma 2.22, X_K is a Moscow space (where K is the countable set A_n). Because of that, there exists a G_δ-subset F_K of the space X_K such that $p_K(x) \in F_K$ and F_K is contained in the closure of $\cup\{p_K(V) : V \in \gamma_n\}$. Now put $P_n = p_K^{-1}(F_K)$, for each $n \in \omega$, and $P = \cap\{P_n : n \in \omega\}$. Clearly, $x \in P$ and P is a G_δ-set in X. Since the sets $\cup\eta$ and P do not depend on coordinates in $A \setminus M$, and the sequence $\{A_n : n \in \omega\}$ is increasing, it follows that P is contained in the closure of U. Hence, X is Moscow. □

Remark 2.28. The condition in Lemma 2.22 and in Theorem 2.27 that X_K is Moscow for each finite set K of coordinates can not be replaced by the weaker assumption that each factor is Moscow. Indeed, $\beta\omega$ is a separable extremally disconnected compactum such that $\beta\omega \times \beta\omega$ is not Moscow (see Example 2.13). However, the space $\beta\omega \times \beta\omega$ is also separable, and therefore, the o-tightness of it is countable.

Corollary 2.29. *Let $\mathcal{F} = \{X_a : a \in A\}$ be a family of compact spaces of the countable tightness such that X_K is a Moscow space, for every finite subset K of A. Then the product space $X = \Pi\{X_a : a \in A\}$ is Moscow.*

Proof. Indeed, by a theorem of Malykhin (see [11]), the product of any finite family of compact spaces of countable tightness has countable tightness (for more general results see [1]). It remains to apply Theorem 2.27. □

Productive classes of Moscow spaces have found curious applications to the problem of power homogeneity of spaces [6]. A space X is called *power homogeneous* if there exists a cardinal number τ such that the space X^τ is homogeneous. In particular, Theorem 2.18 was instrumental in obtaining the next results [6]:

Theorem 2.30. *If a Corson compactum is power homogeneous, then it is first countable.*

Recall that *a Corson compactum* is any compact subspace of the Σ-product of some family of compact metrizable spaces.

Theorem 2.31. *A compact scattered space is power homogeneous if and only if it is countable.*

In the questions of power homogeneity the notion of a weakly Klebanov space, that is a stronger version of the notion of a Moscow space, plays also an important role. See the definition of it, and other results on power homogeneity, involving Moscow and weakly Klebanov spaces, in [6].

3. The Breadth of the Class of Moscow Groups

In this section we consider when a topological group is a Moscow space. It turns out that the class of Moscow groups contains many other important classes of groups. The main metamathematical conclusion to which we arrive is that a topological group is much more often a Moscow space than a topological space in general. To be more precise, there is a long list of cardinal invariants such that the countability restriction on any one of them guarantees that a topological group is a Moscow space. This phenomenon does not take place in the class of general topological spaces. On the other hand, examples of topological groups that are not Moscow where given in [4], [5], and [7]. Recall that one set of such examples can be obtained on the basis of Theorem 2.2.

Recall that a subset U of a space X is said to be *a canonical open subset* of X if U is the interior of its closure.

A point a of a space X is said to be *a point of canonical weak pseudocompactness* [3], if the following condition is satisfied:

(cwp) *for each canonical open subset* U *of* X *such that* $a \in \overline{U}$ *there exists a sequence* $\{A_n : n \in \omega\}$ *of subsets of* U *such that* $a \in \overline{A_n}$*, for each* $n \in \omega$*, and for each indexed family* $\eta = \{O_n : n \in \omega\}$ *of open subsets of* X*, satisfying the condition* $O_n \cap A_n \neq \emptyset$*, the family* η *has an accumulation point in* X*.*

We say that a space X is *canonically weakly Fréchet-Urysohn at a point* $a \in X$, or *κ-Fréchet-Urysohn at* a, if whenever $a \in \overline{U}$, where U is a canonical open subset of G, then some sequence of points of U converges to a. If X is κ-Fréchet-Urysohn at every point of X, we say that X is κ-Fréchet-Urysohn [3]. Obviously, if a space X is canonically weakly Fréchet-Urysohn at a point $a \in X$, then a is also a point of canonical weak pseudocompactness of X.

We call a space X *pointwise canonically weakly pseudocompact* [3], if each point of X is a point of canonical weak pseudocompactness. All κ-Fréchet-Urysohn spaces are pointwise canonically weakly pseudocompact. Notice that every compact topological group is a κ-Fréchet-Urysohn space, since every dyadic compactum is κ-Fréchet-Urysohn (see [11]). On the other hand, it was shown in [3] that all almost metrizable groups and all locally pseudocompact topological groups belong to the class of pointwise canonically weakly pseudocompact groups.

Theorem 3.1. [3] *If a topological group* G *is pointwise canonically weakly pseudocompact, then* G *is a Moscow space.*

Proof. Let U be a canonical open subset of G. It is enough to show that if $e \in \overline{U}$, then there exists a (closed) G_δ-subset P of G such that $e \in P \subset \overline{U}$. So let $e \in \overline{U}$. Fix subsets A_n of U such as in the condition (cwp) (where $a = e$).

Let us define a sequence $\{V_n : n \in \omega\}$ of open neighbourhoods of e, and a sequence $\{a_n : n \in \omega\}$ of points in U such that $a_n \in A_n$. First, choose a_0 to be any point of A_0, and let V_0 be an open neighbourhood of e such that $a_0 V_0 \subset U$.

Suppose now that an open neighbourhood V_k of e is already defined, for some $k \in \omega$. Then we let a_{k+1} to be any point of $A_{k+1} \cap V_k$. Now let V_{k+1} be any symmetric open neighbourhood of e such that $V_{k+1}^2 \subset V_k$ and $a_{k+1}V_{k+1} \subset U$. The recursive definition is complete.

By condition (cwp), the indexed family $\eta = \{a_n V_{n+1} : n \in \omega\}$ has a point of accumulation in G. We denote by H the set of all accumulation points of η in G. Thus, H is not empty.

Put $P = \cap\{V_n : n \in \omega\}$. From the construction it is clear that P is a (closed) subgroup of G.

Claim 1. $H \subset P$.

Indeed, take any $a \in H$, and fix $m \in \omega$. We have to show that $a \in V_m$. Put $k = m + 2$. There exists $n > k$ such that $aV_k \cap a_n V_{n+1} \neq \emptyset$. Then, for some $b \in V_{n+1}$ and $c \in V_k$, $ac = a_n b$. Hence $a = a_n bc^{-1} \in V_{n-1}V_{n+1}V_k \subset V_{m+2}^3 \subset V_{m+1}^2 \subset V_m$. Therefore, $a \in V_m$, for each $m \in \omega$. It follows that $a \in P$.

Claim 2. $aP = P$, for any $a \in H$.

Indeed, this is so, since $H \subset P$ and P is a subgroup of G.

Claim 3. $P \subset \overline{\cup\{a_n V_n : n \in \omega\}}$.

Fix $a \in H$. Then

$$P = aP \subset \overline{\cup\{a_n V_{n+1} P : n \in \omega\}} \subset \overline{\cup\{a_n V_n : n \in \omega\}}.$$

Since P is a subgroup of G, we have: $e \in P$. Therefore,

$$e \in P \subset \overline{\cup\{a_n V_n : n \in \omega\}} \subset \overline{U}.$$

Since P is a G_δ-set, the proof is complete. □

From Theorem 3.1 and Proposition 1.3 we obtain the next result from [3]:

Theorem 3.2. *Every dense subspace of any pointwise canonically weakly pseudocompact topological group is a Moscow space. In particular, if a topological group G satisfies at least one of the following conditions, then it is a Moscow space:*

1) G is pseudocompact;

2) G is totally bounded;

3) G is locally bounded;

4) G is a subgroup of an almost metrizable group;

5) G is a dense subgroup of a κ-Fréchet-Urysohn group.

Recall that *a right topological (a semitopological) group* is a group with a topology on it such that the multiplication on the right (on the right and on the left) is continuous. A *paratopological group* is a group with a topology such that the multiplication is jointly continuous.

The following cardinal invariant of tightness type for topological groups was defined in [5].

Let G be a right topological group, and $U \subset G$. A subset A of G is called *an ω-deep subset of U* if there exists a G_δ-subset P of G such that $e \in P$ and $AP \subset U$. We say that *the g-tightness* $t_g(G)$ of a right topological group G is countable (and write $t_g(G) \leq \omega$), if for each canonical open subset U of G and each $x \in \overline{U}$ there exists an ω-deep subset A of U such that $x \in \overline{A}$.

Now it is clear how to define the notion of a τ-deep subset A of U, where τ is any infinite cardinal, and how to define the g-tightness of any right topological group G.

Before we state our main result on right topological groups of countable g-tightness (Theorem 3.8), we present several almost obvious statements, showing how large is the class of these objects.

Proposition 3.3. *If G is a semitopological group of countable tightness, then the g-tightness of X is also countable.*

Proof. Clearly, if G is a semitopological group, and U is an open subset of G, then every one-point subset of U is ω-deep in U. □

It remains to apply the next obvious assertion:

Proposition 3.4. *The union of any countable family of ω-deep subsets of U is an ω-deep subset of U.*

Proposition 3.5. *Let G be a paratopological group such that the Souslin number of G is countable. Then the g-tightness of G is countable.*

Proof. Indeed, let U be an open subset of G. Let γ be a maximal disjoint family of non-empty ω-deep open subsets of U. Since G is a paratopological group, the set $A = \cup\gamma$ is dense in U. Therefore, $x \in \overline{A}$, for any $x \in \overline{U}$. It remains to observe that γ is countable and, therefore, A is an ω-deep subset of U, by Proposition 3.4. □

The next two statements are obvious.

Proposition 3.6. *Let G be an extremally disconnected semi-topological group. Then the g-tightness of G is countable.*

Proposition 3.7. *Let G be a right topological group of countable pseudocharacter. Then the g-tightness of G is countable.*

Theorem 3.8. *Every right topological group G of countable g-tightness is a Moscow space.*

Proof. Take any open subset U of G, and any $x \in \overline{U}$. Obviously, we may assume that U is a canonical open subset of G. Since $t_g(G) \leq \omega$, there exists an ω-deep subset A of U such that $x \in \overline{A}$. Now we can fix a G_δ-subset P of G such that $e \in P$ and $AP \subset U$. Then $x \in xP \subset \overline{AP} \subset \overline{U}$, and xP is a G_δ-subset of G. Thus, G is a Moscow space. □

Together Theorems 3.1 and 3.8 cover very large classes of topological groups (see below). It is really amazing how many topological conditions, which are innocently weak in the general case of arbitrary topological spaces, turn out to be sufficient for a topological group to be a Moscow space.

Let us say that *the κ-tightness* of a space X does not exceed the cardinal number τ (notation : $\kappa t(X) \leq \tau$) if for every canonical open set U in X and each point a in the closure of U there exists a subset B of U such that $a \in \overline{B}$ and $|B| \leq \tau$. Clearly, for every κ-Fréchet-Urysohn space X the κ-tightness of X is countable. The proof of the next statement is obvious.

Proposition 3.9. *If G is a paratopological group of countable o-tightness, then the g-tightness of G is also countable.*

Proposition 3.10. *If G is a κ-metrizable paratopological group, then the g-tightness of G is countable.*

Proof. Indeed, it is a result of Tkačenko [20] that the o-tightness $ot(X)$ of any κ-metrizable space is countable. Now arguing as in the proof of Proposition 3.5 we arrive at the conclusion that the g-tightness of G is countable.

Corollary 3.11. *Every dense subspace of any topological group G of countable g-tightness is a Moscow space. In particular, if a topological group G satisfies at least one of the following conditions, then it is Moscow:*

1) The pseudocharacter of G is countable, that is, each point in G is a G_δ-set;

2) The tightness of G is countable;

3) The Souslin number of G is countable;

4) G is a dense subgroup of a topological group F such that the κ-tightness of F is countable;

5) G is extremally disconnected;

6) G is κ-metrizable;

7) G is a subgroup of a topological group F such that F is a k-space;

8) The o-tightness of G is countable.

The sufficiency of 7), and 8) was established in [20] and [21], and the sufficiency of 4) and 5) was noticed in [3].

Another device, allowing to build new examples of Moscow groups from the Moscow groups already in existence, is provided by the following statement, which follows from Theorem 3.2, Corollary 3.11, and Theorem 1.6. Recall that, for a topological group G, ρG is the Rajkov completion of G and $\rho_\omega G$ is the G_δ-closure of G in ρG. It is a standard fact that ρG is a topological group containing G as a dense subgroup, and it is easy to see that $\rho_\omega G$ is also a topological group [3].

Theorem 3.12. [7] *Let G be a topological group satisfying at least one of the following conditions:*

1) The tightness of G is countable;

2) G is κ-metrizable;

3) G is a subgroup of an almost metrizable group;

4) G is a dense subgroup of a topological group H such that the κ-tightness of H is countable;

5) G is a k-space.

Then $\rho_\omega G$ (and any subgroup between G and $\rho_\omega G$) is a Moscow group.

It is shown in [5] that the product of two topological groups of countable g-tightness need not be a group of countable g-tightness. However, there are very large productive classes of Moscow groups, as we will presently see.

Note that every totally bounded topological group is a dense subspace of a compact group. Therefore, we have the next corollary to Theorem 3.2:

Corollary 3.13. *The product of any family of dense subspaces of totally bounded topological groups is a Moscow space.*

Whether a similar assertion holds for topological groups with the countable Souslin number is much less clear. Here we have only a consistency result.

Corollary 3.14. *Assume Martin's Axiom and the negation of the Continuum Hypothesis ($MA + \neg CH$). Then the product of any family of paratopological groups with the countable Souslin number is a Moscow space.*

Proof. It is well known that under ($MA + \neg CH$) the product of any family of topological spaces with the countable Souslin number is a space with the countable Souslin number (see [15]). Clearly, the product of any family of paratopological groups is again a paratopological group. It remains to refer to the fact that every paratopological group with the countable Souslin number is a Moscow space (see Proposition 3.5 and Theorem 3.8). □

We can further extend the reach of the above theorems using the following approach.

A space X is called *a groupy* space if it can be represented as a dense subspace of a paratopological group. From Theorem 3.8 and Proposition 3.5, using the fact that the Souslin number of a dense subspace coincides with the Souslin number of the whole space, we obtain:

Theorem 3.15. *Every groupy space with the countable Souslin number is Moscow.*

Arguing as in the proof of Corollary 3.14 and making use of Theorem 3.15, we come to the following conlusion:

Corollary 3.16. *Assume Martin's Axiom and the negation of the Continuum Hypothesis* $(MA + \neg CH)$*. Then the product of any family of groupy spaces with the countable Souslin number is a Moscow space.*

Problem 3.17. Can we drop the assumption $(MA + \neg CH)$ in Corollary 3.16?

Though the answer to the last question is unknown, we have the next result in ZFC:

Theorem 3.18. *The product of any family of separable groupy spaces is a Moscow space.*

Proof. Indeed, the product of any family of separable spaces is a space with the countable Souslin number [11]. It remains to observe that a topological group with the countable Souslin number is a Moscow space and every dense subspace of a Moscow space is a Moscow space. □

Here is a more general result. Let us call a space X *k-separable* if it contains a dense σ-compact subspace.

Theorem 3.19. *The product of any family of k-separable groupy spaces is a Moscow space.*

Proof. Indeed, the product of any family of k-separable spaces is a k-separable space [8]. On the other hand, M.G. Tkačenko established [22] that the Souslin number of every k-separable topological group is countable. Since every topological group with the countable Souslin number is a Moscow space and every dense subspace of a Moscow space is a Moscow space, we are done. □

A large productive class of Moscow spaces can be described as follows. Recall that ω_1 (or $\aleph_1$) is said to be *a precaliber* of a space X if every uncountable family of open subsets of X contains a centered uncountable subfamily. If ω_1 is a precaliber of X, then the Souslin number of X is, obviously, countable. The converse is true if we assume $(MA + \neg CH)$. It is well known that if X is a dense subspace of the product of any family of separable metrizable spaces, then ω_1 is a precaliber of X (see [11]). Here is the basic fact we need: if ω_1 is a precaliber of X_α for each $\alpha \in A$, then ω_1 is a precaliber of the product of all spaces X_α (see [11]). From this fact, arguing as above, we immediately obtain the following result:

Theorem 3.20. *If $\{X_\alpha : \alpha \in A\}$ is a family of groupy spaces, and ω_1 is a precaliber of X_α, for each $\alpha \in A$, then the product space $\Pi\{X_\alpha : \alpha \in A\}$ is Moscow.*

Problem 3.21. Let G be a topological group of countable tightness. Is then $G \times G$ a Moscow group?

Notice that we still do not have a ZFC-example of a topological group G of countable tightness such that the tightness of $G \times G$ is not countable.

Problem 3.22. Let G be an extremally disconnected topological group. Is then $G \times G$ a Moscow group?

Problem 3.23. Let G be an extremally disconnected topological group and B a compact group. Is then $G \times B$ a Moscow group?

Note, that Theorem 3.15 can be generalized as follows.

Theorem 3.24. *If the o-tightness of a groupy space X is countable, then X is a Moscow space.*

Proof. Let G be a paratopological group such that X is a dense subspace of G. Now we need the next lemma which generalizes a result of M.G. Tkačenko proved in [23]. □

Lemma 3.25. *If X is a dense subspace of a homogeneous space Z and the o-tightness of X is countable, then the o-tightness of Z is also countable.*

Proof. Indeed, let γ be a family of open sets in Z and $z \in Z$, $z \in \overline{\cup\gamma}$. Since Z is homogeneous, we may assume that $z \in X$. Since X is dense in Z, we have $z \in \overline{\cup\eta}$, where $\eta = \{U \cap X : U \in \gamma\}$. Since $ot(X) \leq \omega$, there exists a countable subfamily ξ of γ such that z is in the closure of $\cup\xi$. Thus, $ot(Z) \leq \omega$.

We continue the proof of Theorem 3.24. By Lemma 3.25, $ot(G) \leq \omega$. Since G is a paratopological group, it follows that G is a Moscow space. Therefore X, as a dense subspace of G, is also a Moscow space. □

Problem 3.26. (I.V. Yaschenko) Is it true that every Moscow topological group has countable g-tightness?

In connection with Problem 3.21, we formulate one more general theorem which easily follows from the results already obtained, and present a few more open questions.

Theorem 3.27. *Suppose $\mathcal{F} = \{G_a : a \in A\}$ is a family of paratopological groups such that the o-tightness of the space $G_K = \Pi\{G_a : a \in K\}$ is countable, for every finite subset K of A, and let $G = \Pi\{G_a : a \in A\}$ be their topological product. Then the space G is Moscow.*

Proof. Indeed, by Lemma 2.23, the o-tightness of the space G is countable. Since G is a paratopological group, it follows from Proposition 3.9 and Theorem 3.8 that the space G is Moscow. □

Theorem 3.27 and Problem 3.21 make the following questions especially interesting.

Problem 3.28. Let G be a topological group of countable tightness. Is then the o-tightness of $G \times G$ countable? Is the g-tightness of $G \times G$ countable?

Problem 3.29. Let G and H be two topological groups of countable o-tightness. Is the o-tightness of $G \times H$ countable?

To know when the product of topological groups is a Moscow group is important, in particular, for the theory of the Hewitt-Nachbin completion. The next theorem, which demonstrates it, was established in [5].

Theorem 3.30. [5] *Let $G = \Pi\{G_\alpha : \alpha \in A\}$ be the product of topological groups G_α such that the space G is Moscow and $|G|$ is Ulam non-measurable. Then $\nu(\Pi\{G_\alpha : \alpha \in A\}) = \Pi\{\nu G_\alpha : \alpha \in A\}$.*

Corollary 3.31. [7] *Let $\mathcal{F} = \{G_\alpha : \alpha \in A\}$ be a family topological groups G_α such that the cardinality of the product group $G = \Pi\{G_\alpha : \alpha \in A\}$ is Ulam non-measurable. Then the formula*

$$\nu(\Pi\{G_\alpha : \alpha \in A\}) = \Pi\{\nu G_\alpha : \alpha \in A\}$$

holds if at least one of the following conditions is satisfied:

1) Every group in $\mathcal{F}$ is totally bounded;

2) Every group in $\mathcal{F}$ is k-separable;

3) ω_1 is a precaliber of every space in $\mathcal{F}$;

4) The Souslin number of the product space $\Pi\{G_\alpha : \alpha \in A\}$ is countable;

5) The Souslin number of every group in $\mathcal{F}$ is countable, and $(MA + \neg CH)$ is satisfied;

6) Every group in $\mathcal{F}$ is κ-metrizable;

7) The tightness of the product space $\Pi\{G_\alpha : \alpha \in A\}$ is countable;

8) The g-tightness of the product group $\Pi\{G_\alpha : \alpha \in A\}$ is countable;

9) Each finite subproduct of the product has countable tightness;

10) Every group in $\mathcal{F}$ has a countable network.

For the case of two factors, the following result, which does not make use of the assumption that the cardinality of the product is Ulam non-measurable, follows easily from Theorem 5.6 in [7].

Theorem 3.32. *If G and H are topological groups such that $G \times H$ is a Moscow space, then $\mu(G \times H) = \mu G \times \mu H$, where μG and μH are the Dieudonné completions of G and H, respectively.*

The proofs of Theorems 3.30 and 3.32 are based upon the following theorem established in [5], which also shows the importance of the notion of a Moscow space for the theory of topological groups.

Theorem 3.33. [5] *Let G be a Moscow topological group. Then the operations on G can be extended to the Dieudonné completion μG of G in such a way that μG becomes a topological group containing G as a topological subgroup.*

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