Topology Proceedings

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E-mail:	topolog@auburn.edu
ISSN:	0146-4124

TOPOLOGY PROCEEDINGS
Volume 25, Summer 2000
Pages 519–527

ASPECTS OF THE EMBEDDABILITY ORDERING IN TOPOLOGY

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Abstract

We consider the collection of topologies definable on a cardinal α, ordered by homeomorphic embeddability. We investigate aspects of this quasi-order and in particular, using a construction method, we show that the embeddability ordering does not form an upper semi-multilattice.

1. Introduction

The embeddability ordering is one of the most natural notions in topology and is defined as follows:

Definition 1.1. *Let α be a cardinal number. Let $T(\alpha) = \{\mathfrak{T}_i : i \in I\}$ be the set of (homeomorphism classes of) topologies definable on α. We say that $(\alpha, \mathfrak{T}_i)$ embeds into $(\alpha, \mathfrak{T}_j)$ if and only if $(\alpha, \mathfrak{T}_i)$ is homeomorphic to a subspace of $(\alpha, \mathfrak{T}_j)$. We define a quasi-order on $T(\alpha)$ as follows:*

$\mathfrak{T}_i \leq \mathfrak{T}_j$ if and only if $(\alpha, \mathfrak{T}_i) \hookrightarrow (\alpha, \mathfrak{T}_j)$, that is, $(\alpha, \mathfrak{T}_i)$ embeds into $(\alpha, \mathfrak{T}_j)$.

Note 1.2. Let us first note that if α is finite then $T(\alpha)$ is simply an antichain. Hence we will only consider infinite α.

Theorem 1.3. $(T(\alpha), \leq)$ *is a quasi-ordered set but not a partially ordered set.*

* The research of the first author was supported by a distinction award scholarship from the Department of Education for Northern Ireland.

Mathematics Subject Classification: 06A06, 54B99.

Key words: Quasi-order, embeddability ordering.

Proof. It is clear that reflexivity and transitivity are satisfied. It remains to show that the order lacks anti-symmetry. For this we will first exhibit two topologies on the set of natural numbers which are pairwise embeddable but not homeomorphic. Consider the following diagrams in which the basic open sets are indicated by enclosing rectangles:

0	1 2	3 4	5 6	7 8	9 10	11 12	13 14	....

0 1	2 3	4 5	6 7	8 9	10 11	12 13	....

A suitable embedding map in both cases is $g(x) = x + 1$. This simple example extends for any cardinal number α by taking the disjoint union of each of these two spaces with the trivial space on α-many points.

Note 1.4. It is easy to see that the following spaces are *minimal* in that any subspace of size α is in fact homeomorphic to the original space. Indeed in the case $\alpha = \aleph_0$ they form a *support* for $T(\alpha)$ in that any countable space contains a copy of (at least) one of them.
(i) $(\alpha, \mathcal{T}_\uparrow)$, whose open sets are precisely the increasing subsets of α,
(ii) $(\alpha, \mathcal{T}_0)$, the trivial space ,
(iii) $(\alpha, \mathcal{D})$, the discrete space,
(iv) $(\alpha, \mathcal{C})$, the cofinite space, and
(v) $(\alpha, \mathcal{T}_\downarrow)$, whose open sets are precisely the decreasing subsets of α.

Definition 1.5. Given a quasi-ordered set $(E, \leq)$, with $b \in A \subseteq E$,

- b is maximal in A if and only if $(x \in A, x \geq b \Rightarrow x = b)$;
- b is weakly maximal in A if and only if $(x \in A, x \geq b \Rightarrow x \leq b)$;

- minimal and weakly minimal elements are dually defined;
- b is maximum in A if and only if $x \leq b$ for all $x \in A$;
- A is cofinal in $(E, \leq)$ if and only if for all $x \in E$, there exists $a \in A$ such that $x \leq a$;
- For $x \in E$ and $y \in E$, an element $z \in E$ is a supremum of x and y if and only if: (i) $x \leq z$ and $y \leq z$, and (ii) ($x \leq w$ and $y \leq w$) implies that $z \leq w$;
- E is said to form an upper semi-multilattice (see, for example, [1]) if and only if for every $x, y, z \in E$ with $x \leq z$ and $y \leq z$, there exists a minimal upper bound w for x and y with $w \leq z$.

Note 1.6.

- Let $(\alpha, \mathcal{I}(x))$ denote the topology on α whose (non-empty) open subsets are precisely those which contain x. Let P be any homeomorphic invariant that is possessed by $(\alpha, \mathcal{I}(x))$ but not by the discrete space (on α-many points); then we have that $(\alpha, \mathcal{I}(x))$ is minimal in P. For example, $(\alpha, \mathcal{I}(x))$ is a minimal member of the family of connected topologies on α.
- Let $(\alpha, \mathcal{E}(x))$ denote the topology on α whose open subsets are precisely those to which x does not belong (together with α). So if P is any homeomorphic invariant that is possessed by $(\alpha, \mathcal{E}(x))$ but not by the discrete space (on α-many points), then we have that $(\alpha, \mathcal{E}(x))$ is minimal in P. For example, $(\alpha, \mathcal{E}(x))$ is a minimal member of the family of compact topologies on α.
- Let $F = \{0, 1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space: i.e. F is the Sierpiński space. Then $F^{\aleph_0}$ (i.e. the Cartesian product $\prod_{n \in \mathbb{N}} F_n$ where $F_n = F$ for every $n \in \mathbb{N}$) is a maximal member of the family of (T_0 plus completely separable) topologies on $\mathfrak{c}$.

McCluskey and McMaster in [2] showed the following:

Theorem 1.7. *Let A be a discrete space of infinite cardinality α, then there is a subset P of $\beta A \setminus A$ such that*
(i) card$(P) = 2^{2^\alpha}$ and
(ii) whenever $p_1 \neq p_2$ in P then neither $A \cup \{p_1\} \hookrightarrow A \cup \{p_2\}$ nor $A \cup \{p_2\} \hookrightarrow A \cup \{p_1\}$ is true.

This demonstrates the existence of an antichain of cardinality 2^{2^α} in $T(\alpha)$ of the form $\{\alpha \cup \{p_\delta\} : \delta < 2^{2^\alpha}\}$, where each $\alpha \cup \{p_\delta\}$ is a subspace of the Stone-Čech compactification of α with the discrete topology.

Theorem 1.8. *There does not exist a maximum member of $T(\alpha)$.*

Proof. Suppose that $\mathcal{T}$ was maximum in $T(\alpha)$. Theorem 1.7 gives us 2^{2^α} non-homeomorphic topologies on α, not all of which can embed into $\mathcal{T}$. □

Corollary 1.9. *There does not exist a weakly maximal member of $T(\alpha)$.*

Proof. Suppose that $\mathcal{T} \in T(\alpha)$ was weakly maximal. Let $\mathcal{T}^*$ be another member of $T(\alpha)$. We have that $(\alpha, \mathcal{T})$ embeds into the disjoint union of $(\alpha, \mathcal{T})$ with $(\alpha, \mathcal{T}^*)$. Since $\mathcal{T}$ is weakly maximal then it will contain a copy of $(\alpha, \mathcal{T}^*)$. That is to say, any weakly maximal member of $T(\alpha)$ would be maximum. Hence by Theorem 1.8 such a space cannot exist. □

Theorem 1.10. *If $A \subseteq T(\alpha)$ is cofinal and up-directed in $T(\alpha)$ then there are no elements weakly maximal in A.*

Proof. Suppose $\mathcal{T}^*$ was weakly maximal in A. Let $\mathcal{T} \in T(\alpha)$. Now A is cofinal in $T(\alpha)$, therefore there exists $\mathcal{T}^\square \in A$ such that $\mathcal{T} \leq \mathcal{T}^\square$. However, A is also up-directed, so there exists $\mathcal{T}^\bullet \in A$ such that $\mathcal{T}^\square \leq \mathcal{T}^\bullet$ and $\mathcal{T}^* \leq \mathcal{T}^\bullet$. But then we have $\mathcal{T}^\bullet \leq \mathcal{T}^*$ (since $\mathcal{T}^*$ is weakly maximal in A). Hence we have

$\mathcal{T} \leq \mathcal{T}^{\square} \leq \mathcal{T}^{\bullet} \leq \mathcal{T}^{*} \Rightarrow \mathcal{T} \leq \mathcal{T}^{*}$. That is, $\mathcal{T}^{*}$ is maximum in $T(\alpha)$: a contradiction, since by Theorem 1.8 there does not exist a maximum member of $T(\alpha)$. Hence there are no elements weakly maximal in A. □

Example 1.11. Let A denote the set of compact topologies on α. It is easy to show that A is cofinal and up-directed in $T(\alpha)$. Hence there does not exist a weakly maximal member of A.

Theorem 1.12. *$T(\alpha)$ is up-directed but no pair of incomparable topologies has a supremum in $T(\alpha)$.*

Proof. Consider $(\alpha_1, \mathcal{T}_1)$ and $(\alpha_2, \mathcal{T}_2)$ where α_1 and α_2 are disjoint copies of α, and $\mathcal{T}_1$ and $\mathcal{T}_2$ are incomparable members of $T(\alpha)$. Let us suppose that they have a supremum which we will call $\mathcal{T}$ defined on α. Then $\mathcal{T}$ must embed into the following spaces:
(i) the space generated by their disjoint union, by the map g say, and
(ii) $\mathcal{T}^{\star} = \{G \in \mathcal{T}_1\} \cup \{\alpha_1 \cup H : H \in \mathcal{T}_2\}$, by the map h.
Consider the first space. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are incomparable we must have that $(g(\alpha) \cap \alpha_1) \neq \emptyset$ and $(g(\alpha) \cap \alpha_2) \neq \emptyset$. We see that $g(\alpha)$ can be expressed as the union of these two sets, so we have that $g(\alpha)$ is disconnected. Hence the supremum, $\mathcal{T}$ is disconnected. Consider the second space. We again have that $(h(\alpha) \cap \alpha_1) \neq \emptyset$ and $(h(\alpha) \cap \alpha_2) \neq \emptyset$. Suppose that $h(\alpha)$ was disconnected. Then it can be expressed as the union of two disjoint non-empty open sets G_1 and $G_2 \in \mathcal{T}^{\star}_{h(\alpha)}$. Now let us say that G_1 meets $(h(\alpha) \cap \alpha_2)$ which means that $(\alpha_1 \cap h(\alpha)) \subset G_1$. Then we must have that G_2 meets $(h(\alpha) \cap \alpha_2)$ (otherwise $G_2 = \emptyset$) which also means that $(\alpha_1 \cap h(\alpha)) \subset G_2$. This then gives $(G_1 \cap G_2) \neq \emptyset$, a contradiction. So in fact we have that $h(\alpha)$ is connected, which in turn gives the supremum, $\mathcal{T}$, to be connected: another contradiction. So we now have that any two incomparable members of $T(\alpha)$ cannot have a supremum. □

2. Ordersum Topology

We now define a topology on the union of a collection of pairwise disjoint topological spaces indexed by a partially ordered set.

Definition 2.1. Let $\{(X_i, \mathcal{T}_i) : i \in I\}$ be a family of non-empty pairwise disjoint topological spaces indexed by a partially ordered set I. Let X denote $\bigcup_{i\in I} X_i$. We define the ordersum topology $(X, \mathcal{T}_X)$ (where $X = \bigcup_{i\in I} X_i$) as follows: we say that a basic neighbourhood of a point $x \in X$ is of the form $G \cup \bigcup_{i>i_x} X_i$ where $x \in X_{i_x}$ and G is a $\mathcal{T}_{i_x}$ neighbourhood of x.

Note 2.2. (i) If I is an antichain then the ordersum topology is simply the disjoint union of this family of topological spaces.
(ii) If, for each $i \in I$, $|X_i| \le \alpha$ and $|I| = \alpha$ then $\mathcal{T}_X \in T(\alpha)$. Similarly if, for each $i \in I$, $|X_i| = \alpha$ and $|I| \le \alpha$ then $\mathcal{T}_X \in T(\alpha)$.
Let us say that a partially ordered set I is connected if and only if it cannot be expressed as the disjoint union of two (non-empty) partially ordered sets I_1 and I_2 such that for each element $i_1 \in I_1$, i_1 is incomparable with each element in I_2.
(iii) The ordersum topology is connected if and only if I is connected, $(|I| \ge 2)$.
(iv) The ordersum topology is T_0 if and only if $(X_i, \mathcal{T}_i)$ is T_0 for each $i \in I$.
(v) The ordersum topology is T_1 if and only if I is an anti-chain and $(X_i, \mathcal{T}_i)$ is T_1 for each $i \in I$. (Similarly for T_2, T_3 and T_4.)

Theorem 2.3. *$(T(\alpha), \le)$ does not form an upper semi-multi-lattice.*

Proof. The proof of this theorem will require us to exhibit two members of $T(\alpha)$ with an upper bound such that, no matter what subspace of this upper bound we take which is still an upper bound to our first two spaces, we can always find a strictly smaller subspace which is still an upper bound to our original two spaces.

By Theorem 1.7 we have the existence of a sequence $\{\alpha \cup \{p_n\} : n \in \mathbb{N}\}$ of incomparable topologies on α-many points. We note from their construction that these spaces are Hausdorff.

Our first space, $\mathcal{T}_1$, will be the ordersum of these spaces indexed by the natural numbers. (See Figure 1.)

Our second space, $\mathcal{T}_2$, will be the ordersum of $\aleph_0$-many copies of $(\alpha, \mathcal{T}_0)$, indexed by the natural numbers, where $\mathcal{T}_0$ denotes the trivial topology and we have disjoint copies of α. (See Figure 1.)

Our upper bound to these two spaces will be $\mathcal{T}_3$, again an ordersum, formed by placing a trivial space between $\alpha \cup \{p_n\}$ and $\alpha \cup \{p_{n+1}\}$ for each $n \in \mathbb{N}$. (See Figure 1.)

Now, what is a typical subspace $\mathcal{T}_4$ of $\mathcal{T}_3$ which contains a copy of both $\mathcal{T}_1$ and $\mathcal{T}_2$? First it contains a copy of $\mathcal{T}_2$. So suppose that a trivial space was mapped into $\alpha \cup \{p_n\}$ for some n. Since $\alpha \cup \{p_n\}$ is Hausdorff we have a contradiction. Suppose that a trivial space was mapped into two different levels. We would then have that it contains a copy of the two-point Sierpiński space: again a contradiction. So we have that a trivial space must be mapped into a trivial space. Hence $\mathcal{T}_4$ must contain $\aleph_0$-many trivial spaces.

Next, $\mathcal{T}_4$ contains a copy of $\mathcal{T}_1$. Suppose that $\alpha \cup \{p_n\}$ was mapped to two different levels in $\mathcal{T}_3$. Then by restricting the topology we would have that $\alpha \cup \{p_n\}$ contains a copy of the Sierpiński space: a contradiction. So we have that $\alpha \cup \{p_n\}$ must be mapped into $\alpha \cup \{p_m\}$. However, because the spaces $\alpha \cup \{p_n\}$ where chosen to be incomparable, m must be equal to n.

So, continuing, we have that if $\mathcal{T}_4$ is a subspace of $\mathcal{T}_3$ which contains a copy of $\mathcal{T}_1$ and a copy of $\mathcal{T}_2$, then it must be as shown in Figure 1, where A_n denotes the image of $\alpha \cup \{p_n\}$ for each $n \in \mathbb{N}$.

We can now remove the first trivial space from $\mathcal{T}_4$ and be left with the subspace $\mathcal{T}_5$ as shown. Notice that $\mathcal{T}_5$ contains a copy of $\mathcal{T}_1$ and $\mathcal{T}_2$. It obviously embeds into $\mathcal{T}_4$ but can $\mathcal{T}_4$ be embedded into $\mathcal{T}_5$? If so, A_m would be mapped into A_m, and A_{m+1} into A_{m+1}. The trivial space in between would then give a similar contradiction as in the above arguments.

Hence we have exhibited two topologies on α which have an upper bound which in turn has no minimal upper bound below it.

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha \cup \{p_n\} & (\alpha, \mathcal{T}_0) & \alpha \cup \{p_3\} & A_{m+1} & (\alpha, \mathcal{T}_0) \\
\vdots & \uparrow & \uparrow & \uparrow & \vdots \\
\alpha \cup \{p_4\} & (\alpha, \mathcal{T}_0) & (\alpha, \mathcal{T}_0) & (\alpha, \mathcal{T}_0) & A_{m+1} \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\alpha \cup \{p_3\} & (\alpha, \mathcal{T}_0) & \alpha \cup \{p_2\} & A_m & A_m \\
\uparrow & \uparrow & \uparrow & \vdots & \vdots \\
\alpha \cup \{p_2\} & (\alpha, \mathcal{T}_0) & (\alpha, \mathcal{T}_0) & A_2 & A_2 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\alpha \cup \{p_1\} & (\alpha, \mathcal{T}_0) & \alpha \cup \{p_1\} & A_1 & A_1 \\
 & & & & \\
\mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5
\end{array}
$$

□

Fig. 1. Ordersum topologies

Corollary 2.4. *$(T(\alpha), \leq)$ does not form a lower semi-multi-lattice.*

Proof. A similar construction, using the ordersum topology, can generate topologies with the required properties. □

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