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ON NAGATA SPACES

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Abstract

A problem of J. Nagata is solved by a characterization of Nagata spaces.

Introduction

All (topological) spaces considered below are assumed to be T_1 and regular. The symbol X denotes a topological space. For definitions of concepts used here without definition, we refer the reader to [Gr].

Recall that a *network* of X is a family $\mathcal{L}$ of subsets of X such that, for every open subset G of X and for every point $x \in G$, there exists $L \in \mathcal{L}$ such that $x \in L \subset G$. The family $\mathcal{L}$ is a *k-network* provided that, for every open set G and for every compact set $K \subset G$, there exists a finite subfamily $\mathcal{L}'$ of $\mathcal{L}$ such that $K \subset \bigcup \mathcal{L}' \subset G$. The family $\mathcal{L}$ is a *π-base* provided that $\mathcal{L}$ consists of nonempty open sets and, for every non-empty open set U, there exists $L \in \mathcal{L}$ such that $L \subset U$.

A subset E of X is called *σ-discrete* (*F_σ-discrete*) if E is the union of countably many (closed and) relatively discrete subsets of X. If X is perfect (i.e., every open subset is an F_σ-set), then every σ-discrete subset of X is F_σ-discrete.

A family $\mathcal{L}$ of subsets of X is called *interior-preserving* provided that we have $\bigcap_{L\in\mathcal{L}'} \mathrm{Int}L = \mathrm{Int}\bigcap_{L\in\mathcal{L}'} L$ for every $\mathcal{L}' \subset \mathcal{L}$.

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In [Na], Nagata calls a family $\mathcal{L}$ of subsets of X a *pip-family* (*pseudo-interior-preserving*) if $\bigcap_{L\in\mathcal{L}'} L \subset \operatorname{Int}\bigcap_{L\in\mathcal{L}'} \operatorname{Cl}L$ for every $\mathcal{L}' \subset \mathcal{L}$. Note that every interior-preserving family consisting of open sets is pip. However, an arbitary interior-preserving family may fail to be pip: a singleton family $\{L\}$ is always interior-preserving, but it is pip only when L is *preopen*, i.e., when $L \subset \operatorname{Int}\operatorname{Cl}L$.

A *Nagata space* is a first countable stratifiable space ([Ce]). By a result of [Itō], Nagata spaces coincide with first countable M_1-spaces. In [Na], Nagata proved that a regular space is metrizable if, and only if, the space has a k-network which is both σ-closure-preserving and σ-pip, and he asked whether a regular space is metrizable if there is such a network on X (see Problem 1 of [Na]). In this note, we characterize Nagata spaces by the existence of a σ-closure-preserving, σ-pip network. As a consequence, the above question of Nagata is answered negatively.

Main Result

Lemma 1. *Every paracompact semimetrizable space has a dense metrizable σ-discrete subspace.*

Proof. Let X be a paracompact semimetrizable space, and let d be a semimetric on X such that, for every $x \in X$, the sets $B(x,n) = \{y \in X : d(y,x) < \frac{1}{n}\}$, for $n \in \mathbf{N}$, form a neighbourhood base of x in X. For every $n \in \mathbf{N}$, let D_n be a subset of X which is maximal among all subsets having the property that $d(x,y) \geq \frac{1}{n}$ for any two distinct points x, y of the subset. Then we have that $D_n \cap B(x,n) \neq \emptyset$ for all $n \in \mathbf{N}$ and $x \in X$, and it follows that the set $D = \bigcup_{n=1}^{\infty} D_n$ is dense in X. For every $n \in \mathbf{N}$, the set D_n is relatively discrete, because we have that $B(x,n) \cap D_n = \{x\}$ for each $x \in D_n$. Since the semimetrizable space X is perfect, the σ-discrete subset D of X is F_σ-discrete.

Hence we can write D in the form $D = \bigcup_{n=1}^{\infty} D'_n$ so that each D'_n is discrete and closed in X; we may also assume that the sets D'_n, for $n \in \mathbf{N}$, are mutually disjoint.

For every $n \in \mathbf{N}$, it follows from paracompactness of X that there exists a discrete open family $\{G_x : x \in D'_n\}$ such that $x \in G_x$ for every $x \in D'_n$. It is easy to see that the family $\{B(x,n) \cap G_x \cap D : x \in D \text{ and } n \in \mathbf{N}\}$ is a σ-discrete base of the subspace D. As a consequence, D is metrizable. □

Lemma 2. *The following conditions are mutually equivalent for a metacompact semimetrizable space X:*
(1) X has a point-finite π-base.
(2) X has an interior-preserving π-base.
(3) X has a dense σ-discrete G_δ-subset.

Proof. The equivalence of (1) and (2) is a consequence of Corollary 4.12 of [Ju].
(1)⇒ (3): Assume that $\mathcal{B}$ is a point-finite π-base for X. Denote by $\mathcal{L}$ the partition of X generated by $\mathcal{B}$, i.e., $\mathcal{L} = \{\bigcap(\mathcal{B})_x \setminus \bigcup(\mathcal{B} \setminus (\mathcal{B})_x) : x \in X\}$, where $(\mathcal{B})_x = \{B \in \mathcal{B} : x \in B\}$. For each $L \in \mathcal{L}$, pick a point x_L from the set L. Let $E = \{x_L : L \in \mathcal{L}\}$ and note that we have $(\mathcal{B})_x \neq (\mathcal{B})_y$ whenever x and y are distinct points of E. We prove that E is a dense σ-discrete G_δ-subset of X.

To show that E is dense in X, let O be a non-empty open subset of X. Since $\mathcal{B}$ is a π-base, there exists $B \in \mathcal{B}$ such that $B \subset O$. Take $x \in B$ and let $L \in \mathcal{L}$ be such that $x \in L$. Then $L \cap B \neq \emptyset$, and it follows from the definition of the sets of the family $\mathcal{L}$ that we must have $L \subset B$. Take $y \in E \cap L$. Then have that $y \in L \subset B \subset O$ and hence that $E \cap O \neq \emptyset$.

Next we show that E is a G_δ-set. For each $x \in E$, let $\langle V_n(x) \rangle_{n=1}^{\infty}$ be a decreasing sequence of open neighbourhoods of x such that $V_1(x) \subset \bigcap(\mathcal{B})_x$ and $\bigcap_{n=1}^{\infty} V_n(x) = \{x\}$. Set $V_n = \bigcup_{x \in E} V_n(x)$. We show that $\bigcap_{n=1}^{\infty} V_n \subset E$. Let $x \in \bigcap_{n=1}^{\infty} V_n$. For every $n \in \mathbf{N}$, there exists $z_n \in E$ such that $x \in V_n(z_n)$.

Let $A = \{z_n : n \in \mathbf{N}\}$. To show that $x \in E$, it suffices to show that $x \in A$. Note that we have $x \in V_1(a) \subset \bigcap(\mathcal{B})_a$ and therefore $(\mathcal{B})_a \subset (\mathcal{B})_x$ for every $a \in A$; it follows, since the family $(\mathcal{B})_x$ is finite and since $(\mathcal{B})_a \neq (\mathcal{B})_b$ for any two distinct points of E, that the set A is finite. Now we easily see that $x \in A$: if this were not the case, there would exist $n \in \mathbf{N}$ such that $x \in X \setminus V_n(a)$ for every $a \in A$ and then we would have that $x \notin V_n(z_n)$ – a contradiction. We have shown that $\bigcap_{n=1}^{\infty} V_n \subset E$; since the converse inclusion holds trivially, we have shown that E is a G_δ-set.

It remains to show that E is σ-discrete. For every $n = 0, 1, 2, ...$, let $E_n = \{z \in E : |(\mathcal{B})_z| = n\}$, and note that we have $\bigcap(\mathcal{B})_z \cap E_n = \{z\}$ for every $z \in E_n$; as a consequence, the set E_n is relatively discrete in X. As the union of the sets E_n, $n = 0, 1, ...$, the set E is σ-discrete.

(3)$\Rightarrow$ (1): Let F be a σ-discrete dense G_δ set of X. Since the semimetrizable space X is perfect, the σ-discrete subset F is F_σ-discrete, and hence we can write $F = \bigcup_{n=1}^{\infty} F_n$ so that each F_n is discrete and closed in X. We may assume that $F_0 = \emptyset$ and, for each $n \in \mathbf{N}$, $F_n \subset F_{n+1}$. Also, we can write $F = \bigcap_{n=1}^{\infty} G_n$ so that, for every $n \in \mathbf{N}$, the set G_n open and $G_{n+1} \subset G_n$. Since X is metacompact there exists, for every $n \in \mathbf{N}$, a family $\{W(n,z) : z \in F_n\}$ of open subsets of X such that, for every $x \in X$, the set $\{z \in F : x \in W(n,z)\}$ is finite, and $W(n,z) \cap F_n = \{z\}$ for every $z \in F_n$. For every $z \in F$, denote by n_z the least number n such that $z \in F_n$, and let $\{V_n(z) : n \in \mathbf{N}\}$ be a decreasing open neighborhood base of z such that the following conditions are satisfied:

(a) $V_1(z) \subset W(n_z, z) \cap G_{n_z}$ and

(b) $V_1(z) \subset \bigcap\{V_n(y) : n \in \mathbf{N}, y \in F_{n_z-1} \text{ and } z \in V_n(y)\}$.

We prove that the family

$$\mathcal{C} = \{V_1(z) : z \in F\} \cup \{\{x\} : x \text{ is isolated in } X\}$$

is a point-finite π-base of X. Let G be a nonempty open subset of X. Let z be a point of the non-empty set $G \cap F$ and let

$n > n_z$ be such that $V_n(z) \subset G$. Set $Q = V_n(z) \setminus \bigcup_{k<n} F_k$, and note that Q is an open set which is non-empty unless the point z is isolated. For each $q \in Q \cap F$ we have that $z \in F_{n_q - 1}$ and $q \in V_n(z)$, and this means that $V_1(q) \subset V_n(z) \subset G$. It follows from the foregoing that the family $\mathcal{C}$ is a π-base of X.

For every $n \in \mathbf{N}$, since $\{W(n,z) : z \in F_n\}$ is a point-finite family of mutually distinct sets and since $V_1(z) \subset W(n,z)$ for every $x \in F_n \setminus \bigcup_{k<n} F_k$, also the family $\{V_1(x) : x \in F_n\}$ is point-finite. To show point-finiteness of $\mathcal{C}$ at a point y of X, we consider two cases. If $y \in F$, then $y \notin V_1(z)$ for any $z \in F \setminus \bigcup_{k \le n_y} F_k$, and hence the family $\{V_1(x) : x \in F\}$ is point-finite at y. If $y \notin F$, then there exists $m \in \mathbf{N}$ such that $y \notin G_n$ for any $n > m$; in this case we have that $y \notin V_1(z)$ for any $z \in F \setminus \bigcup_{k \le m} F_k$ and, again, the family $\{V_1(x) : x \in F\}$ is point-finite at y. By the foregoing, the π-base $\mathcal{C}$ is point-finite. □

Remark. Note that the full assumptions on X above were only needed to prove that (2) ⇒ (1). The given proof of (1) ⇒ (3) works for any X whose singleton subsets are G_δ-sets, and the proof of (3) ⇒ (1) for any X which is metacompact, perfect and first countable.

Proposition 1. *A space X is a Nagata space if, and only if, X has a network which is both σ-closure-preserving and σ-pip.*

Proof. Sufficiency. If $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a network of X such that each $\mathcal{F}_n$ is closure-preserving and pip, then it is easy to see that $\{\mathrm{Int}\mathrm{Cl}F : F \in \mathcal{F}\}$ is a σ-closure-preserving base of X, so that X in an M_1-space; moreover, for each $x \in X$, the family $\{\mathrm{Int} \bigcap \{\mathrm{Cl}F : F \in (\mathcal{F}_n)_x\} : n \in \mathbf{N}\}$ is a countable neighborhood base of x, so that X is a Nagata space.

Necessity. Assume that X is a Nagata space. Then X is paracompact and semimetrizable. By Lemma 1, X has a metrizable F_σ-discrete dense subset M. By Lemma 2, the subspace M of X has a point-finite π-base $\mathcal{L}$. Since X is a first countable M_1-space, we can give open neighborhood bases $\{B_{x,n} : n \in \mathbf{N}\}$ for points $x \in X$ in such a way that, for every $n \in \mathbf{N}$, the family $\mathcal{B}_n = \{B_{x,n} : x \in X\}$ is closure-preserving.

For all $x \in X$ and $n \in \mathbf{N}$, let $B'_{x,n} = \bigcup\{L \in \mathcal{L} : L \subset B_{x.n}\} \cup \{x\}$. Set $\mathcal{F}_n = \{B'_{x,n} : x \in X\}$ for each $n \in \mathbf{N}$, and let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Since $\{B_{x,n} : x \in X \text{ and } n \in \mathbf{N}\}$ is a base of X, the family $\mathcal{F}$ is a network. To complete the proof, we show that each $\mathcal{F}_n$ is closure-preserving and pip. Let $n \in \mathbf{N}$ be given.

Since $\mathcal{B}_n$ is closure-preserving, we can prove the same property for $\mathcal{F}_n$ by showing that $\mathrm{Cl}B'_{x,n} = \mathrm{Cl}B_{x,n}$ for each $x \in X$. So let $x \in X$. Since $B'_{x,n} \subset B_{x,n}$, it suffices to show that $\mathrm{Cl}B_{x,n} \subset \mathrm{Cl}B'_{x,n}$. Let $y \in \mathrm{Cl}B_{x,n}$ and let U be an open neighborhood of y. Then $U \cap B_{x,n} \cap M$ is a nonempty open set of the subspace M. Since $\mathcal{L}$ is a π-base of M, there is an $L \in \mathcal{L}$ such that $L \subset U \cap B_{x,n} \cap M$. We have that $L \subset B'_{x,n}$ and it follows that $U \cap B'_{x,n} \neq \emptyset$. By the foregoing, we have that $y \in \mathrm{Cl}B'_{x,n}$. We have shown that $\mathrm{Cl}B_{x,n} \subset \mathrm{Cl}B'_{x,n}$ and hence that $\mathrm{Cl}B_{x,n} = \mathrm{Cl}B'_{x,n}$.

It remains to prove that $\mathcal{F}_n$ is pip. Let $A \subset X$. To show that $\bigcap_{x\in A} B'_{x,n} \subset \mathrm{Int}\bigcap_{x\in A} \mathrm{Cl}B'_{x,n}$, let $y \in \bigcap_{x\in A} B'_{x,n}$. Let $A' = A \setminus \{y\}$, and note that $\bigcap(\mathcal{L})_y \subset \bigcap_{x\in A'} B'_{x,n}$. Since $\mathcal{L}$ is point-finite, the set $\bigcap(\mathcal{L})_y \cap M$ is open in M. Let U be an open subset of X such that $U \cap M = \bigcap(\mathcal{L})_y \cap M$ and, if $(\mathcal{L})_y = \emptyset$, then $U = X$; note that U is a neighbourhood of y in X. We have that $U \cap M \subset \bigcap_{x\in A'} B'_{x,n}$ and it follows by the density of M that $U \subset \mathrm{Cl}\bigcap_{x\in A'} B'_{x,n} \subset \bigcap_{x\in A'} \mathrm{Cl}B'_{x,n}$. We have shown that the set $\bigcap_{x\in A'} \mathrm{Cl}B'_{x,n}$ is a neighbourhood of y in X. The proof that $y \in \mathrm{Int}\bigcap_{x\in A} \mathrm{Cl}B'_{x,n}$ is now completed by observing that since we have $\mathrm{Cl}B_{y,n} = \mathrm{Cl}B'_{y,n}$, also the set $\mathrm{Cl}B'_{y,n}$ is a neighbourhood of y in X. □

Remark : Let $\mathcal{F}$ be a family of preopen subsets of a space X. It is easy to see that if $\mathcal{F}$ is point-finite, then $\mathcal{F}$ is pip. Hence, if a space has a preopen network which is both σ-closure-preserving and σ-point-finite, then the space is a Nagata space. Note that if a space has a σ-locally finite preopen network $\mathcal{F}$, then the space has a σ-locally finite base $\{\mathrm{IntCl}F : F \in \mathcal{F}\}$, and hence the space is metrizable. Does there exist an equally simple characterization of spaces with a preopen, σ-closure-preserving and σ-point-finite network? Is every space with such a network metrizable? Does every Nagata space have such a network? In the following, we show that the last two questions have negative answers.

Example 1. A Nagata space which does not have a point-countable preopen network.

Let X be the upper half plane $\{(x,y) : x, y \in \mathbf{R} \text{ and } y \geq 0\}$ with the topology in which each (x,y) with $y > 0$ is isolated and, for a point $(x,0)$, the sets

$$W(x,n) = \{(z,y) : 0 < |z-x| < \frac{1}{n} \text{ and } 0 \leq y < \frac{1}{n}\} \cup \{(x,0)\}\ ,$$

for $n = 1, 2, 3, \ldots$, form a neighborhood base. This is a well-known example of a Nagata space (compare with Example 9.1 of [Ce]).

Let $\mathcal{F}$ be a preopen network of X. We show that $\mathcal{F}$ is not point-countable. For each $(x,0) \in X$, let $F_x \in \mathcal{F}$ be such that $(x,0) \in F_x \subset W(x,1)$. Note that, for every $x \in R$, since the set F_x is preopen and all the points $(z,y) \in X$ with $y \neq 0$ are isolated, there exists $k_x \in \mathbf{N}$ such that

$$W(x,k_x) \setminus \{(z,0) : z \in \mathbf{R}\} \subset F_x.$$

Hence we have that $W(x,k_x) \setminus \{(z,0) : z \in \mathbf{R}\} \subset F_x \subset W(x,1)$ for every $x \in \mathbf{R}$, and it follows that the family $\{F_x : x \in \mathbf{R}\}$ contains uncountably many distinct sets and also that each set F_x intersects the set

$$A = \{(x,y) \in X : x \text{ and } y \text{ are rational and } y > 0\}.$$

Since the set A is countable, the uncountable subfamily $\{F_x : x \in \mathbf{R}\}$ of $\mathcal{F}$ is not point-countable on A. □

The following Proposition shows that a space may fail to be metrizable even if it has a network which is both σ-closure-preserving and σ-disjoint.

Proposition 2. *Every Nagata space can be embedded in a space which has a σ-closure-preserving, σ-disjoint preopen network.*

Proof. Let X be a Nagata space. Denote by X_d the discrete space on X, and let Y be the metrizable space X_d^ω. Fix a point $\bar{y} \in Y$, and set $Y' = Y \setminus \{\bar{y}\}$. For every $x \in X$, let

$$D_x = \{(z_n) \in Y' : \text{ there exists } m \in \mathbf{N} \text{ such that } z_n = x \text{ for every } n \geq m\}\,.$$

Note that $\{D_x : x \in X\}$ is a disjoint family of dense subsets of Y. We topologize the set $Z = X \times Y$ by requiring that each point in $X \times Y'$ has the same neighbourhoods as in the Cartesian product $X_d \times Y$ and each point in $X \times \{\bar{y}\}$ has the the same neighbourhoods as in the Cartesian product $X \times Y$. Note that the map $x \mapsto (x, \bar{y})$ embeds X in Z.

Since X is a Nagata space, we can give points $x \in X$ open neighbourhood bases $\{B(x,n) : n \in \mathbf{N}\}$ in such a way that, for each $n \in \mathbf{N}$, the family $\{B(x,n) : x \in X\}$ is closure-preserving. Let $\{V_k : k \in \mathbf{N}\}$ be an open neighbourhood base of $\bar{y}$ in Y. For all $n, k \in \mathbf{N}$, let $\mathcal{F}_{n,k} = \{(B(x,n) \times (V_k \cap D_x)\,) \cup \{(x,\bar{y})\} : x \in X\}$, and note that $\mathcal{F}_{n,k}$ is a closure-preserving and disjoint family of preopen subsets of Z. Let $\mathcal{B}$ be a σ-discrete base for the (metrizable and open) subspace $X \times Y'$ os Z. Then the family $\mathcal{B} \cup \bigcup\{\mathcal{F}_{n,k} : n, k \in \mathbf{N}\}$ is a network of Z with all the required properties. □

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