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FORCING COUNTABLY COMPACT GROUP TOPOLOGIES ON A LARGER FREE ABELIAN GROUP

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Abstract

We present a method to obtain countably compact group topologies that make a group without non-trivial convergent sequences from the weakest form of Martin's Axiom, improving constructions due to van Douwen and Tkachenko. We force the existence of such topology on the free Abelian group of size $2^{\mathfrak{c}}$, providing a partial answer to a question of Dikranjan and Shakmatov.

1. Introduction

The classification of the groups which may carry a compact group topology was set by Hewitt in 1944 and solved by the efforts of Kaplansky, Harison and Hulanicki in the next decade. The counterpart of Halmos' problem for pseudocompact groups was faced by many authors among them Comfort, van Mill, Remus, Dikranjan and Shakmatov.

Recently, Dikranjan and Tkachenko obtained the classification of countable compactness for groups of size at most continuum using Martin's Axiom. To carry out this classification,

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the technique Tkačenko used in [12] was essential. The problem one encounters to classify groups of larger cardinality is that Tkachenko's countably compact group topology was only possible on the free Abelian group of size continuum.

This arouses greater interest on the following question due to Dikranjan and Shakmatov [7]:

For what cardinals κ, is there a countably compact group topology on the free Abelian group of size κ?

It is well known that compact groups must contain non-trivial convergent sequences. In particular, every topological group which is ω-bounded (countable subsets are compact) contains non-trivial convergent sequences. Sirota [11] gave examples of pseudocompact groups (in ZFC) which do not have non-trivial convergent sequences. More results concerning such topological groups can be found in [10].

A countably compact group without non-trivial convergent sequences was first obtained by Hájnal and Juhász [8] under CH. Few years later, van Douwen [5] obtained one from Martin's Axiom (MA). Those groups were used by van Douwen to construct under MA two countably compact groups whose product is not countably compact, answering in the negative a question from [4]. In that paper, van Douwen asked for ZFC examples of (1) *two countably compact groups whose square is not countably compact* and (2) *a countably compact group without non-trivial convergent sequences.*

Hart and van Mill [9] manage to obtain an example for (1) from $MA_{countable}$. However, their example did not touch question (2), that is, MA remained the weakest assumption to obtain a countably compact group without non-trivial convergent sequences.

Tkachenko [2] asked for a ZFC *group topology on the free Abelian group of size $\mathfrak{c}$ that makes it countably compact*, after obtaining such group topology under CH [12]. The second author showed that such group topology can be obtained from MA(*σ-centered*) [15].

In this note, we construct two examples of group topologies on free Abelian groups. We first construct from $MA_{countable}$ a group topology on the free Abelian group of size $\mathfrak{c}$ that makes it countably compact and without non-trivial convergent sequences. We then force a group topology on the free Abelian group of size $2^{\mathfrak{c}}$ that makes it countably compact and such that all its infinite subsets have $2^{\mathfrak{c}}$ accumulation points.

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2. An Example from $MA_{countable}$

The example is constructed, as in [8], [12], [5], [9] or [15], by induction on $\mathfrak{c}$ and at each stage a function is defined to obtain the new coordinates for the elements that will be in the group.

In van Douwen's example [5], the domain of the function constructed by MA increased throughout the induction to $\mathfrak{c}$. To work with just $MA_{countable}$, Hart and van Mill [9] use an ω-bounded group that takes care of many of the accumulation points, making it possible to worry only about the accumulation points of a fixed countable group. The use of an ω-bounded group makes the group have many non-trivial convergent sequences. The idea to overcome this came from elementary submodels. We shall deal with countable many elements using $MA_{countable}$ but those points change at each stage.

Example 1. ($MA_{countable}$) There exists a group topology on the free Abelian group of size $\mathfrak{c}$ that makes it countably compact and without non-trivial convergent sequences.

Denote by $\mathbb{T}$ the unitary circle group with additive notation. We will construct a family $X = \{x_\alpha : \alpha < \mathfrak{c}\} \subseteq {}^{\mathfrak{c}}\mathbb{T}$ which will be free and the subgroup of ${}^{\mathfrak{c}}\mathbb{T}$ generated by X with the subspace topology will be countably compact and without non-trivial convergent sequences.

We define below two enumerations that will be needed for bookkeeping.

Definition 2. Let $\{h_\xi : \omega \leq \xi < \mathfrak{c}\}$ be an enumeration of all functions h such that $dom\, h = \omega$ and h is one to one; for each $n \in \omega$ $h(n)$ is a function whose domain is a finite non-empty subset of $\mathfrak{c}$ and its image is a subset of $\mathbb{Z} \setminus \{0\}$. Furthermore, for each $\xi \in [\omega, \mathfrak{c})$, $\bigcup_{n\in\omega} dom\, h_\xi(n)$ is a subset of ξ.

The sequence associated to h_ξ will be the sequence

$$\{ \sum_{\mu \in dom\, h_\xi(n)} h_\xi(n)(\mu) x_\mu : n \in \omega\}.$$

Thus, the h_ξ's will bookkeep all injective sequences in the group generated by X which do not contain 0.

Definition 3. Let $\{j_\xi : \omega \leq \xi < \mathfrak{c}\}$ be an enumeration of all functions j such that the domain of j is a non-empty finite subset of $\mathfrak{c}$ and its image is a subset of $\mathbb{Z} \setminus \{0\}$. Furthermore, for each $\xi \in [\omega, \mathfrak{c})$, $dom\, j_\xi \subseteq \xi$.

Note that X will be free if $\sum_{\mu \in dom\, j_\xi} j_\xi(\mu) x_\mu \neq 0 \in {}^{\mathfrak{c}}\mathbb{T}$ for each $\xi \in [\omega, \mathfrak{c})$. For this, it suffices that $\sum_{\mu \in dom\, j_\xi} j_\xi(\mu) x_\mu(\xi) \neq 0 \in \mathbb{T}$.

We are ready to detail the construction.

An infinite subset of any countably compact space without non-trivial convergent sequences has size at least $\mathfrak{c}$. Thus, without loss of generality we can assign different accumulation points to different sequences. We will then promise that x_ξ will be an accumulation point of the sequence associated to h_ξ. If we show that X is free, we show, in particular, that $\{x_\xi : \xi < \mathfrak{c}\}$ is faithfully indexed, thus, we are promising $\mathfrak{c}$ many accumulation points for each non-trivial convergent sequence. Thus, at each inductive stage we have to witness that the combination associated to j_ξ is non-zero and keep the promises made by the accumulation points.

The idea is to separate the accumulation points in two types: countably many that $dom\, j_\xi$ depend on to be defined and the others.

Definition 4. By induction, define $S(n) = \omega$ and let $S(\xi) = \{\xi\} \cup \bigcup\{S(\mu) : \mu \in \bigcup_{n\in\omega} dom\, h_\xi(n)\}$. Given $F \subseteq \mathfrak{c}$, let $S(F) = \bigcup_{\mu\in F} S(\mu)$

The sequences and accumulation points that affect $dom\, j_\xi$ are contained in $S(dom\, j_\xi)$.

Lemma 5. *$S(\xi)$ is countable for any $\xi < \mathfrak{c}$ and if $\mu \in S(\xi)$ then $\bigcup_{n\in\omega} dom\, h_\xi(n) \subseteq S(\xi)$.*

The induction.

At stage ω, we define $x_n \restriction \omega \in {}^{\omega}\mathbb{T}$ arbitrarily. Suppose that at stage $\alpha < \beta$, the following inductive conditions are satisfied:

i) $x_\xi \restriction \alpha$ is defined for each $\xi < \alpha < \beta$;

ii) $\sum_{\mu\in dom\, j_\xi} j_\xi(\mu)x_\mu(\xi) \neq 0$ for each $\xi \in [\omega, \alpha)$;

iii) $x_\xi \restriction \alpha$ is an accumulation point of

$$\{\sum_{\mu\in dom\, h_\xi(n)} h_\xi(n)(\mu)x_\mu \restriction \alpha : n \in \omega\} \text{ for each } \xi \in [\omega, \alpha).$$

Let us show that the induction holds for β.

If β is a limit ordinal, define $x_\xi \restriction \beta = \bigcup_{\xi<\alpha<\beta} x_\xi \restriction \alpha$. Clearly in this case, the inductive conditions hold.

If $\beta = \alpha + 1$, first define $x_\alpha \restriction \alpha$ as any accumulation point of the sequence $\{\sum_{\mu\in dom\, h_\alpha(n)} h_\alpha(n)(\mu)x_\mu \restriction \alpha : n \in \omega\}$.

We will now construct, using $MA_{countable}$, a function $\phi : S(dom\, j_\alpha) \longrightarrow \mathbb{T}$ which is going to be used to define $x_\mu(\alpha)$ for each $\mu \in S(dom\, j_\alpha)$.

Definition 6. Let $\mathcal{B}$ a countable base of $\mathbb{T}$ consisting of non-empty open subarcs and for convenience, containing $\mathbb{T}$. An element of $\mathbb{Q}$ will be a function f where $dom\, f \in [S(dom\, j_\alpha)]^{<\omega}$ and $ran\, f \subseteq \mathcal{B}$, $dom\, j_\alpha \subseteq dom\, f$ and $0 \notin \overline{\sum_{\mu\in dom\, j_\alpha} j_\alpha(\mu)f(\mu)}$.

Given $f, g \in \mathbb{Q}$, $f \leq g$ if $dom\, f \supseteq dom\, g$ and for each $\mu \in dom\, g$ either $f(\mu) = g(\mu)$ or $\overline{f(\mu)} \subseteq g(\mu)$.

Clearly $\mathbb{Q}$ is a countable partial order. We need to define some sets before we define the dense subsets of the partial order.

Definition 7. For each $F \subseteq \alpha$ finite, $n \in \omega$ and $\nu \in S(dom\, j_\alpha)$, let $E(\nu, F, n) =$

$$\{m \in \omega : \forall \theta \in F \, || \sum_{\mu \in dom\, h_\nu(m)} h_\nu(m)(\mu) x_\mu(\theta) - x_\nu(\theta)|| < \frac{1}{n+1}\},$$

where $|| \cdot ||$ is the usual norm in $\mathbb{R}^2$ restricted to $\mathbb{T}$.

Lemma 8. *For each $F \in [\alpha]^{<\omega}$, $n, k \in \omega$ and $\nu \in S(dom\, j_\alpha)$ the set $\{f \in \mathbb{Q} : \exists m \in E(\nu, F, n) \setminus k \wedge dom\, h_\nu(m) \subseteq dom\, f \wedge \sum_{\mu \in dom\, h_\nu(m)} h_\nu(m)(\mu) f(\mu) \subseteq f(\nu) \wedge diam(f(\nu)) < \frac{1}{n+1}\}$ is dense in $\mathbb{Q}$.*

Proof. Follows from the lemma below. ☐

Lemma 9. *Let I be a set of indexes and h be a one to one function of domain ω such that $dom\, h(n) \in [I]^{<\omega} \setminus \{\emptyset\}$ and $ran\, h(n) \subseteq \mathbb{Z} \setminus \{0\}$. Let U be an open subset of $\mathbb{T}$, $A \in [\omega]^\omega$ and $f : I \longrightarrow \mathcal{B}$ where $\{i \in I : \underline{f(i)} \neq \mathbb{T}\}$ is finite. Then there exists $m \in A$, $g : I \longrightarrow \mathcal{B}$ with $\overline{g(i)} \subseteq h(i)$ and $\{i \in I : g(i) \neq \mathbb{T}\}$ finite such that $\sum_{i \in dom\, h(m)} h(m)(i) g(i) \subseteq U$.*

Proof. See [12] or [15]. ☐

Applying the lemma above, let $\mathbb{G}$ be a generic subset of $\mathbb{Q}$. Define $\phi(\nu) \in \bigcap_{f \in \mathbb{G} \wedge \nu \in dom\, f} \overline{f(\nu)}$. Define $x_\nu(\alpha) = \phi(\nu)$ for each $\nu \in S(dom\, j_\alpha)$. Clearly condition $ii)$ is satisfied and conditions $i)$ and $iii)$ are satisfied for $\nu \in S(dom\, j_\alpha)$. Now, we define $\phi \restriction \beta \setminus S(dom\, j_\alpha)$ by induction. Suppose that $\xi \in \beta \setminus S(dom\, j_\alpha)$, $\phi \restriction \xi$ is already defined, $S(dom\, j_\alpha) \cup \xi$ satisfy the inductive hypothesis and $\phi(\xi)$ is not defined yet. By hypothesis, $x_\xi \restriction \alpha$ is an accumulation point of $\{\sum_{\mu \in dom\, h_\xi(n)} h_\xi(n)(\mu) x_\mu \restriction \alpha : n \in \omega\}$

and $\phi(\mu)$ is already defined for each $\mu \in \bigcup_{n\in\omega} h_\xi(n)$. Since $\mathbb{T}$ is compact, there exists $a \in \mathbb{T}$ such that $(x_\xi \restriction \alpha, a)$ is an accumulation point of the sequence $\{(\sum_{\mu\in dom\, h_\xi(n)} h_\xi(n)(\mu)x_\mu \restriction \alpha, \sum_{\mu\in dom\, h_\xi(n)} h_\xi(n)(\mu)\phi(\mu)) : n \in \omega\}$. Then $x_\xi(\alpha) = \phi(\xi) = a$ satisfies the inductive hypothesis for ξ at stage β.

Now, let $x_\mu = \bigcup_{\mu<\eta<\mathfrak{c}} x_\mu \restriction \eta$ for each $\mu < \mathfrak{c}$. Clearly the group generated by $X = \{x_\xi : \xi < \mathfrak{c}\}$ is as required in Example 1.

Example 10. ($MA_{countable}$) There exists a countably compact group topology on the free Abelian group of size $\mathfrak{c}$ that makes its square not countably compact.

Proof. To make the square of the group not countably compact, it suffices to work with a sequence of pairs. So the method presented here can be used to modify the example from [14] that used Martin's Axiom.

3. Forcing

Throughout this section, we will assume that κ is an uncountable regular cardinal with $\kappa^\omega = \kappa$.

The following enumeration will be used to enumerate all one to one sequences in the group we are constructing.

Definition 11. Let $\{h_\xi : \omega \leq \xi < \kappa\}$ be an enumeration of all functions h such that $dom\, h = \omega$, h is one to one; for each $n \in \omega$ $h(n)$ is a function whose domain is a finite non-empty subset of κ and its image is a subset of $\mathbb{Z} \setminus \{0\}$. Furthermore, for each $\xi \in [\omega, \kappa)$, $\bigcup_{n\in\omega} dom\, h_\xi(n)$ is a subset of ξ.

Define $S(\xi)$ by induction as in Definition 4.

The partial order. An element of $\mathbb{P}$ will be of the form $p = (\alpha_p, D_p, f_p)$ where $\alpha_p \in \omega_1$; $D_p \in [\kappa]^\omega$ and $f_p : D_p \longrightarrow {}^{\alpha_p}\mathbb{T}$ satisfy

(1) $D_p \supseteq \bigcup_{\xi\in D_p} S(\xi)$;

(2) $f_p(\xi)$ is an accumulation point of the sequence

$$\{ \sum_{\mu \in dom\ h_\xi(n)} h_\xi(n)(\mu) f_p(\mu) : n \in \omega\}$$

for every $\xi \in D_p \setminus \omega$;

(3) $\{f_p(\xi) : \xi \in D_p\}$ is free; that is, $\sum_{\xi \in dom\, j} j(\xi) f_p(\xi) \neq 0 \in {}^{\alpha_p}\mathbb{T}$, for each function j whose domain is a finite subset of D_p and its range is $\mathbb{Z} \setminus \{0\}$.

Given $p, q \in \mathbb{P}$, we say that $p \leq q$ if and only if $\alpha_p \geq \alpha_q$, $D_p \supseteq D_q$ and $f_p(\xi) \restriction \alpha_q = f_q(\xi)$ for each $\xi \in D_q$.

We will use the following facts to construct our example:

Lemma 12. *Let $\mathbb{P}$ be as above. Then the following are satisfied:*

Fact 0. $\mathbb{P}$ *is a p. o.*

Fact 1. $\mathbb{P}$ *is countably closed.*

Fact 2. *(CH)* $\mathbb{P}$ *is ω_2-cc.*

Fact 3. $\mathcal{D}_\xi := \{p \in \mathbb{P} : \xi \in D_p\}$ *is dense open in* $\mathbb{P}$, $\forall \xi < \kappa$.

Fact 4. $\mathcal{H}_\alpha = \{p \in \mathbb{P} : \alpha_p \geq \alpha\}$ *is dense open in* $\mathbb{P}$, $\forall \alpha < \omega_1$

Theorem 13. *Let V be a model of $GCH + \kappa > \omega_1$, κ regular. Let $\mathbb{P}$ be the partial ordering above defined and let $\mathbb{G}$ be a generic filter over $\mathbb{P}$. Then in $V[\mathbb{G}]$, there exists a group topology on the free Abelian group of size $\kappa = (2^{\mathfrak{c}})^{V[\mathbb{G}]}$ which makes it countably compact and without non-trivial convergent sequences.*

Proof of Theorem 13.

From Facts 1 and 2, $(\omega_1 < \kappa = 2^{\omega_1})^{V[\mathbb{G}]}$ and the countable subsets of V are the same in V and $V[\mathbb{G}]$.

For each $\alpha < 2^{\mathfrak{c}}$, let $x_\alpha = \bigcup\{f_p(\alpha) : p \in \mathbb{G} \wedge \alpha \in dom\, f_p\}$.

From Facts 3 and 4, x_α is a function in $2^{\mathfrak{c}}$ for each $\alpha < 2^{\mathfrak{c}}$.

We claim that $X = \{x_\alpha : \alpha < 2^{\mathfrak{c}}\}$ is free. Indeed, given a function $j : F \longrightarrow \mathbb{Z} \setminus \{0\}$, where F is a finite subset of $2^{\mathfrak{c}}$, there exists $p \in \mathbb{G}$ such that $F \subseteq dom\, p$. From condition (3) in the ordering, there exists $\beta < \alpha_p$ such that $\sum_{\xi \in dom\, j} j(\xi) x_\xi(\beta) = \sum_{\xi \in dom\, j} j(\xi) f_p(\xi)(\beta) \neq 0 \in \mathbb{T}$.

It suffices now to show that the group G generated by X is countably compact. Let $\{y_n : n \in \omega\}$ be a sequence in G. Without loss of generality, we may assume that $y_n \neq 0$ for any $n \in \omega$ and that $y_n \neq y_m$ whenever $n \neq m$. Then, there exists a one to one function $h : \omega \longrightarrow [\mathbb{Z}\setminus\{0\}]^{<\omega}$ such that $y_n = \sum_{\mu \in dom\, h(n)} (\mu) x_\mu$ for each $n \in \omega$. As no new countable subsets of V are added, h is an element of V, thus, there exists $\beta \in [\omega, 2^{\mathfrak{c}})$ such that $h_\beta = h$. Now, given $\alpha < \mathfrak{c}$, there exists $p \in \mathbb{G}$ such that $\beta \in D_p$ and $\alpha_p > \alpha$. Therefore, $x_\beta \restriction \alpha = f_p(\beta) \restriction \alpha$ is an accumulation point of $\{y_n \restriction \alpha : n \in \omega\} = \{\sum_{\mu \in dom\ h_\beta(n)} h_\beta(n)(\mu) f_p(\mu) \restriction \alpha : n \in \omega\}$. Since $\mathfrak{c}$ is a limit ordinal, it follows that x_β is an accumulation point of $\{y_n : n \in \omega\}$. $\square$

The following lemma will be needed to prove Lemma 12.

Lemma 14. *Let $D \in [\kappa]^\omega$, $\beta < \omega_1$ and $f : D \longrightarrow {}^\beta\mathbb{T}$ be such that $f(\xi)$ is the limit of a sequence $\{\sum_{\mu \in dom\ h_\xi(n)} h_\xi(n)(\mu) f(\mu) : n \in A_\xi\}$ with $A_\xi \in [\omega]^\omega$ for every $\xi \in D \setminus \omega$. If F is a non-empty finite subset of $D$and $j : F \longrightarrow \mathbb{Z} \setminus \{0\}$, then there exists a function $\phi : D \longrightarrow \mathbb{T}$ such that $\phi(\xi)$ is a limit of a subsequence of $\{\sum_{\mu \in dom\ h_\xi(n)} h_\xi(n)(\mu)\phi(\mu) : n \in A_\xi\}$ for each $\xi \in D \setminus \omega$ and $\sum_{\mu \in F} j(\mu)\phi(\mu) \neq 0 \in \mathbb{T}$.*

Proof. The space ${}^\beta\mathbb{T}$ has countable weight, so it suffices to apply Lemma 8 which in this case holds in ZFC. $\square$

Proof of Lemma 12

Fact 0 is trivial. In order to prove prove fact 1, let $\{p_n = (\alpha_n, D_n, f_n) : n \in \omega\}$ be a decreasing chain in $\mathbb{P}$. Let $\alpha = sup_{n \in \omega} \alpha_n$, $D = \bigcup_{n \in \omega} D_n$ and $f(\beta) = \bigcup_{n \in \omega \wedge \beta \in D_n} f_n(\beta)$. Then $p = (\alpha, D, f) \in \mathbb{P}$ and $p \leq p_n$ for each $n \in \omega$.

For fact 2, let $\{p_\xi : \xi < \omega_2\}$ be a subset of $\mathbb{P}$. Applying CH and the Δ-system lemma on $\{D_\xi : \xi < \omega_2\}$, there exists $E \in [\kappa]^{\leq\omega}$ and $I \subseteq \omega_2$ of size ω_2 such that $D_\xi \cap D_\mu = E$ whenever $\xi, \mu \in I$ are distinct. Since $\alpha_\xi < \mathfrak{c} < \omega_2$, we may also assume that there exists $\alpha < \mathfrak{c}$ such that $\alpha_\xi = \alpha$ for each $\xi \in I$. Furthermore, $| \ {}^E({}^\alpha\mathbb{T})| \leq \mathfrak{c} = \omega_1 < \omega_2$, we can also assume that there exists f such that $f_\xi \restriction E = f$ for each $\xi \in I$. We claim that if $\eta, \nu \in I$ then p_η and p_ν are compatible.

Indeed, let $f = f_\eta \cup f_\nu$, $D = D_\eta \cup D_\nu$ and $\alpha = \alpha_\eta = \alpha_\nu$. Note that conditions (1) and (2) in the definition of the partial order $\mathbb{P}$ are satisfied for D and f. Let $\{j_n : n \in \omega\}$ be an enumeration of all functions whose domain is a finite subset of D and the range is $\mathbb{Z} \setminus \{0\}$. Since α is countable, there exists $A^0_\xi \in [\omega]^\omega$ for each $\xi \in D \setminus \omega$ such that $f(\xi)$ is the limit of a sequence $\{\sum_{\mu \in dom\ h_\xi(n)} h_\xi(n)(\mu) f(\mu) : n \in A^0_\xi\}$.

Applying Lemma 14, we can define by induction $\phi_n : D \longrightarrow \mathbb{T}$ and a $\subseteq$-decreasing chain $\{A^n_\xi : n \in \omega\}$ for each $\xi \in D$ such that $\phi_n(\xi)$ is the limit of the sequence

$$\{ \sum_{\mu \in dom\ h_\xi(n)} h_\xi(n)(\mu)\phi_n(\mu) : n \in A^n_\xi\}$$

and $\sum_{\mu \in F} j_n(\mu)\phi_n(\mu) \neq 0 \in \mathbb{T}$. Define $D_q = D$, $\alpha_q = \alpha + \omega$ and let $f_q : D \longrightarrow {}^{\alpha_q}\mathbb{T}$ such that $f_q(\xi) \restriction \alpha = f(\xi)$ and $f_q(\xi)(\alpha+n) = \phi_n(\xi)$ for each $\xi \in D \setminus \omega$ and $n \in \omega$. Clearly $q = \langle \alpha_q, D_q, f_q \rangle \in \mathbb{P}$ and q extends both p_η and p_ν.

For fact 3, let p an element of $\mathbb{P}$ and $\xi \in \kappa$ arbitrary. If $\xi \in D_p$ we are done. So, we suppose that $\xi \notin D_q$. We can find a countable subset D of κ such that $D_p \cup \{\xi\} \subseteq D$ and $D \supseteq \bigcup_{\xi \in D} S(\xi)$. We define $f : D \longrightarrow^{\alpha_p} \mathbb{T}$ by induction as follows: set $f \restriction D_p = f_p$ and let $\xi \in D \setminus D_p$ be the least for which $f(\xi)$ has not been defined. Then the sequence $\{\sum_{\mu \in dom\ h_\xi(n)} h_\xi(n)(\mu) f(\mu) : n \in \omega\}$ is already defined. Thus, let $f(\xi) \in {}^{\alpha_p}\mathbb{T}$ be an accumulation point for this sequence. Now, D and f satisfy conditions (1) and (2) from the definition of the partial order. Applying the same argument from fact 2, we can obtain $q \in \mathbb{P}$ extending p with $D_q = D \ni \xi$ and $\alpha_q = \alpha_p + \omega$.

For fact 4, given $p \in \mathbb{P}$ with $\alpha_p < \alpha$ let f whose domain is D_p and for each $\beta \in D_p$, let $f(\beta) \restriction \alpha_p = f_p(\beta)$ and $f \restriction [\alpha_p, \alpha) = 0$. $\square$

Note: By standard closing-off arguments, it is possible to obtain countably compact group topologies on each infinite free Abelian group of size $\lambda = \lambda^\omega \leq 2^{\mathfrak{c}}$.

Recently the method of forcing was used to construct a countably compact topological group of size $\aleph_\omega$ [16], answering in the negative a question of van Douwen [6]. In that paper, van Douwen showed that under the Generalized Continuum Hypothesis, the size of a countably compact topological group cannot have countable cofinality.

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