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TOPOLOGICALLY CONVEX SPACES

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Abstract

A notion of topologically convex spaces is introduced. For this class of spaces it is extended the Schauder-Tychonoff Theorem stating that each continuous compact selfmap of a convex subset of a locally convex topological vector space has a fixed point. Some questions related to the Schauder's conjecture are raised.

Most of numerous papers devoted to extensions of the Schauder-Tychonoff Theorem apply approximative methods. In this paper we present a combinatorial-topological method of proving Schauder-Tychonoff type fixed point theorems which enables us to replace algebraic structures by some topological conditions. The main tool of this paper is some kind of the Sperner Lemma which is called here to be Lemma on Indexed Covering. Other applications of this Lemma are given in [5,6].

Some notations which appears in this paper are taken from [3] and [8].

We shall use notation $[p_0, \dots, p_n]$ for n-dimensional geometric simplex spanned by vertices p_i, where the points $p_0, \dots, p_n$ are affinely independent. Each point $x \in [p_0, \dots, p_n]$, $x = \sum t_i \cdot p_i$, $\sum t_i = 1$, $t_i \geq 0$, is uniquely determined by its barycentric coordinates t_i. A k-dimensional simplex spanned by any $k+1$ of the vertices p_i of a simplex $S = [p_0, \dots, p_n]$ is called a *k-face* of S. The union of all k-faces of the simplex S is called the *k-skeleton*

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of S and the $(n-1)$-skeleton of an n-dimensional simplex S is said to be its geometric *boundary* ∂S;

$$\partial S := \bigcup_{i=0}^{n} [p_0, \ldots, \hat{p}_i, \ldots, p_n], \quad \text{where} \quad S = [p_0, \ldots, p_n]$$

Any continuous map $\sigma : [p_0, \ldots, p_n] \longrightarrow X$ into topological space X is said to be a *singular simplex* contained in X. The following lemma can be obtained from the Brouwer fixed point theorem (cf. [1,4]).

Lemma on Indexed Covering. *Let* $\{U_0, \ldots, U_n\}$ *be an open covering of a topological space* X *and* $\sigma : [p_0, \ldots, p_n] \longrightarrow X$ *a singular simplex. Then there exists a sequence* $0 \leq i_0 < \ldots < i_k \leq n$ *of indices such that* $\sigma[p_{i_0}, \ldots, p_{i_k}] \cap U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$

Proof. Let us put $S := [p_0, \ldots, p_n]$ and $A_i := \sigma^{-1}(U_i)$ for $i = 0, \ldots, n$. The sets A_i are open in S. Define a continuous map $f : S \to S$;

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} \frac{d_i(x)}{d(x)} \cdot p_i, \\ \text{where } d_i(x) :&= \inf\{||x-y|| : y \in S \setminus A_i\}, \\ d(x) &= \sum_{i=0}^{n} d_i(x) \end{aligned}$$

Since the sets A_i form an open covering of the simplex S, we infer that $d(x) > 0$ for each point $x \in S$. According to the Brouwer Fixed Point Theorem there exists a point $a \in S$ such that $f(a) = a$. This means that

$$d_i(a) = t_i(a) \cdot d(a) \quad \text{for each} \quad i = 0, \ldots, n$$

Since the sets A_i are open and $d(a) > 0$ we infer that

$$t_i(a) > 0 \quad \text{if and only if } a \in A_i \quad \text{for each} \quad i = 0, \ldots, n.$$

Now, let us put $\{i_0, \ldots, i_k\} = \{i \leq n : t_i(a) > 0\}$. Then, from the above we get

$$a \in [p_{i_0}, \ldots, p_{i_k}] \cap A_{i_0} \cap \ldots \cap A_{i_k}.$$

This completes the proof. □

Definition. A subset Y of a topological space X is said to be *a topologically convex* subspace of X if for each finite covering $\mathcal{W} = \{W_0, \ldots, W_m\}$ of Y, $Y \subset W_0 \cup \ldots \cup W_m$, with open in X sets, there exist a finite relatively open covering $\mathcal{U} = \{U_0, \ldots, U_n\}$ of Y and a singular simplex $\sigma : [p_0, \ldots, p_n] \to X$ with $\sigma(p_i) \in U_i$ such that , for each sequence $0 \leq i_0 < \ldots < i_k \leq n$,

$$U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset \text{ implies } U_{i_0} \cup \ldots \cup U_{i_k} \cup \sigma[p_{i_0}, \ldots, p_{i_k}] \subset W_j \text{ for some } j \leq m.$$

Main Theorem. *If $g : X \to X$ is a continuous map from a Hausdorff space X into itself and $\overline{g(X)}$ is a topologically convex compact subspace of X, then g has a fixed point.*

Proof. Suppose, contrary to our claim, that $g(x) \neq x$ for each $x \in X$. Since X is a Hausdorff space hence for each $x \in X$ there exists an open neighbourhood $W_x \in \mathcal{B}$ of x such that

$$W_x \cap g(W_x) = \emptyset \tag{1}$$

Let us put $Y := \overline{g(X)}$. The set Y is compact and therefore from the family $\{W_x : x \in Y\}$ one can choose a finite subfamily $\mathcal{W} = \{W_0, ..., W_m\}$ such that

$$Y \subset W_0 \cup \ldots \cup W_m. \tag{2}$$

Since Y is assumed to be a topologically convex subspace of X, there exist a relatively open covering $\mathcal{U} = \{U_0, ..., U_n\}$ of Y

and a singular simplex $\sigma : [p_0, \ldots, p_n] \to X$ with $\sigma(p_i) \in U_i$ and there exists a sequence $0 \le i_0 < \ldots < i_k \le n$ such that $U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$ implies

$$U_{i_0} \cup \ldots \cup U_{i_k} \cup \sigma[p_{i_0}, \ldots, p_{i_k}] \subset W_j \text{ for some } j \le m. \tag{3}$$

The family $\{g^{-1}(U_i) : i = 0, ..., n\}$ is an open covering of X and according to the Lemma on Indexed Covering there exist a set $I = \{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$ and a point $w \in X$ such that

$$w \in \sigma[p_{i_0}, ..., p_{i_k}] \cap g^{-1}(U_{i_0}) \cap .... \cap g^{-1}(U_{i_k}) \tag{4}$$

From the above we have $g(w) \in U_{i_0} \cap ... \cap U_{i_k}$. Since $\sigma(p_i) \in U_i$, we infer from (3) that there exists $W_j \in \mathcal{W}$ such that

$$U_{i_0} \cup \ldots \cup U_{i_k} \cup \sigma[p_{i_0}, \ldots, p_{i_k}] \subset W_j. \tag{5}$$

Thus from (5) and (4) we get $w, g(w) \in W_j$, contradicting (1). □

Theorem 1. *Let X be a convex subset of a Hausdorff locally convex vector space. Then each compact subset $Y \subset X$ is a topologically convex subspace of X.*

Proof. Let Y be a compact subset of X. Without loss of generality we may assume that $\mathcal{W} = \{W_0, \ldots, W_m\}$ is a covering of Y with convex and open in X sets. Since Y is compact Hausdorff space, there is a relatively open covering $\mathcal{U} = \{U_0, \ldots, U_n\}$ of Y which is a star-refinement of $\mathcal{W}$ (cf. Engelking [3], p. 377) i.e., for each $y \in Y$ there is $j \le m$ such that

$$st(y, \mathcal{U}) := \bigcup \{U \in \mathcal{U} : y \in U\} \subset W_j$$

Now, fix an arbitrary n-dimensional geometric simplex $[p_0, \ldots, p_n]$ and define a singular simplex $\sigma : [p_0, \ldots, p_n] \to X$ to be an affine map:

$$\sigma(\sum_{i=0}^{n} t_i p_i) = \sum_{i=0}^{n} t_i \sigma(p_i) \text{ with } \sigma(p_i) \in U_i \text{ for each } i \leq m.$$

Assume that $U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$ and choose a point $y \in U_{i_0} \cap \ldots \cap U_{i_k}$. Then according to definition of star-point refinement, there is $j \leq m$ such that

$$U_{i_0} \cup \ldots \cup U_{i_k} \subset st(y, \mathcal{U}) \subset W_j$$

Since $\sigma(p_{i_0}), \ldots, \sigma(p_{i_k}) \in W_j$, W_j is convex, we infer that

$$\sigma[p_{i_0}, \ldots, p_{i_k}] = conv\{\sigma(p_{i_0}), \ldots, \sigma(p_{i_k})\} \subset W_j.$$

Thus we have obtained that

$$U_{i_0} \cup \ldots \cup U_{i_k} \cup \sigma[p_{i_0}, \ldots, p_{i_k}] \subset W_j$$

This completes the proof. □

A map $g : X \to Y$ between Hausdorff spaces is said to be *compact* if $\overline{g(X)}$ is a compact subset of Y.

From the above theorem we obtain the

Schauder-Tychonoff Theorem. *Each continuous compact selfmap of a convex subset of a Hausdorff locally convex vector space has a fixed point.*

Let us recall definition of covering dimension, dim X, of a normal topological space X: dim $X \leq s$ provided that for each open finite covering $\mathcal{W}$ there exists an open finite covering $\mathcal{U}$ of order $\leq s$, ord $\mathcal{U} \leq s$, being a refinement of $\mathcal{W}$ (i.e., for each $U \in \mathcal{U}$ there is $W \in \mathcal{W}$ such that $U \subset W$ and for each $x \in X$; $\text{ord}(\mathcal{U}, x) := |\{U \in \mathcal{U} : x \in U\}| \leq s + 1$).

Theorem 2. *Let X be a convex subset of a Hausdorff vector space E. Then each compact subset $Y \subset X$ of finite dimension is a topologically convex subspace of X.*

Proof. Let $s := \dim Y$ and let $\mathcal{B}(0)$ be a base of neighbourhoods of the point $0 \in E$ satisfying;

$$(1) \qquad t\,V \subset V \text{ for each } t \in [0,1] \text{ and } V \in \mathcal{B}(0).$$

Fix a finite covering $\mathcal{W} = \{W_0, \ldots, W_m\}$ of Y with open in X sets. Using compactness arguments we can find a finite covering $\mathcal{A} = \{A_0, \ldots, A_l\}$ of Y with sets of the form;

$$(2) \qquad A_i = y_i + V_i, \text{ where } y_i \in Y \text{ and } V_i \in \mathcal{B}(0),$$

which is a refinement of the covering $\mathcal{W}$. Continuity of the operation "+" implies that we may assume, in addition, that for each $i \le l$ there is $j \le m$ such that

$$(3) \; y_i + z_0 + \ldots + z_s \in W_j \text{ for each } s\text{-sequence } z_0, \ldots, z_s \in V_i.$$

Since $\dim Y = s$, we can choose a relatively open covering $\mathcal{U} = \{U_0, \ldots, U_n\}$ of Y of order $\le s$ which is a star-refinement of the covering $\mathcal{A}$.

Let $\sigma : [p_0, \ldots, p_n] \to X$ be an affine map;

$$(4) \qquad \sigma\left(\sum_{i=0}^{n} t_i p_i\right) = \sum_{i=0}^{n} t_i \sigma(p_i) \text{ with } \sigma(p_i) \in U_i \text{ for } i \le n.$$

Assume that $U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$ and choose $y \in U_{i_0} \cap \ldots \cap U_{i_k}$. Since $\operatorname{ord} \mathcal{U} \le s$ we infer that $k \le s$. According to our construction there are $i \le l$ and $j \le m$ such that

$$(5) \qquad U_{i_0} \cup \ldots \cup U_{i_k} \subset st(y, \mathcal{U}) \subset A_i \subset W_j.$$

Since $\sigma(p_i) \in U_i$ we get

$$\sigma(p_{i_0}), \dots, \sigma(p_{i_k}) \in A_i. \tag{6}$$

Let us verify that

$$\sigma[p_{i_0}, \dots, p_{i_k}] = conv\{\sigma(p_{i_0}), \dots, \sigma(p_{i_k})\} \subset W_j. \tag{7}$$

Fix $y \in \sigma[p_{i_0}, \dots, p_{i_k}]$. Then $y = \sum_{r=0}^{k} t_r \sigma(p_{i_r})$, $\sum_{r=0}^{k} t_r = 1$, $t_r \geq 0$. According to (6) and (2); $\sigma(p_{i_r}) = y_i + x_r$, where $x_r \in V_i$. Thus

$$y = \sum_{r=0}^{k} t_r(y_i + x_r) = y_i + \sum_{r=0}^{k} t_r x_r.$$

In view of (1), $t_r x_r \in V_i$. Since $k \leq s$, we infer from (3) that $y \in W_j$. Thus we have proved that $U_{i_0} \cup \dots \cup U_{i_k} \cup \sigma[p_{i_0}, \dots, p_{i_k}] \subset W_j$. □

A topological space X is said to be ∞-*connected*, $X \in C^\infty$, if each continuous map $f : \partial S \to X$ from the boundary of an n-dimensional simplex into X, $n = 1, 2, \dots$, has a continuous extension $F : S \to X$, $F|\partial S = f$.

The condition $X \in C^\infty$ is equivalent to the following statement (cf. Spanier [6], Th.1.3.12):

> (a) Each continuous map $f : \partial Q \to X$ from the boundary of a ball $Q \subset R^n$, $n = 1, 2, \dots$, is homotopic to a constant map,
> (b) Each continuous map $f : \partial Q \to X$ from the boundary of a ball Q has a continuous extension over Q.

A space X is said to be *contractible* if the identity map $id_X : X \to X$ is homotopic to a constant map i.e., there is a continuous map $H : X \times [0, 1] \to X$ such that $H(x, 0) = x$ and $H(x, 1) = c$ for each $x \in X$.

Each contractible space is ∞-connected. (cf. Spanier [7], Th. 1.3.13).

Any vector topological Hausdorff space E has a neighbourhood base $\mathcal{B}(0)$ at $0 \in E$ such that

$$t\,V \subset V \text{ for each } t \in [0,1].$$

From the above it follows that any topological vector space E has a base consisting of open ∞-connected (contractible) sets.

Indeed, sets of the form $U = x_0 + V$, where $V \in \mathcal{B}(0)$, $x_0 \in E$ are contractible, because the continuous map $H : U \times [0,1] \to U$, $H(x,t) := x_0 + tx$ is a homotopy between the identity map id_U and the constant map x_0.

Similarly, it is easy to observe that each convex subset of E is a contractible space and moreover it has a base consisting of ∞-connected (contractible) relatively open sets. Unfortunately we do not know if such a base is closed under finite intersections. If E is locally convex then the answer is "yes" because we can assume that the sets $U = x_0 + V$, $V \in \mathcal{B}$ are convex.

An affirmative answer to this question would solve the Schauder problem (Problem 54 in the *Scottish Book* [2]), whether a continuous selfmap of a compact convex subset of any topological vector space has a fixed point.

A topological ∞-connected space X is said to be *perfectly ∞-connected* if it has a base $\mathcal{B}$ which is closed under finite intersections and the base consists of ∞-connected sets i.e.,

(a) $X \in \mathcal{B}$,

(b) $U_1, \ldots, U_n \in \mathcal{B}$ implies $U_1 \cap \ldots \cap U_n \in \mathcal{B}$,

(c) each set $U \in \mathcal{B}$ is ∞-connected.

The following theorem is a generalization of Theorem 1.

Theorem 3. *Each compact subset Y of a Hausdorff perfectly ∞-connected space X is a toplogically convex subspace of X.*

Proof. Let $Y \subset X$ be a compact subset. Without loss of generality we may assume that $\mathcal{W} = \{W_0, \dots, W_m\}$ is a covering of Y, $Y \subset W_0 \cup \dots \cup W_m$, with ∞-connected and open in X sets, and in addition, we may assume the finite family $\mathcal{W}$ is closed under intersections. Choose $\mathcal{U} = \{U_0, ..., U_n\}$ to be a finite covering of Y with relatively open sets U_i and being a star-refinement of $\mathcal{W}$.

Define $\mathcal{W}^* := \{X\} \cup \mathcal{W}$ and fix an arbitrary n-dimensional simplex $S := [p_0, \dots, p_n]$. For each $I \subset \{0, \dots, n\}$ let $W_I \in \mathcal{B}$ be the ∞-connected set:

$$W_I := \bigcap \{W \in \mathcal{W}^* : \bigcup \{U_i : i \in I\} \subset W\}. \tag{1}$$

and denote by S_I the face of the simplex S:

$$S_I := [p_{i_0}, \dots, p_{i_k}], \quad \text{where} \quad I = \{i_0, \dots, i_k\}. \tag{2}$$

We shall describe by induction (on the k-skeleton of S) a continuous map $\sigma : S \to X$ such that

$$\sigma(S_I) \subset W_I \quad \text{ for each } I \subset \{0, \dots, n\}. \tag{3}$$

step 0. Choose points $x_i \in U_i$ for each $i = 0, \dots, n$ and set $\sigma(p_i) := x_i$.

step 1. For each 2-elements set $I = \{i, j\} \subset \{0, \dots, n\}$ choose a continuous map $\sigma : [p_i, p_j] \to W_I$ such that $\sigma(p_i) = x_i$ and $\sigma(p_j) = x_j$ e.i., σ is a continuous extension of the map $\sigma|\partial[p_i, p_j]$. The facts $\sigma(p_i) \in U_i$, $\sigma(p_j) \in U_j$, $U_i \cup U_j \subset W_I$ and W_I is ∞-connected imply that such a choice of σ is possible.

step $(k+1)$, $k < n$. Assume that we have defined a continuous map σ on the k-skeleton of the simplex S. We shall extend continuously the map σ over the $(k+1)$-skeleton of S such that the condition (3), $\sigma(S_I) \subset W_I$, holds and $\sigma|S_I$ is an extension of $\sigma|\partial S_I$ for $|I| = k+1$. According to the inductive assumption;

$$\bigcup\{\sigma(S_J) : J \subset I, |J| = k\} \subset W_I, \text{ where } |I| = k+1$$

and to the assumption that W_I is ∞-connected it is possible to carry out such a construction.

The n-th step completes the construction of the singular simplex σ.

Assume that $U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$ and choose a point $y \in U_{i_0} \cap \ldots \cap U_{i_k}$. Then according to definition of star-refinement there is $j \leq n$ such that $U_{i_0} \cup \ldots \cup U_{i_k} \subset st(y, \mathcal{U}) \subset W_j$, where $I = \{i_0, \ldots, i_k\}$. From (1) and (3) it follows that $W_I \subset W_j$. In consequence we obtain $U_{i_0} \cup \ldots \cup U_{i_k} \cup \sigma[p_{i_0}, \ldots, p_{i_k}] \subset W_j$. $\square$

Corollary. *Each continuous compact map $g : X \to X$ from a Hausdorff perfectly ∞-connected space into itself has a fixed point.*

Remark. Investigating the notion of topologically convex subspace it is possible to establish the fixed-point property of the Schauder-Tychonoff type for each contractible space X which has a triangulation induced by a simplicial complex K (not necessarily finite, cf. Spanier [7], Chapter 3). Namely, one can prove that each compact subset $Y \subset X$ is a topologically convex subspace of X.

The following Corollary is a topological generalization of the Schauder-Tychonoff Theorem

Corollary. *If a contractible Hausdorff space X has a base which is closed under finite intersections and consists of contractible sets, then any continuous compact selfmap of X has a fixed point.*

A positive answer to one of the following problems would solve Schauder's conjecture.

Problem 1. Let $Y \subset X$ be a compact subset of a convex set X contained in a Hausdorff vector space E. Is Y a topologically convex subspace of X ?

Problem 2. Is a convex subset of a topological vector space a perfectly ∞-connected space?

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