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ON THE BOREL CLASS OF MULTIVALUED FUNCTIONS OF TWO VARIABLES

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Abstract

In the paper we give a sufficient conditions causing a multivalued functions defined on the product of metrizable spaces with values in a perfectly normal topological one to be of Baire class 1. Obtained Corollary is a multivalued analog of Kempisty theorem [6].

1. Introduction

Various results were publisched about the Borel classification of multivalued functions depending on the one variable (cf. [1], [2], [3], [7], [8], [9]). Obviously each multivalued function of two variables $x \in X$ and $y \in Y$ may be treated as a multivalued function of a single variable $(x, y) \in X \times Y$. The essential difference is the possibility of formulation of hypotheses concerning multivalued functions in terms of its sectionwise properties. S. Kempisty in his paper [6] has shown that if a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is upper semicontinuous with respect to one of its variable and lower semicontinuous with respect to other one, then f is of Borel class 1. In this paper we generalize this theorem into the case of multivalued functions in possible general abstract spaces.

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2. Preliminaries

Let T and Z be two nonempty sets and let $\Phi : T \to Z$ be a multivalued function, i.e. Φ denotes a mapping such that $\Phi(t)$ is a nonempty subset of Z for $t \in T$. Then two inverse images of subset $G \subset Z$ may be defined:

$\Phi^+(G) = \{\, t \in T : F(t) \subset G\}$ and $\Phi^-(G) = \{\, t \in T : F(t) \cap G \neq \emptyset\}$.

The following relations hold betwen these inverse images:

(1) $\Phi^-(G) = T \setminus \Phi^+(Z \setminus G)$ and $\Phi^+(G) = T \setminus \Phi^-(Z \setminus G)$.

We also define the image of a subset $U \subset T$ under Φ by the formula

$$\Phi(U) = \bigcup_{u \in U} \Phi(u).$$

Let $\mathcal{P}_0(Z)$ denote the family of all nonempty subsets of Z. A multivalued map may be treated as a function $\Phi : T \to \mathcal{P}_0(Z)$. Then

$$\Phi^{-1}(\mathcal{G}) = \{t \in T : \Phi(t) \in \mathcal{G}\} \quad \text{for } \mathcal{G} \subset \mathcal{P}_0(Z).$$

Let $(T, \mathcal{T}(T))$ and $(Z, \mathcal{T}(Z))$ be two topological spaces. A multivalued function $\Phi : T \to Z$ is called $\mathcal{T}(T)$-*upper* (resp. $\mathcal{T}(T)$-*lower*) *semicontinuous* at a point $t \in T$ if

$$\forall G \in \mathcal{T}(Z)(\Phi(t) \subset G \Rightarrow t \in \mathrm{Int}\Phi^+(G))$$

$$(\text{resp. } \forall G \in \mathcal{T}(Z)(\Phi(t) \cap G \neq \emptyset \Rightarrow t \in \mathrm{Int}\Phi^-(G))).$$

F is called $\mathcal{T}(T)$-*continuous* at t if it is simultaneously $\mathcal{T}(T)$-upper and $\mathcal{T}(T)$-lower semicontinuous at t.

A multivalued function Φ being $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) semicontinuous at each point $t \in T$ is said to be $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) semicontinuous.

It is clear that a multivalued function Φ is $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) semicontinuous if and only if $\Phi^+(G) \in \mathcal{T}(T)$ (resp. $\Phi^-(G) \in \mathcal{T}(T)$) whenever $G \in \mathcal{T}(Z)$.

We denote the following families of sets:

$\mathcal{C}(Z) = \{A \in \mathcal{P}_0(Z) : A \text{ is closed}\}$ and

$\mathcal{K}(Z) = \{A \in \mathcal{P}_0(Z) : A \text{ is compact}\}$.

Let $\mathcal{T}_V(Z)$ be the Vietoris topology in the space $\mathcal{C}(Z)$.

Given any countable ordinal number α, let $\sum_\alpha(T)$ and $\Pi_\alpha(T)$ denote the additive and multiplicative class α, respectively, in the Borel hierarchy of subsets of the topological space $(T, \mathcal{T}(T))$.

Definition 1. A multivalued function $F : T \to Z$ belongs to the $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) Borel class α if $F^-(G) \in \sum_\alpha(T)$ (resp. $F^+(G) \in \sum_\alpha(T)$), whenever $G \in \mathcal{T}(Z)$.

A multivalued function $F : T \to Z$ with compact values will be said of Borel class α if

$F^{-1}(\mathcal{G}) \in \sum_\alpha(T)$ for any $\mathcal{G} \in \mathcal{T}_V(Z)$.

Let us remark, that a multivalued function belonging to $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) class 0 is $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) semicontinuous.

We shall use B_α to denote the Borel class α and LB_α and UB_α to denote the $\mathcal{T}(T)$-lower and $\mathcal{T}(T)$-upper Borel classes α, respectively.

The classes LB_α and UB_α have been considered by Kuratowski in [7] and by Garg in [2].

Let $(T, \mathcal{T}(T))$ be a perfect space. It is known (see [2], Theorem 1.1 and Theorem 1.2) that

(2) If $F : T \to \mathcal{K}(Z)$ and Z is second countable, then $F \in B_\alpha$ if and only if $F \in LB_\alpha \cap UB_\alpha$.

(3) If $(Z, \mathcal{T}(Z))$ is a perfect space and $F : T \to Z$ is a multivalued function with closed values, then

(a) If $F \in UB_\alpha$, then $F \in LB_{\alpha+1}$.

(b) If F is compact-valued and $F \in LB_\alpha$, and if Z is further normal, then $F \in UB_{\alpha+1}$.

A sequence $\Phi_n : T \to Z$ of multivalued functions with closed values will be said converging to a multivalued function $\Phi : T \to Z$ if the sequence $(\Phi_n(t))_{n\in\mathbb{N}}$ converges to $\Phi(t)$ in the Vietoris topology on the hyperspace of nonempty, closed subsets of Z (cf. [5]).

The following assertion is known (see [2], Theorem 3.1 (b)).

(4) Suppose $(Z, \mathcal{T}(Z))$ is perfectly normal and let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of closed-valued multivalued functions $\Phi_n : T \to Z$ which converges to a multivalued function $\Phi : T \to Z$. If Φ is compact valued and $\Phi_n \in LB_\alpha$ for $n \in \mathbb{N}$, then $\Phi \in UB_{\alpha+1}$.

Let us remark that

(5) If Z is the real line $\mathbb{R}$, then a multivalued function $\Phi : T \to \mathbb{R}$ defined by formula $\Phi(t) = [f(t), g(t)] \subset \mathbb{R}$ is of $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) Borel class α if and only if g is of $\mathcal{T}(T)$-lower (resp. $\mathcal{T}(T)$-upper) and f is of $\mathcal{T}(T)$-upper (resp. $\mathcal{T}(T)$-lower) class α in the Young classification.

In fact, for $a < b$ we have

$$\Phi^-((a,b)) = \{t \in T : f(t) < b\} \cap \{t \in T : g(t) > a\}$$

and

$$\Phi^+((a,b)) = \{t \in T : f(t) > a\} \cap \{t \in T : g(t) < b\}.$$

3. Main results

Let us recall that if $F : X \times Y \to Z$ is a multivalued function, then multivalued function $F_x : Y \to Z$ defined by $y \to F(x, y)$ is called x-section of F and multivalued function $F^y : X \to Z$ defined by $x \to F(x, y)$ is called its y-section.

Lemma 1. *Let $(X, \mathcal{T}(X))$ be a topological space, (Y, d) a metric space and $(Z, \mathcal{T}(Z))$ a third topological space. Let $F : X \times Y \to Z$ be a multivalued function with $\mathcal{T}(X)$-lower semicontinuous y-sections. Then for each $n \in \mathbb{N}$ the multivalued function*
$F_n : X \times Y \to Z$ defined by formula

$$F_n(u, v) = F(u, B(v, 2^{-n})) = \bigcup_{y \in B(v, 2^{-n})} F(u, y)$$

is $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-lower semicontinuous ($B(v, r)$ denotes an open ball in Y).

Proof. Consider a sequence $(F_n)_{n \in N}$ of multivalued functions $F_n : X \times Y \to Z$ such that

(6) $F_n(x, y) = \bigcup_{v \in B(y, 2^{-n})} F(x, v)$ for $n \in N$.

Let $n \in N$ be fixed. We will show that

(7) F_n is a $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-lower semicontinuous multivalued function.

Let us fix an arbitrary point $(x, y) \in X \times Y$ and an open set $G \subset Z$ such that $F_n(x, y) \cap G \neq \emptyset$. According to (6) there exists a point $v \in B(y, 2^{-n})$ such that $F(x, v) \cap G \neq \emptyset$. Since the v-section of F is $\mathcal{T}(X)$-lower semicontinuous at x, there is an open neighbourhood $U(x)$ of x such that $F(u, v) \cap G \neq \emptyset$ whenever $u \in U(x)$.

Observe that there exists a real number $r > 0$ such that

(8) $F(u, v) \subset F(u, B(y_0, 2^{-n})) = F_n(u, y_0)$ for all $u \in U(x)$ and $y_0 \in B(y, r)$.

Indeed. Let $r = 2^{-n} - d(v, y)$. Then $r > 0$ because $v \in B(y, 2^{-n})$. Thus for each $t \in B(y, r)$ we have:

(9) $d(t, v) \leq d(t, y) + d(y, v) < r + 2^{-n} - r = 2^{-n}$.

Then by (9) the following implication holds :

$t \in B(y,r) \Longrightarrow v \in B(t, 2^{-n})$.

Therefore (8) is true on $U(x) \times B(y,r)$. According to (6) and (8) $F_n(u,v) \cap G \neq \emptyset$ whenever $(u,v) \in U(x) \times B(y,r) = V(x,y)$. The set $V(x,y)$ is an open neighbourhood of the point $(x,y) \in X \times Y$. Therefore (7) has been shown and the proof of Lemma 1 is finished. □

We turn now our attension to the well known result on real functions of two real variables, namely on Borel class 1 of functions with semicontinuous sections (see [6], p. 240). We extend this result into the multivalued case in more general spaces.

Theorem 1. *Let $(X, \mathcal{T}(X))$ be a toplogical space, $(Y, \mathcal{T}(Y))$ a metrizable one and $(Z, \mathcal{T}(Z))$ a perfectly normal topological space. If $F : X \times Y \to Z$ is a $\mathcal{T}(Z)$-compact-valued multivalued function with $\mathcal{T}(X)$-lower semicontinuous y-sections and $\mathcal{T}(Y)$-upper semicontinuous x-sections, then F belongs to the $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-upper Borel class 1.*

Proof. Let (F_n) be the sequence of multivalued functions given in Lemma 1. Observe that $F(x,y) \subset F_n(x,y)$ for any $n \in N$. Therefore $F(x,y) \subset \bigcap_{n \in N} \mathrm{cl}(F_n(x,y))$. We will show that the inverse inclusion is also true.

Let $(x,y) \in X \times Y$ and let us suppose that $z \in Z \setminus F(x,y)$. The set $F(x,y)$ is closed. Therefore there exist an open set $G \subset Z$ and an open neighbourhood $W(z)$ of the point z such that $F(x,y) \subset G$ and $W(z) \cap G = \emptyset$. The x-section of F is $\mathcal{T}(Y)$-upper semicontinuous at the point y. Therefore there exsists a number $m \in N$ such that $F(x,v) \subset G$ for each $v \in B(y, 2^{-m})$. Thus $W(z) \cap F(x,v) = \emptyset$ for each $v \in B(y, 2^{-m})$.

Then $W(z) \cap F_m(x,y) = \emptyset$ and then $W(z) \cap \mathrm{cl}(F_m(x,y)) = \emptyset$. Hence $z \in Z \setminus \bigcap_{n \in N} \mathrm{cl}(F_n(x,y))$ and the inclusion $F(x,y) \supset \bigcap_{n \in N} \mathrm{cl}(F_n(x,y))$ is true.

Therefore we have shown, that

(10) $F(x,y) = \bigcap_{n\in N} \mathrm{cl}(F_n)(x,y)$ whenever $(x,y) \in X \times Y$, where $\mathrm{cl}(F_n)(x,y) = \mathrm{cl}(F_n(x,y))$.

Now consider the sequence $(\mathrm{cl}(F_n))_{n\in N}$. All the multivalued functions F_n are $\mathcal{T}(X)\otimes\mathcal{T}(Y)$-lower semicontinuous by Lemma 1. The multivalued functions $\mathrm{cl}(F_n)$ have $\mathcal{T}(Z)$-closed values and, as it is easy to see, they are also $\mathcal{T}(X)\otimes\mathcal{T}(Y)$-lower semicontinuous.

By (10) the sequence $(\mathrm{cl}(F_n))_{n\in N}$ converges to F because F is compact-valued. Since F has compact values and $\mathrm{cl}(F_n)$ are $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-lower semicontinuous, F belongs to the $\mathcal{T}(X)\otimes\mathcal{T}(Y)$-upper Baire class 1 by (4) and Theorem 1 has been proved. □

The following theorem shows that the topological spaces X and Y in Theorem 1 cannot be both entirely arbitrary.

Theorem 2. *Assume CH. There are topological spaces $(X,\mathcal{T}(X)), (Y,\mathcal{T}(Y))$ and $(Z,\mathcal{T}(Z))$ and a multivalued function $F : X \times Y \to Z$ with $\mathcal{T}(Z)$-compact values whose all y-sections are $\mathcal{T}(X)$-lower semicontinuous and all x-sections are $\mathcal{T}(Y)$-upper semicontinuous, but F belongs neither to any upper nor lower Borel class on $X \times Y$.*

Proof. Let $X = Y = \mathbb{R}$ be the real line with Euclidean topology. Let $\mathcal{I}$ be the ideal of the sets of the first category in $\mathbb{R}$ and let $A \subset \mathbb{R}$. Following Hashimoto (see [4, p. 6]) let $A^* = \{p : \forall U(p), U(p)\cap A \notin \mathcal{I}\}$, where $U(p)$ means a neighbourhood of p. Putting the $*$-closure $\widetilde{A}$ of A by $\widetilde{A} = A\cup A^*$ defines a new topology on $\mathbb{R}$, which is called the $*$-topology. The set $\mathbb{R}$ has now two topologies on it (the notation on $*$-topology is specified by $*$):

(11) A subset $G \subset \mathbb{R}$ is $*$-open if and only if G is the difference of an open set and a set belonging to $\mathcal{I}$ (see [4, Corollary 1]).

Let $Z = \mathbb{R}$ be the real line with usual Euclidean topology. Denote by M the subset of the plane having all y-sections

$M^y = \{x \in X : (x,y) \in M\}$ countable and all x-sections $M_x = \{y \in Y : (x,y) \in M\}$ with countable complements. Such a set M exists by CH (see [12, p. 179]). Let us define $F : X \times Y \to Z$ by putting

$$F(x,y) = \begin{cases} [-2,5], & \text{if } (x,y) \in M, \\ [0,1], & \text{if } (x,y) \in \mathbb{R}^2 \setminus M. \end{cases}$$

Then for any open set $G \subset \mathbb{R}$, $F_x^+(G)$ and $F^{y-}(G)$ are $*$-open (see (11)), thus all the y-sections of F are $*$-lower semicontinuous and all the x-sections of F are $*$-upper semicontinuous. On the other hand $F^-((2,3)) = F^-([2,3]) = M$. It is enough to show that the set M cannot have the Baire property. Suppose on the contrary. Then by Theorem 15.4 (see [9, p. 57]), the set M is of the first category on $\mathbb{R}^2$ because all y-sections of M are countable.Thus by Theorem 15.1 (see [9, p. 56]) all x-sections of M, except of a set of the first category, are first category in $\mathbb{R}$ which gives a contradiction. Thus F is as in the statement.□

Lemma 2. *Let $(T, \mathcal{T}(T))$ be a topological space and let $(Z, \mathcal{T}(Z))$ be an uniformizable one. Let $F : T \to Z$ be a multivalued function. Consider the following statements:*

(i) *F belongs to the $\mathcal{T}(T)$-lower Borel class α;*

(ii) *there exists a family $\{d_i : i \in I\}$ of pseudometrics on Z, generating the topology $\mathcal{T}(Z)$ such that for every $i \in I$ and $z \in Z$ the real function*

(12) *$g_{iz} : T \to \mathbb{R}$ defined by formula $g_{zi}(t) = d_i(z, F(t)) = \inf\{d_i(z,w) : w \in F(t)\}$ is in the $\mathcal{T}(T)$-upper class α in the Young classification;*

(iii) *for each $G \in \mathcal{T}(Z)$ there is a countable index set $I_G \subset I \times Z \times (0,\infty)$ (depending on G) such that G is representable as an union*

(13) $G = \bigcup\{B_i(z,r) : (i,z,r) \in I_G\}$,

where $B_i(z,r) = \{w \in Z : d_i(w,z) < r\}$ *denotes* d_i-*pseudoball centered at* z *with radius* r.

Then the following implications hold:
(i)$\Rightarrow$ (ii) *and* (iii)$\wedge$ (ii) $\Rightarrow$ (i).

Proof. First we will show that (i) implies (ii). Since $(Z, \mathcal{T}(Z))$ is uniformizable, there exists a family $\{d_i : i \in I\}$ of pseudometrics on Z, generating the topology $\mathcal{T}(Z)$. Fix $i \in I$, $z \in Z$, $r > 0$, and observe that

(14) $$F^-(B_i(z,r)) = \{t \in T : F(t) \cap B_i(z,r) \neq \emptyset\} = \\ = \{t \in T : d_i(z, F(t)) < r\} = g_{zi}^{-1}((-\infty, r)).$$

Therefore $F^-(B_i(z,r))$ belongs to the additive Borel class α, if F is in $\mathcal{T}(T)$-lower Borel class α, since $B_i(z,r) \in \mathcal{T}(Z)$. The number r was arbitrary, therefore by (14) the function g_{zi} belongs to the $\mathcal{T}(T)$-upper Young class α.

Now we show that (iii) and (ii) implies (i). Let $\{d_i : i \in I\}$ be a family of pseudometrics as in (ii) and let $G \in \mathcal{T}(Z)$. Since the real functions g_{zi} are in $\mathcal{T}(T)$-upper Young class α, the formula (14) yields

$F^-(B_i(z,r)) = \{t \in T : g_{zi}^{-1}(t) < r\}$ is of additive Borel class α in T for any triplet $(i,z,r) \in I_G$. Therefore by (13) the set $F^-(G) = F^-(\bigcup\{B_i(z,r) : (i,z,r) \in I_G\}) = \bigcup\{F^-(B_i(z,r)) : (i,z,r) \in I_G\}$ is of additive Borel class α, as claimed. □

Lemma 3. *Let* $(Y, \mathcal{T}(Y))$ *and* $(Z, \mathcal{T}(Z))$ *be two topological spaces and let us suppose, that* $F : Y \to Z$ *is a compact-valued multivalued function belonging to the* $\mathcal{T}(Y)$*-upper Borel class* α*. Then for every* $z \in Z$ *end every pseudometric* d *on* Z*, generating a weaker topology than* $\mathcal{T}(Z)$*, the real function* (12) *is of* $\mathcal{T}(Y)$*-lower Young class* α*.*

Proof. Let $z \in Z$ and d be as in the statement. Fix $r > 0$. By hypothesis the set

$\bar{B}(z,r) = \{w \in Z : d(w,z) \le r\}$

is closed in Z, because of $\mathcal{T}(Z) \otimes \mathcal{T}(Z)$-continuity of d. We have

$$g_z^{-1}((r,\infty)) = \{y \in Y : d(z,F(y)) > r\} =$$
$$= Y \setminus \{y \in Y : d(z,F(y)) \le r\} = Y \setminus \{y \in Y : F(y) \cap \bar{B}(z,r) \ne \emptyset\} =$$
$$= Y \setminus F^-(\bar{B}(z,r)) = F^+(Z \setminus \bar{B}(z,r)) \quad \text{(see (1))}.$$

Moreover $F^+(Z \setminus \bar{B}(z,r))$ belongs to the additive Borel class α, since F is in $\mathcal{T}(Y)$-upper Borel class α. This completes the proof of Lemma 3. □

Theorem 3. *Let $(X,\mathcal{T}(X))$ and $(Y,\mathcal{T}(Y))$ be metrizable spaces and $(Z,\mathcal{T}(Z))$ a perfectly normal topological space fulfilling (iii).*

If $F : X \times Y \to Z$ is a compact-valued multivalued function with $\mathcal{T}(X)$-lower semicontinuous y-sections and $\mathcal{T}(Y)$-upper semicontinuous x-sections, then F belongs to the $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-lower Baire class 1.

Proof. The topology $\mathcal{T}(Z)$ being normal, is also completely regular and thus uniformizable. We will show that for every $i \in I$ and $z \in Z$

(15) the real function $g_{zi} : X \times Y \to \mathbb{R}$ defined by formula

$$g_{zi}(x,y) = d_i(z, F(x,y))$$

is $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-upper class 1 in the Young classification.

By the $\mathcal{T}(X)$-lower semicontinuity of F_y and by Lemma 2 (i) $\Rightarrow$ (ii) we conclude that

(16) the function g_{zi} has $\mathcal{T}(X)$-upper semicontinuous y-sections for all $y \in Y$.

By the $\mathcal{T}(Y)$-upper semicontinuity of F_x and Lemma 3 we conclude that

(17) the function g_{zi} has $\mathcal{T}(Y)$-lower semicontinuous x-sections for all $x \in X$.

Now we define a multivalued function $H : X \times Y \to \mathbb{R}$ as follows:

$$H(x,y) = [-\text{arctg}\, g_{zi}(x,y), 5]$$

Multivalued function H has compact values. By (16), (17) and (5) all y-sections of H are $\mathcal{T}(X)$-upper semicontinuous and all x-sections of H are $\mathcal{T}(Y)$-lower semicontinuous. Applying Theorem 1 to the compact valued multivalued function H we know that it belongs to the $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-upper Borel class 1. Using (5) to the multivalued function H we conclude that g_{zi} belongs to the $\mathcal{T}(X) \otimes \mathcal{T}(Y)$-upper Young class 1 and (15) has been shown.

By Lemma 2, (iii) $\wedge$ (ii) $\Rightarrow$ (i), F belongs to the $\mathcal{T}(X)\otimes\mathcal{T}(Y)$-lower Borel class 1 and the proof of Theorem 3 is finished. □

If the x-sections and y-sections of a multivalued functions are both lower or both upper semicontinuous, then its behaviour may by very bad, as the following theorem shows.

Theorem 4. *There exists a multivalued function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with compact values whose all x-sections and y-sections are lower semicontinuous which is neither of any lower Baire class nor of any upper Baire class.*

Proof. Let us decompose the real line onto two disjoint, nonborel subsets A and B such that $\mathbb{R} = A \cup B$. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a multivalued function defined by formula

$$F(x,y) = \begin{cases} [-3,3], & \text{if } x \neq y, \\ [-1,0], & \text{if } x = y \in A, \\ [1,2], & \text{if } x = y \in B. \end{cases}$$

It is easy to check that F is as it was required. □

This example seems to be simpler than the corresponding one from Theorem 2 on p. 88 in [13] (cf. also Example 2 in [11]).

Remark 1. Theorem 4 shows, that there exists a compact-valued multivalued function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, whose y-sections are of lower Borel class 1 and x-sections are of upper Borel class 1 (see (3) (b)), but F is not of upper Borel class 2. Therefore Theorem 1 cannot be generalized into higher classes.

From Theorems 1 and 3 by using (2) we obtain the following Corollary.

Corollary 1. *Let $(X, \mathcal{T}(X))$ and $(Y, \mathcal{T}(Y))$ be metrizable spaces and let $(Z, \mathcal{T}(Z))$ be a second countable, perfectly normal topological one. Let us suppose that $F : X \times Y \to Z$ is a compact-valued multivalued function.*

If all x-sections of F are $\mathcal{T}(Y)$-upper semicontinuous and all y-sections are $\mathcal{T}(X)$-lower semicontinuous, then F belongs to the Borel class 1 (as a single valued function from $X \times Y$ into the hyperspace $\mathcal{K}(Z)$ endowed with the Vietoris topology).

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