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EQUILIBRIUM THEOREM AS THE CONSEQUENCE OF THE STEINHAUS CHESSBOARD THEOREM

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Abstract

Kulpa proved the existence of a stable-like point (Equilibrium Theorem)and applied this theorem to show the existence of rational divisions of bounded Lebegue measurable sets in Euclidean spaces. We present an algorithm for determining on the Euclidean plane the place where the equilibrium points are. For this purpose, we use the Steinhaus chessboard theorem. The existence of market equilibrium is a classical problem in economics (Walras, von Neumann, Nash). The Brouwer fixed point theorem was the main mathematical tool in Nash's paper, for which he won the Nobel prize in economics. The Brouwer Theorem is an easy consequence of Kulpa's Equilibrium Theorem. Hence, an algorithm for determining a fixed point is also given.

1. Introduction

The existence of market equilibrium is a classical problem in economics (Walras, von Neumann [12], Nash [9]). The Brouwer fixed point theorem was the main mathematical tool in Nash's paper [9] for which he won the Nobel prize in economics ([1],

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[8]). In fact, the Brower fixed point theorem is an immediate consequence of the Poincaré theorem announced in 1883 (see [11]).

Poincaré Theorem
"Let $f_1, ..., f_n$ be n continuous functions of n variables $x_1, ..., x_n$; the variable x_i is subjected to vary between the limits a_i and $-a_i$. Let us suppose that for that for $x_i = a_i, f_i$ is constantly positive, and that for $x_i = -a_i, f_i$ is constantly negative; I say there will exist a system of values of x which all $f's$ vanish". In the paper "Parametric extension of the Poincaré theorem" [6] the case where the functions change in time is investigated. The special case of this theorem from the paper mentioned above is the following statement, by Hugo Steinhaus (see [7]); "Consider a chessboard with some "mined square" on. Assume that the king cannot go across the chessboard from the left to the right one with out meeting a mined square. Then the rook can go across the chessboard from upper edge to the lower one movnig exclusively on the mined squares."

A modified version of this theorem will be used for proving Kulpa's theorem. In his paper [5] Kulpa proved the existence of a stable-like point (Equilibrium Theorem) and applied his theorem to show the existence of rational division of bounded Lebegue measurable sets in Euclidean spaces. His theorem gives a generalisation of the Perron- Frobenius Theorem which states that every square matrix $\{a_{ij}\}$ with $a_{ij} > 0$ has at least one non-negative real eigen value. This theorem plays a very important role in economics models (cf. Nikaido [10]). Kulpa's theorem has been investigated, generalised and used in papers [3], [4] by Idzik and Ichiishi.

The aim of this paper is to prove by presenting an algorithm on the Euclidean plane Equilibrium Theorem applying the Steinhaus chessboard theorem (see [7]). However, chessboard will be of simplex shape. An algorithm allowing us to find a stable-like point will be shown. The Brouwer fixed point theorem is an easy consequence of Kulpa's Equilibrium Theorem. Hence, an algorithm for determining a fixed point is also given (see D. Gale [2]).

2. An Algorithm

Consider the plane R^2 with Cartesian coordinates and right hand (counter-clock) orientation. Let D be a simplex with vertices $(0,0),(1,0)$ and $(1,1)$.
Let us fix a natural number $k > 1$ and let

$$Z_k = \{\frac{i}{k} : i \in \{0, ..., k\}\}$$

and denote by

$$D_k^2 = (Z_k \times Z_k) \cap D$$

and by $e_0 = (\frac{1}{k}, 0), e_1 = (0, \frac{1}{k})$ basic vectors.

Definition 1. An ordered set $z = [z_0, z_1, z_2]$ is said to be a simplex iff

$z_1 = z_0 + e_i, z_2 = z_1 + e_{1-i}$ where $i \in \{0, 1\}$.

Any subset $[z_0, z_1], [z_1, z_2]$ and $[z_2, z_0] \subset z$ is said to be a face of the simplex z.

Observation 1. Any face of a simplex z contained in the simplex D is a face of exactly one or two simplexes from D, depending on whether or not lies on the boundary of D.

boundary $D =$ union of segments $[(0,0),(1,0)], [(0,0),(1,1)]$ and $[(1,0),(1,1)]$

Let us consider D_k^2 and let $\mathcal{P}(k)$ be a family of all simplexes in D_2^k. Let $\mathcal{P}(k)$ be a family consisting of all faces of simplexes from $\mathcal{P}(k)$, all vertices of simplexes from $\mathcal{P}(k)$, all simplexes from $\mathcal{P}(k)$ and empty set.

Let $\mathcal{V}(k)$ be a set of all vertices of simplexes from $\mathcal{P}(k)$ and $f : \mathcal{V}(k) \to \{0, 1\}$ be a function defined as the following one

$$f(p) = \begin{cases} 0 & \text{if} \quad p \in [(1,1),(1,0)] \\ 1 & \text{if} \quad p = (0,0). \end{cases}$$

The function f is called a colouring of the partition $\mathcal{P}(k)$.
The face s of the simplex S is called a gate if $f(s) = \{0, 1\}$.

Lemma 1 (Sperner's Lemma for 1 dimensional simplex).

Let $C = \{0, \frac{1}{k}, ..., 1\}$ and $f : C \to \{0, 1\}$ be such that $f(0) = 0$ and $f(1) = 1$. Then there exists $i, 1 \leq i \leq k$ such that $f(\{\frac{i-1}{k}, \frac{i}{k}\}) = \{0, 1\}$. The number of such pairs is odd.

Lemma 2. *Let w be a simplex and $\mathcal{W}$ be the set of vertices of w and $f : \mathcal{W} \to \{0, 1\}$. Then w has an even number of gates.*

Definition 2. Two simplexes w and v from $\mathcal{P}(k)$ are in the relation $\sim$ if $w \cap v$ is a gate.

Definition 3. A subset $\mathcal{S} \subset \mathcal{P}(k)$ is called a chain in $\mathcal{P}(k)$ if $\mathcal{S} = \{w_0, w_1, ..., w_n\}$ and for each $i, i = 0, ..., n-1, w_i \sim w_{i+1}$

From Lemma 2 it follows

Observation 2. For each chain $\{v_1, ..., v_l\} \subset \mathcal{P}(k)$ there exists not more than one v such that $\{v_1, ..., v_l, v\}$ is a chain.

Observation 3. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ are maximal chains in $\mathcal{P}(k)$. Then $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$ or $\mathcal{C}_1 = \mathcal{C}_2$.

Proof. Suppose that $v \in \mathcal{C}_1 \cap \mathcal{C}_2$. Let $\mathcal{C} \subset \mathcal{C}_1 \cap \mathcal{C}_2$ be a maximal chain in $\mathcal{C}_1 \cap \mathcal{C}_2$ to which v belongs, $\mathcal{C} = \{w_1, ..., w_n\}$. There exists not more than one w_0 and not more than one w_{n+1} such that $\{w_0, w_1, ..., w_n, w_{n+1}\}$ is a chain in $\mathcal{P}(k)$. Hence $\mathcal{C}_1 = \mathcal{C}_2$. □

Theorem 1. *For any partition $\mathcal{P}(k)$ of a simplex D and any colouring there exist a chain $\mathcal{C} \in \mathcal{P}(k), \mathcal{C} = \{w_1, ..., w_n\}$ such that $w_1 \cap [(1,0),(1,1)] \neq \emptyset$ and $w_n \cap [(0,0),(1,1)] \neq \emptyset$.*

Proof. By the assumptions there is an odd number of gates in the segment $[(1,0),(1,1)]$. Walking along the segment $[(1,0),(1,1)]$ from the point $(1,1)$ to the point $(1,0)$ we met the first gate. Let v_1 be the simplex to which this face belong. Hence, by lemma 2, there exists a second face of v_1 which is a gate and which is a common face of two simplexes. Suppose that the maximal chain $\{v_1, ...v_l\}$ has been defined. Then

$$(1) v_l \cap [(1,0),(1,1)] \neq \emptyset$$

or

$$(2) v_l \cap [(0,0),(1,0] \neq \emptyset$$

If (2), then end. If (1), then we take the next gate in order, which is between $v_l \cap [(1,0),(1,1)]$ and the point $(1,0)$. Since the number of gates is odd, hence in the end we have the case (2) (see the figure 1).

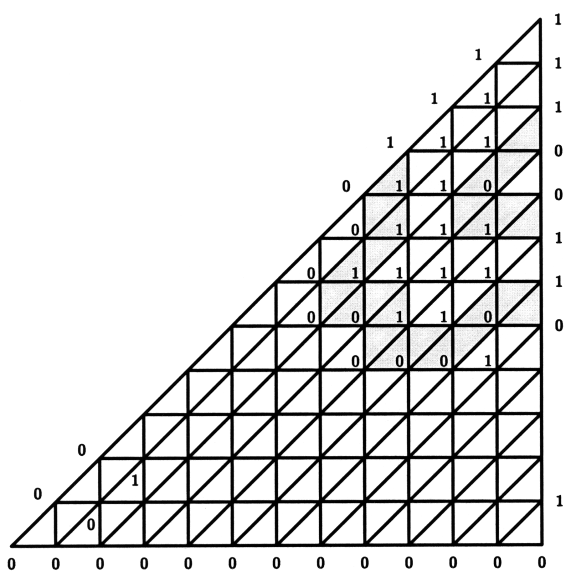

Fig. 1.

$\square$

In the paper [6] the following Lemma has been proved

Lemma 3. *Let $\{A_m : m \in N\}$ be a sequence of connected subsets of a compact metric space X such that some sequence $\{a_n : n \in N\}$ of points $a_n \in A_n\}$ is converging in X. Then the set $A = Ls\{A_n : n \in N\}$ is compact and connected.*

For $x \in R^3$ let $\mid x \mid = \mid x_1 \mid + \mid x_2 \mid + \mid x_3 \mid$ where $x = (x_1, x_2, x_3)$.

Theorem 2. (Kulpa's Equilibrium Theorem for $n = 2$)
Let $S = [s_1, s_2, s_3]$ and $S_i = [..., \hat{s}_i, ...]$
Let $f : S \longrightarrow [0, \infty)^3, f = (f_1, f_2, f_3)$ be a continuous map such that for each $i : f_i(S_i) = 0$.
Then for each continuous function $g : S \longrightarrow [0, \infty)^3$ there exists a point $x \in S$ such that

$$f(x) \mid g(x) \mid = \mid f(x) \mid g(x).$$

Each such point will be called an equilibrium point.

Proof. Let $g = (g_1, g_2, g_3)$. Define $h_i = f_i(x) \mid g(x) - g_i(x) \mid f(x) \mid$.

Claim 1. Each point $x \in h_1^{-1}(0) \cap h_2^{-1}(0)$ is an equilibrium point.
Proof. For $x \in h_1^{-1}(0) \cap h_2^{-1}(0)$ we have

$$f_1(x) \mid g(x) \mid = g_1(x) \mid f(x) \mid$$

and

$$f_2(x) \mid g(x) \mid = g_2(x) \mid f(x) \mid$$

Hence $(f_1(x) + f_2(x)) \mid g(x) \mid = (g_1(x) + g_2(x)) \mid f(x) \mid$ and since $f_1(x) + f_2(x) = \mid f(x) \mid - f_3(x)$ we have $\mid f(x) \mid (g_1(x) + g_2(x) + g_3(x)) - f_3(x) \mid g(x) \mid = (g_1(x) + g_2(x)) \mid f(x) \mid$. Then $g_3(x) \mid f(x) \mid = f_3(x) \mid g(x) \mid$. Hence x is an equilibrium point.

Let $S_1 = [(1,0),(1,1)], S_2 = [(0,0),(1,1)]$ and $S_3 = [(0,0),(1,0)]$.

If $x \in S_1$, then $h_1((0,0)) = f_1((0,0)) \mid g(x) \mid -g_1((0,0))f_1((0,0)) > 0$ and $h_1(x) = f_1(x) \mid g(x) \mid -g_1(x) \mid f(x) \mid < 0$ for $x \in S_1$ Generally, the same for each $i = 1,2,3$.

Let F^i_k be a colouring defined by the function h_i and the partition $\mathcal{P}(k)$ of $\mathcal{S}$ as the following one

$$F^i_k(p) = \begin{cases} 0 & \text{if} \quad h_i(p) \leq 0 \\ 1 & \text{if} \quad h_i(p) > 0 \end{cases}$$

where p is a vertex of some simplex from $\mathcal{P}(k)$. From Theorem it follows that for any partition $\mathcal{P}(k)$ of $\mathcal{S}$ there exists a chain $\mathcal{C}^k_i$ connecting two different from S_i faces of S. By Lemma the set $\mathcal{C}_i = Ls\{C^k_i : k \in N\}$ is a compact, connected set and $\mathcal{C}_i \cap S_j \neq \emptyset$ for $j \neq i, j \in \{1,2,3\}$ and $h_i(\mathcal{C}_i) = 0$. Let $x_1 = sup(\mathcal{C}_1 \cap [(0,0),(1,0)])$ and $x_2 = inf(\mathcal{C}_2 \cap [(0,0),(1,0)])$.

We have $h_1(x_1) = 0$ and $h_2(x_2) = 0$. Suppose that $x_1 < x_2 < (1,0)$. Then $h_1|[(0,0),(1,0)] + h_2|[(0,0),(1,0)] = f_1|g| - g_1(f_1+f_2+f_3) + f_2|g| - g_2(f_1+f_2+f_3) = f_1g_2 + f_1g_3 - g_1f_2 + f_2g_1 + f_2g_3 - g_2f_1 = f_1g_3 + f_2g_3 \geq 0$. From this we have that $h_1(x_2) > 0$. Hence between x_2 and $(1,0)$ we have an odd number of gates. Since $\mathcal{C}_1$ is the first chain which connects $[(0,0),(1,0)]$ and $[(0,0),(1,1)]$, hence there exists a gate between x_1 and x_2 and the chain which connects this gate with a gate which lies between x_2 and $(1,0)$. Hence, by Lemma, we have a connected set $\mathcal{C}$ such that $h_1(\mathcal{C}) = 0$ and $inf\{\mathcal{C} \cap [(0,0),(1,0)]\} < x_2 < sup\{\mathcal{C} \cap [(0,0),(1,0)]\}$. Then $\mathcal{C}_2 \cap S_3$ and $\mathcal{C}_2 \cap S_1$ are in disjoint components of $S \setminus \mathcal{C}$. Hence $\mathcal{C} \cap \mathcal{C}_2 \neq \emptyset$. By the claim we have that each point $x \in \mathcal{C} \cap \mathcal{C}_2$ is an equilibrium point. If $x_2 < x_1$, then $\mathcal{C}_2 \cap S_3$ and $\mathcal{C}_2 \cap S_1$ are in disjoint components of $S \setminus \mathcal{C}_1$. Hence $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$. From Claim it follows that each point x in $\mathcal{C}_1 \cap \mathcal{C}_2$ is an equilibrium point. $\square$

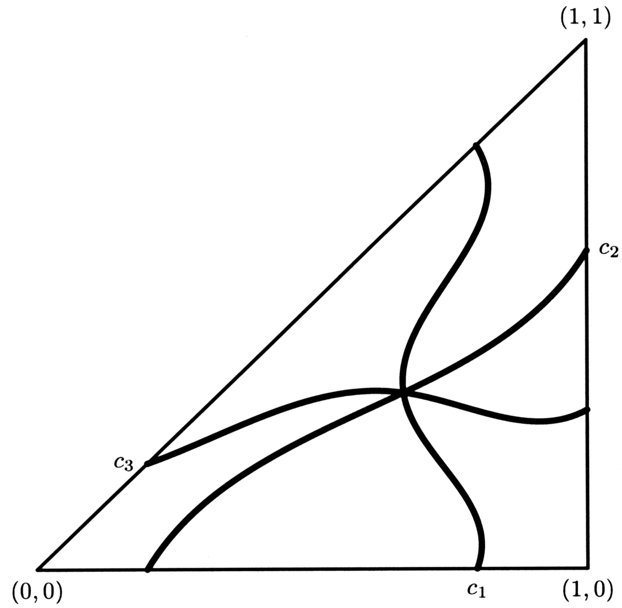

Fig. 2.

Corollary 1. (Brouwer fixed point theorem)
Let $\mathcal{S} = \{x \in \mathcal{R}^3 :\mid x \mid= 1\}$.*If* $g : \mathcal{S} \longrightarrow \mathcal{S}$, *then there exists* $x \in \mathcal{S}$ *such that* $g(x) = x$.

Proof. Let $f : \mathcal{S} \longrightarrow \mathcal{S}$ be the identity map. Since for $x \in \mathcal{S}$ there is $\mid x \mid= 1$, hence, by the equilibrium theorem, there exists point x such that $x \mid g(x) \mid=\mid x \mid g(x)$. Since $\mid x \mid= 1$ and $\mid g(x) \mid= 1$, hence $g(x) = x$. □

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