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CONTINUOUS MAPS OF DENDRITES WITH FINITELY MANY BRANCH POINTS AND NONWANDERING SETS

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ABSTRACT. Let f be a continuous map of a dendrite X with finitely many branch points into itself and $\Omega(f)$ the set of nonwandering points for f. We show the following two results: (1) if $\Omega(f)$ is finite, it always is not the set of periodic points of f and (2) $\Omega(f)$ is contained in the closure of the set of eventually periodic points of f.

1. Introduction

Let f be a continuous map from a dendrite X to itself, $\Omega(f)$ the set of nonwandering points for f, $P(f)$ the set of periodic points of f and $EP(f)$ the set of eventually periodic points of f. And $\overline{A}$ implies the closure of a space A. When X is the interval, in [1], Block examined $\Omega(f)$ and $P(f)$ and showed the following:

(1) if $\Omega(f)$ is finite, we have $\Omega(f) = P(f)$ and

(2) $\Omega(f) \subset \overline{EP(f)}$.

Then, after about 20 years, Hosaka and Kato examined dendrites in [3], and they showed that (1) and (2) hold when X is a tree. And they constructed two dendrites X_1, X_2 and two maps $g_1 : X_1 \to X_1, g_2 : X_2 \to X_2$ such that $\Omega(g_1)$ is finite, that $\Omega(g_1) \neq P(g_1)$, and that $\Omega(g_2) \not\subset \overline{EP(g_2)}$. Since the sets of branch points of X_1

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and X_2 are infinite, Arai asks the following question: When the set of branch points of X is finite, do (1) and (2) hold?

In [3], they proved many lemmas to show the above (1). Lemma 2.6, an important one, is able to be extended from a tree to a dendrite with finitely many branch points.

Theorem 1. (*Invariance of the unstable manifold*) *Let f be a map from a dendrite X with finitely many branch points to itself and p a periodic point of f. If $W(p, f)$ is the unstable manifold of p, then $f(W(p, f)) = W(p, f)$.*

Theorem 1 is proved by the method which is different from the proof of [3, Lemma 2.6]. We should notice that the proof of the above Theorem 1 is more simple and geometrical.

But, for dendrites with finitely many branch points which are not trees, the above (1) doesn't always be true.

Example. *Let S be a subspace $\{re^{i\theta} : n = 1, 2, \cdots, \theta = 2\pi/n$ and $0 \le r \le 1/n\}$ of the complex plane. For each $m > n$, there exists a continuous map $f_{m,n} : S \to S$ such that $|\Omega(f_{m,n})| = m$ and $|P(f_{m,n})| = n$.*

The space S is the easiest space in dendrites with finitely many branch points and the above example is more simple than Example 1.5 in [3].

But, even when X has finitely many branch points, the above (2) holds.

Theorem 2. *Let f be a map from a dendrite X with finitely many branch points to itself. Then $\Omega(f) \subset \overline{EP(f)}$.*

Theorem 2 is proved by using the relation between infinite edges and finitely many branch points, that is to say, we use the property that some infinite sequence of nonwandering points converges a branch point.

2. Notations and definitions

Let X be a compact metric space and f denote a continuous map of X into itself. We denote the n-fold composition of f with itself by $f \circ ... \circ f$. Let f^0 denote the identity map. A point $x \in X$ is a *periodic point of period $n \ge 1$ for f* if $f^n(x) = x$. The least positive integer n for which $f^n(x) = x$ is called the prime period of

x. Especially, $x \in X$ is a fixed point for f if $n = 1$. A point $x \in X$ is an *eventually periodic point of period* n *for* f if there exists $m \geq 0$ such that $f^{n+i}(x)$=$f^i(x)$ for all $i \geq m$. That is, $f^i(x)$ is a periodic point of period n for $i \geq m$. A point $x \in X$ is *nonwandering point for* f if for any open set U containing x there exist $y \in U$ and $n > 0$ such that $f^n(y) \in U$.

We denote the set of fixed points for f, periodic points for f, eventually periodic points for f, and nonwandering points for f by $F(f), P(f), EP(f)$ and $\Omega(f)$, respectively. $\overline{A}$ denotes the closure of a set A. Notice that $P(f) \subset \Omega(f)$, $P(f) \subset EP(f)$, $f(P(f)) \subset P(f)$, $f(\Omega(f)) \subset \Omega(f)$ and $\Omega(f)$ is closed.

An *arc* is any space which is homeomorphic to the closed interval [0,1]. A *continuum* is nonempty, compact and connected metric space. A *graph* is a continuum which can be written as the union of finitely many arcs, any two of which are disjoint or intersect only in one or both of their end points. A *tree* is a graph without circuits, that is, a uniquely arcwise connected graph. A *dendrite* is a locally connected, uniquely arcwise connected continuum. We say subcontinuum A of a continuum X is of order less than or equal to β in X, written $\mathrm{Ord}(A, X) \leq \beta$, provided that for each open subset U of X with $A \subset U$ there exists an open subset V of X such that $A \subset V \subset U$ and $|\mathrm{Bd}(V)| \leq \beta$, where $\mathrm{Bd}(V)$ means the boundary of V. We say that A is of order β in X, written $\mathrm{Ord}(A, X) = \beta$, if $\mathrm{Ord}(A, X) \leq \beta$ and $\mathrm{Ord}(A, X) \not\leq \alpha$ for any cardinal number $\alpha < \beta$. A point $x \in X$ is a *branch point of* X provided that $\mathrm{Ord}(x,X) \geq 3$. Let B $=\{b_1, b_2, ..., b_n\}$ be the set of branch points of a dendrite X. For $x \in X \setminus B$, there exists an open neighborhood V of x such that V is homeomorphic to (0,1) or (0,1].

And the *unstable manifold* $W(p, f)$ for some periodic point p is as follows:

$W(p, f) = \{x \in X |$for any neighborhood $V$of $p, x \in f^n(V)$ for some $n > 0\}$

Let X be a dendrite and Y a subspace of X. We denote the maximal connected set containing Y by $[Y]$. Particularly, if $Y = \{x, y\}$, write $[Y] = [x, y]$.

3. Lemmas

By the proof of [5, Lemma 2.8], we have the following:

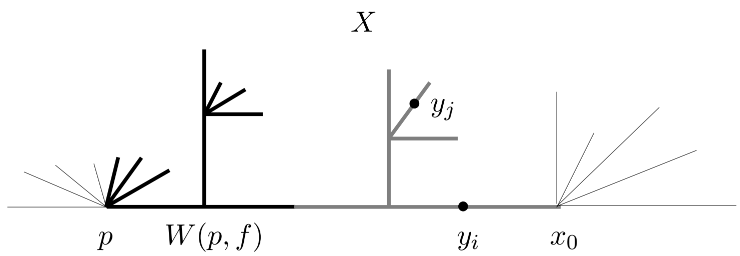

FIGURE 1

Lemma 1. *Let X be a dendrite, f a continuous map from X into itself, and $X \setminus B = \bigcup_{j=1}^{\infty} I_j$. If an open interval $J \subset I_j$ for some $j = 1, 2, \cdots$ satisfies $J \cap P(f) = \emptyset$, then $J \cap f^n(J \cap \Omega(f)) = \emptyset$ for any positive integer n.*

By the proof of [3, Lemma 2.4], we have the following:

Lemma 2. *Let f be a continuous map from a dendrite X to itself and p a fixed point of f. Then $W(p, f)$ is connected.*

Lemma 3. *Let f be a continuous map from a dendrite X with finitely many branch points to itself and p a fixed point of f. Then $f(W(p, f)) = W(p, f)$. (See Figure 1.)*

Proof: By definition, we see that $f(W(p, f)) \subset W(p, f)$. We show that $f(W(p, f)) \supset W(p, f)$. It suffices to show that $f^{-1}(z) \cap W(p, f) \neq \emptyset$ for each $z \in W(p, f)$. We suppose that $f^{-1}(z) \cap W(p, f) = \emptyset$ for some $z \in W(p, f)$.

Since $z \in W(p, f)$, there exists an increasing sequence $n_1, n_2, \cdots$ and $x_i \in X (i = 1, 2, \cdots)$ such that $f^{n_i}(x_i) = z$ for each $i = 1, 2, \cdots$ and $x_i \to p (i \to \infty)$. We notice that $Y = \{f^{n_i - 1}(x_i) : i = 1, 2, \cdots\} \subset f^{-1}(z)$. We suppose that $|\{i : y = f^{n_i - 1}(x_i)\}| = \infty$ for some $y \in f^{-1}(z)$. Since p is a fixed point of f, we have $y \in W(p, f)$ and a contradiction. We may assume that $f^{n_i - 1}(x_i) \neq f^{n_j - 1}(x_j) (i \neq j)$. Moreover, we may assume that $y_i = f^{n_i - 1}(x_i) \to x_0 (i \to \infty)$ and all y_i are contained in some small neighborhood of x_0.

Since $f^{-1}(z)$ is closed, we have $x_0 \in f^{-1}(z)$. Note that if u is any point which is in infinitely many of the intervals $[p, y_i]$, then $u \in$

$W(p,f)$. Since x_0 is a limit point of Y, at least one point of $\{x_0\}\cup Y$ must be such a point u, contradicting that $f^{-1}(z)\cap W(p,f)=\emptyset$. $\square$

Lemma 4. *Let f be a continuous map from a dendrite X with finitely many branch points to itself and p a point of X with $f^n(p) = p (n > 1)$. Then $f(W(p,f^n)) = W(f(p),f^n)$.*

Proof: By definition, we have $f(W(p,f^n)) \subset W(f(p),f^n)$. Thinking of p as $f^k(p)$ $(k = 1,2,\cdots,n)$, we have $f(W(f^k(p),f^n)) \subset W(f^{k+1}(p),f^n)$. We see that $f^n(W(f(p),f^n)) \subset$
$f^{n-1}(W(f^2(p),f^n)) \subset \cdots \subset f(W(f^n(p),f^n)) = f(W(p,f^n))$. Since $f(p)$ is a fixed point of f^n, by Lemma 3, we have $f^n(W(f(p),f^n) = W(f(p),f^n)$ and $W(f(p),f^n) \subset f(W(p,f^n))$. We conclude that $f(W(p,f^n)) = W(f(p),f^n)$. $\square$

4. Proofs

Proof of Theorem 1: Let p be an n-periodic point of f. We have
$f(W(p,f))$
$= f(W(p,f^n)) \cup f(W(f(p),f^n)) \cup \cdots \cup f(W(f^{n-1}(p),f^n))$ (by [3, Lemma 2.5])
$= W(f(p),f^n) \cup W(f^2(p),f^n) \cup \cdots \cup W(f^n(p),f^n)$ (by Lemma 4)
$= W(p,f)$ (by [3, Lemma 2.5]). $\square$

In [3, Example 1.5], for each point $p \in P(g_1)$ we have $f(W(p,f)) = W(p,f)$.

Question. Let f be a map from a dendrite X to itself and p a periodic point of f. Do we have $f(W(p,f)) = W(p,f)$?

Example. *Let S be a subspace $\{re^{i\theta} : n = 1,2,\cdots,\theta = 2\pi/n$ and $0 \le r \le 1/n\}$ of the complex plane. Take integers $m > n$. We construct a continuous map $f_{m,n} : S \to S$ such that $|\Omega(f_{m,n})| = m$ and $|P(f_{m,n})| = n$.*

First, we construct a continuous map $f : S \to S$ such that $\Omega(f) = \{(0,0),(1/2,0)\}$ and $P(f) = \{(0,0)\}$. Denote $I_n = \{re^{2\pi i/n} : 0 \le r \le 1/n\} \subset S$, $J_n = \{(x,0) : 1/2 + 1/2n \le x \le 1/2 + 1/2(n-1)\}$ for each $n = 2,3,\cdots$ and $J = \{(x,0) : 1/2 < x \le 1\} = \bigcup_{n=2}^{\infty} J_n$. (See Figure 2.)

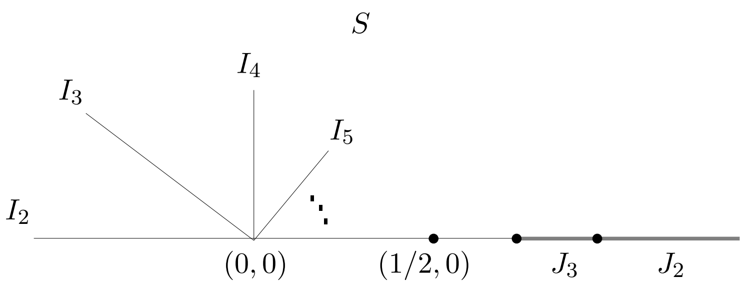

FIGURE 2

Define $f(\{(x, 0) : 0 \leq x \leq 1/2$ or $x = 1/2 + 1/2n$ for each $n = 2, 3, \cdots\}) = \{(0, 0)\}$, $f(I_n) = I_{n-1}$ for each $n > 2$, $f(I_2) = \{(x, 0) : 0 \leq x \leq 1/2\}$ and $f(J_n) = I_n$ for each $n = 2, 3, \cdots$. Since $f^n(I_n) = \{(0, 0)\}$ for each $n = 2, 3, \cdots$, we have $\Omega(f) \cap I_n = \{(0, 0)\}$ for each $n = 2, 3, \cdots$. And we see that $\Omega(f) \cap \{(x, 0) : 0 < x \leq 1/2\} = \{(1/2, 0)\}$.

Since $f^m(J) \cap J = \emptyset$ for each m, we have $\Omega(f) \cap I_n = \emptyset$. We conclude that $\Omega(f) = \{(0, 0), (1/2, 0)\}$ and $P(f) = \{(0, 0)\}$.

There exists a continuous map $g : [0, 1] \to [0, 1]$ such that $\Omega(g) = P(g) = \{0, 1\}$. (See Figure 3.)

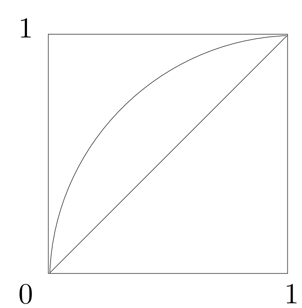

FIGURE 3

Denote the space $S \cup_{(0,0)=0} [0, 1]$ attached by a point (0,0) of S and a point 0 of [0,1]. We see that $S \cup_{(0,0)=0} [0, 1]$ is homeomorphic to S. Define $f_{3,2} = f \cup g : S \cup_{(0,0)=0} [0, 1] \to S \cup_{(0,0)=0} [0, 1]$. We have $|\Omega(f_{3,2})| = 3$ and $|P(f_{3,2})| = 2$.

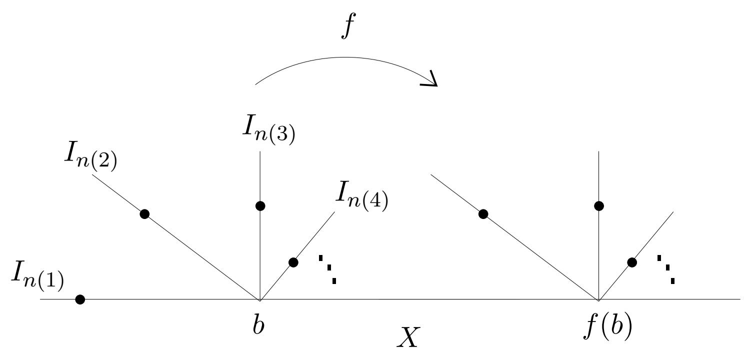

FIGURE 4

Denote the space $S \cup_{(0,0)=(0,0)} S$ attached by a point (0,0) of S and a point (0,0) of another space S. We see that $S \cup_{(0,0)=(0,0)} S$ is homeomorphic to S. Define $f_{3,1} = f \cup f : S \cup_{(0,0)=(0,0)} S \to S \cup_{(0,0)=(0,0)} S$. We have $|\Omega(f_{3,1})| = 3$ and $|P(f_{3,1})| = 1$.

By the above, we have a continuous map $f_{m,n} : S \to S$ such that $|\Omega(f_{m,n})| = m$ and $|P(f_{m,n})| = n$.

Proof of Theorem 2: We suppose that $\Omega(f) \not\subset \overline{EP(f)}$, i.e., $V \cap \Omega(f) \neq \emptyset$, where $V = X \setminus \overline{EP(f)}$. Let x be an element of $V \cap \Omega(f)$ and W the component of V containing x. Since V is open, W is a neighborhood of x. Since $x \in \Omega(f)$, there exists a positive integer n such that $f^n(W) \cap W \neq \emptyset$. Denote $g = f^n$ and $T = \bigcup_{i=0}^{\infty} g^i(W)$ which is connected containing x. We see that $Y = \{g^i(x) : i = 0, 1, \cdots\} \subset T \supset g(T)$, that $T \cap EP(f) = \emptyset$, and that $\overline{T}$ is a dendrite.

Let B be the set of branch points of X. By [3, Theorem 1.2], we may assume that $\bigcup_{j=1}^{\infty} I_j = \overline{T} \setminus B$, where each I_j is a component of $\overline{T} \setminus B$. If there exist distinct integers i_1, i_2 and $j = 0, 1, \cdots$ such that $g^{i_1}(x), g^{i_2}(x) \in I_j$, by Lemma 1, then $I_j \cap P(f) \neq \emptyset$ and we have a contradiction. We may assume that $|Y \cap I_j| \leq 1$ for each j. This shows that $\overline{Y} \setminus Y \subset B \cap T$. Since $g(Y) \subset Y$ and $g(\overline{Y}) \subset \overline{Y}$, we have $g(\overline{Y} \setminus Y) \subset \overline{Y} \setminus Y$.

We have $n(1) < n(2) < \cdots$ and $b \in B \cap T$ that $|Y \cap I_{n(j)}| = 1$ for each j and that $\{b\} = \bigcap_{j=1}^{\infty} \overline{I_{n(j)}}$. Since B is finite, we have $b \notin Y$ and $b \in EP(f)$. And since $|Y \cap I_{n(j)}| = 1$ for each j

and $\{b\} = \bigcap_{j=1}^{\infty} \overline{I_{n(j)}}$, we have $b \in T$. This contradicts because $T \cap EP(f) = \emptyset$. (See Figure 4.) □

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