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STRONG MONOTONE AND NESTED NORMALITY

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Abstract. Possible strengthenings of monotone normality are examined. *Strong monotone normality*: For each pair (H, K) of disjoint closed sets in X there is an open set $G(H, K)$ such that (i.) $H \subset G(H, K) \subset \overline{G(H, K)} \subset X \backslash K$; (ii.) for disjoint closed sets H' and K' with $H \subset H'$ and $K' \subset K$, $G(H, K) \subset G(H', K')$; (iii.) $G(H, K) \cap G(K, H) = \phi$; (iv.) if H' is a closed set and $H' \subset G(H, K)$, then $G(H', K) \subset G(H, K)$. *Nested normality* requires properties (i.), (ii.), and (iv.). Metric spaces, compact monotonically normal spaces, and linearly ordered topological spaces are among the strongly monotonically normal spaces. Strong monotone normality is preserved by closed maps among other things. Using the fact that every stratifiable space can be embedded (as a perfect retract) in an M_1-space, one can prove that if every M_1-space is strongly monotonically normal, then so is every stratifiable space. It is not known whether every M_1-space is strongly monotonically normal. A version of the Dugundji Extension Theorem is proved for nestedly normal spaces.

The concept of monotone normality was first studied by C. J. R. Borges in his dissertation where he showed that all stratifiable spaces have the property (see his paper, *On Stratifiable Spaces* [1]). In 1973, the first major paper [17] on "monotone normality" by Heath, Lutzer, and Zenor (who named the property) appeared. In

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the same year, Borges [3] gave some additional characterizations of monotone normality. Another major work is Moody's dissertation [23] in which acyclic monotone normality is defined and many relationships among acyclic monotone normality, K_0-spaces, K_1-spaces, and monotone normality are sorted out. (See also [24] and Moody's papers with Reed, Roscoe, and Collins [25][26] .) *Strong monotone normality* is an ostensible strengthening of monotone normality that was defined by Heath and Zenor [18][19].

Definition 1. [17][3] [2] A T_1 space X is *monotonically normal* if it satisfies one of the following equivalent conditions:
(a.) there is a function G which assigns to each ordered pair (H, K) of disjoint closed subsets of X an open set G(H, K) such that:
i.) $H \subset G\,(H, K) \subset \overline{G\,(H, K)} \subset X \backslash K$;
ii.) if (H', K') is a pair of disjoint closed sets such that $H \subset H'$ and $K \supset K'$, then $G\,(H, K) \subset G\,(H', K')$.
(b.) there is a function G which assigns to each ordered pair (H, K) of mutually separated subsets of X an open set G(H, K) such that:
i.) $H \subset G\,(H, K) \subset \overline{G\,(H, K)} \subset X \backslash K$;
ii.) if (H', K') is a pair of mutually separated sets such that $H \subset H'$ and $K \supset K'$, then $G\,(H, K) \subset G\,(H', K')$.
(c.) there is a function G which assigns to each ordered pair (p, C), with C closed and p not an element of C, an open set $G\,(p, C)$ such that:
i.) $p \in G\,(p, C) \subset X \backslash C$;
ii.) if D is closed and $p \notin C \supset D$, then $G\,(p, C) \subset G\,(p, D)$;
iii.) if p and q are distinct points of X, then

$$G\ (p, \{q\})\ \cap G\,(q, \{p\}) = \phi.$$

All of the functions G above are called *monotone normality operators.* In both (a.) and (b.), we may assume that

$$G\,(H, K) \cap G\,(K, H) = \phi.$$

Recall that acyclic monotone normality is a strengthening of monotone normality due to Moody, Reed, Roscoe, and Collins [25].

Definition 2. A T_1 space X is *acyclically monotonically normal* if there is an operator M which assigns to each $x \in X$ and each open set U containing x, an open set $M(x,U)$ containing x which satisfies:

i.) $M(x,U) \subset M(x,U')$ whenever $x \in U \subset U'$;

ii.) if x and y are distinct points of X, then

$$M(x, X\setminus\{y\}) \cap M(y, X\setminus\{x\}) = \phi;$$

iii.) $\cap_{i=0}^{n-1} M(x_i, X\setminus\{x_{i+1}\}) = \phi$ if $x_0, ..., x_{n-1}$ are distinct, $x_n = x_0$, and $n \geq 2$.

The function M is called an *acyclic monotone normality operator.*

We are concerned with another strengthening of monotone normality, called strong monotone normality, due to Heath and Zenor [18][19].

Definition 3. A T_1 space X is *strongly monotonically normal (SMN)* if there is a function G which assigns to every ordered pair (H, K) of disjoint closed sets in X an open set $G(H, K)$ such that:

i.) $H \subset G(H,K) \subset \overline{G(H,K)} \subset X\setminus K$;

ii.) for disjoint closed sets H' and K' with $H \subset H'$ and $K' \subset K$, $G(H,K) \subset G(H',K')$;

iii.) $G(H,K) \cap G(K,H) = \phi$;

iv.) if H' is a closed set and $H' \subset G(H,K)$, then

$$G\left(H', K\right) \subset G(H,K).$$

The function G is called a *strong monotone normality operator.*

Question 1. Is strong monotone normality really a strengthening of monotone normality? We have been unable to find an example of a space that is monotonically normal but not strongly monotonically normal. Rudin's [28] example of a monotonically normal space that is not an acyclically monotonically normal space (nor a K_0-space) is also strongly monotonically normal.

Question 2. Is property iii.) in Definition 3 necessary? In Definition 1, it follows from the other properties and may here, too.

Towards answering these and other questions, we consider the properties separately.

Definition 4. Let X be a T_1 space and N a function that assigns to each closed set H and each open neighborhood U of H an open neighborhood $N(H,U)$ of H.

(A.) N is a *normality operator* if $H \subset N(H,U) \subset \overline{N(H,U)} \subset U$.
(B.) N is *separating* if, whenever $H \cap K = \phi$,

$$N(H, X\backslash K) \ \cap N(K, X\backslash H) = \phi.$$

(C.) N is *monotone* if, whenever $H \subset K$ and $U \subset V$, $N(H,U) \subset N(K,V)$.
(D.) N is *nested* if, whenever $H \subset N(K,V)$ and $U \subset V$, $N(H,U) \subset N(K,V)$.
(E.) N is *Hausdorff* if, whenever $x \neq y$,

$$N(\{x\}, X\backslash\{y\}) \ \cap N(\{y\}, X\backslash\{x\}) = \phi.$$

We should note that properties (A.) and (E.) imply (C.).

Theorem 5. *The following are equivalent in a T_1 space X:*
(a.) X is strongly monotonically normal.
(b.) There is a function G which assigns to each ordered pair (H,K) of mutually separated subsets of X an open set $G(H,K)$ such that:
i.) $H \subset G(H,K) \subset \overline{G(H,K)} \subset X\backslash K$;
ii.) if H' and K' are mutually separated sets in X such that $H \subset H'$ and $K' \subset K$, then $G(H,K) \subset G(H',K')$;
iii.) $G(H,K) \cap G(K,H) = \phi$;
iv.) if $H' \subset G(H,K)$, then $G(H',K) \subset G(H,K)$.
(c.) There is a function G which assigns to each ordered pair (p,K), with K a closed set in X and $p \notin K$, an open neighborhood $G(p,K)$ of p that misses K such that:
i.) if H, K are closed, $H \subset K$, and $p \notin K$ then

$$G(p,K) \subset G(p,H);$$

ii.) if p and q are distinct points of X, then

$$G(p,\{q\}) \ \cap G(q,\{p\}) = \phi;$$

iii.) if $q \in G(p,K)$, then $G(q,K) \subset G(p,K)$.
(d.) X admits a Hausdorff nested normality operator.
(e.) X admits a nested separating normality operator.

Proof:

$(a.) \Rightarrow (c.)$ Define $G(p,K) = G'(\{p\},K)$, where G' is a function satisfying the definition of strong monotone normality. The result follows directly.

$(c.) \Rightarrow (b.)$ Let G' satisfy $(c.)$. Given separated sets H and K in X, define $G(H,K) = \cup\left\{G'\left(p,\overline{K}\right) : p \in H\right\}$. Clearly $(b.)$ is satisfied.

The other implications are routine. □

The functions in Theorem 5 $(a.)-(e.)$ are called *strong monotone normality operators* on the space X. We may distinguish between them by designating them "$(a.)\,,(b.)\,,(c.)\,,(d.)\,$, or $(e.)$ of Theorem 5."

Definition 6. A T_1 space X is *nestedly normal (NN)* if there is a function N that assigns to each closed set H and each open neighborhood U of H an open neighborhood $N(H,U)$ of H such that:

i.) $H \subset N(H,U) \subset \overline{N(H,U)} \subset U$;

ii.) for closed sets H and K and open sets U and V with $H \subset K$ and $U \subset V$, $N(H,U) \subset N(K,V)$;

iii.) for closed sets H and K and (respective) open neighborhoods U and V, with $H \subset N(K,V)$ and $U \subset V$, $N(H,U) \subset N(K,V)$.

Theorem 7 includes the obvious dual characterization of nested normality in terms of disjoint closed sets. The result shows that Question 2 is asking if nested normality implies strong monotone normality.

Theorem 7. *The following are equivalent in a T_1 space X:*

$(a.)$ *X is nestedly normal.*

$(b.)$ *There is a function G which assigns to every ordered pair (H,K) of disjoint closed sets in X an open set $G(H,K)$ such that:*

i.) $H \subset G(H,K) \subset \overline{G(H,K)} \subset X\backslash K$;

ii.) for disjoint closed sets H' and K' with $H \subset H'$ and $K' \subset K$, $G(H,K) \subset G(H',K')$;

iii.) if H' is a closed set and $H' \subset G(H,K)$, then

$$G\left(H',K\right) \subset G(H,K).$$

$(c.)$ *There is a function G which assigns to every ordered pair (H,K) of mutually separated sets in X an open set $G(H,K)$ such that:*

i.) $H \subset G(H,K) \subset \overline{G(H,K)} \subset X\backslash K$;

ii.) for disjoint closed sets H' and K' with $H \subset H'$ and $K' \subset K$, $G(H,K) \subset G(H',K')$;

iii.) if H' is a closed set and $H' \subset G(H,K)$, then

$$G\left(H',K\right) \subset G(H,K).$$

Proof:
$\underline{(a.) \Rightarrow (b.)}$ Let $G(H,K) = N(H, X\backslash K)$.
$\underline{(b.) \Rightarrow (a.)}$ Let $N(H,U) = G(H, X\backslash U)$.
$\underline{(b.) \Rightarrow (c.)}$ Let $G(H,K) = \cup\left\{G\left(\{x\}, \overline{K}\right) : x \in H\right\}$.
$\underline{(c.) \Rightarrow (b.)}$ Since disjoint closed sets are separated, this is immediate. □

Both functions N and G will be referred to as *nested normality operators* on the space X.

Theorem 8. *If X is a linearly ordered topological space, then X is strongly monotonically normal.*

Proof: Essentially the same as for monotone normality [17].

Definition 9. A topological space X is *non-archimedean* if X has a rank 1 base. A base $\mathcal{B}$ is a *rank 1 base* if whenever B_1 and B_2 are elements of $\mathcal{B}$ and $B_1 \cap B_2 \neq \phi$, then either $B_1 \subset B_2$ or $B_2 \subset B_1$.

Theorem 10. *Any non-archimedean space X is strongly monotonically normal.*

Proof: Let $\mathcal{B}$ be a non-archimedean base for X, and let K be a closed set in X and $p \notin K$. Then $G(p,K) = \cup\{B \in \mathcal{B} | p \in B \text{ and } B \cap K = \phi\}$, which obviously satisfies $(c.)$ of Theorem 5. □

The matter of preservation by closed maps and subspaces seems to be critical in the investigation of all types of normality and of generalized metric spaces. Slaughter [32] showed that closed images of metric spaces (Lasnev spaces) are $\mathbf{M}_1$-spaces, but if it could be shown that closed maps preserve $\mathbf{M}_1$-spaces – or that $\mathbf{M}_1$-spaces are hereditary – the very long-standing question of whether stratifiable spaces are $\mathbf{M}_1$-spaces would be answered. Stratifiability, monotone normality, and acyclic monotone normality have been shown to be hereditary and preserved by closed maps. It is easily shown that strong monotone normality and nested normality are preserved by closed maps.

Theorem 11. *If $f : X \to Y$ is a closed continuous map from the strongly monotonically normal (nestedly normal) space X onto a T_1 space Y, then Y is strongly monotonically normal (nestedly normal).*

An example would be the quotient space obtained from $\mathbf{R} \times \mathbf{R}$ by identifying the points on the y-axis. A direct proof [5] shows some of the difficulties that may be encountered and shows some of the details of what is going on in a Lasnev space. In view of Slaughter's result [32], this may give a clue as to whether or not M_1-spaces are strongly monotonically normal.

Proto-metrizable spaces were introduced by Nyikos [27]. This class contains all non-archimedean spaces and all metrizable spaces, and many of the metrization theorems for non-archimedean spaces also hold for proto-metrizable spaces. Fuller [8] proved every proto-metrizable space is the image of a non-archimedean space under a perfect mapping. Hence, all proto-metrizable spaces are strongly monotonically normal.

Definition 12. A topological space X is *proto-metrizable* if it is paracompact and has an ortho-base $\mathcal{B}$. A base $\mathcal{B}$ for X is an *ortho-base* if whenever $\mathcal{B}' \subset \mathcal{B}$ and $x \in \cap\, \mathcal{B}'$, then either $\cap\, \mathcal{B}'$ is open or $\mathcal{B}'$ is a base for the neighborhoods of x.

Corollary 13. *If X is a proto-metrizable space, then X is strongly monotonically normal.*

Corollary 14. *Any metric space is strongly monotonically normal.*

Thus, Lasnev spaces (closed images of metric spaces) are strongly monotonically normal.

In view of the result by Heath and Junilla [14] that every stratifiable space is the image of an M_1-space under a perfect retraction, we get the following corollary.

Corollary 15. *M_1-spaces are strongly monotonically normal if and only if stratifiable spaces are strongly monotonically normal.*

Rudin recently showed [31] that a compact monotonically normal space is the continuous image of a compact LOTS. This gives us the following corollary.

Corollary 16. *If a monotonically normal space X is compact, then X is strongly monotonically normal.*

Question 3. What other topological properties might be necessary and/or sufficient for all such monotonically normal spaces to be strongly monotonically normal (or acyclically monotonically normal)?

Question 4. Which other classes of monotonically normal spaces are strongly monotonically normal (or acyclically monotonically normal)?

Question 5. In particular, are M_0-spaces [14] (or M_1-spaces or stratifiable spaces) strongly monotonically normal?

We have two partial results, using the weaker property of strongly monotonically T_2, in theorems 18 and 20. Although Theorem 18 could be a corollary to Theorem 20, we include both proofs, which are very different. It is possible that the proof of Theorem 18 might be helpful in answering Question 5.

Definition 17. [6] A topological space X is *monotonically T_2* if there is a function $g : X \times X \to \mathcal{T}_X$ assigning to each ordered pair (x, y) of distinct points in X an open neighborhood, $g(x, y) \subseteq X$, of x such that:

i.) $g(x, y) \cap g(y, x) = \phi$;

ii.) if $x \in \overline{\cup \{g(y, x) : y \in M\}}$, then $x \in \overline{M}$.

We will call such a g a *monotone T_2 operator* on X.

We say that X is *strongly monotonically T_2* if we add:

iii.) if $z \in g(x, y)$, then $g(z, y) \subseteq g(x, y)$.

Theorem 18. *If X is an M_0-space (X has a σ-closure preserving base of clopen sets), then X is strongly monotonically T_2.*

Proof: Let $\mathcal{C} = \cup_{i\in\omega}\mathcal{U}_i$ be a σ-closure preserving base of clopen sets for X. Ordering these, there is a κ such that for $\mathcal{C} = \{C_\alpha : \alpha < \kappa\}$, for all $\beta < \kappa$, we have that $\cup_{\alpha<\beta}C_\alpha$ is clopen. For $p \neq q$, define: $g(p, q) = C_{\alpha_{p,q}} \backslash \cup \{C_\beta : p \notin C_\beta, \beta < \alpha_{p,q}\}$, where $C_{\alpha_{p,q}}$ is the first member of $\mathcal{C}$ that contains p and misses q. Clearly, $g(p, q) \cap g(q, p) = \phi$ for all $p \neq q$. If $z \in g(p, q)$, then $C_{\alpha_{z,q}} = C_{\alpha_{p,q}}$, so $g(z, q) \subset g(p, q)$.

Now suppose that $p \notin \overline{M}$. Then we must show that $p \notin \overline{\cup \{g(x, p) : x \in M\}}$. Since $\mathcal{C}$ is a base, there is a $C_\gamma \in \mathcal{C}$ such that $p \in C_\gamma \subset X\backslash M$. Then for all $x \in M$, either $\alpha_{x,p} > \gamma$, in which case we have that $C_\gamma \cap g(x, p) = \phi$, or $g(x, p) \subset C_{\alpha_{x,p}} \subset \cup \{C_\lambda : p \notin C_\lambda, \lambda < \gamma\}$. Note that $\alpha_{x,p} = \lambda$ for some $\lambda < \gamma$. Then we have that $p \in (X\backslash \cup \{C_\lambda : p \notin C_\lambda, \lambda < \gamma\}) \cap$

$C_\gamma \subset \cup\{g(x,p) : x \in M\}$. Thus, $p \notin \overline{\cup\{g(x,p) : x \in M\}}$ and X is strongly monotonically T_2. □

Lemma 19. *The space X is stratifiable if and only if there is a function σ that assigns to each closed set H and each $n \in \omega$ an open set $\sigma(H,n)$ such that:*

i.) $H = \cap_{n\in\omega}\sigma(H,n) = \cap_{n\in\omega}\overline{\sigma(H,n)}$;

ii.) $\sigma(H,n+1) \subset \sigma(H,n)$;

iii.) if $K \subset \sigma(H,n)$, *then* $\sigma(K,n) \subset \sigma(H,n)$.

Proof: Since X is stratifiable and hence an M_2-space, there is a σ-closure preserving quasi-base of closed sets $\{G_i : i \in \omega\}$. We define $\sigma(H,n) = X \backslash \cup\{g \in G_i : g \cap H = \phi, i \leq n\}$. □

The function σ will be called a *nested stratification* of X.

Theorem 20. *If X is stratifiable, then X is strongly monotonically T_2.*

Proof: Let σ be a nested stratification of X. Since X is stratifiable, there is a coarser metric topology $\mathcal{T}$ on X. Let $\mathcal{G} = \{G_i\}$ be a development on $\mathcal{T}$ such that each G_i is locally finite. If $x \neq y$ are in X, let $n(x,y)$ denote the first integer such that $st\left(x, G_{n(x,y)}\right) \cap st\left(y, G_{n(x,y)}\right) = \phi$. Let $u(x,y) = \cap\{g \in G_j : x \in g,\, j \leq n(x,y)\}$ and $w(x,y) = u(x,y) \cap \sigma(\{x\}, n(x,y))$. Note that if $z \in u(x,y)$, then $n(z,y) \geq n(x,y)$ and $u(z,y) \subset u(x,y)$. Thus, we have $w(x,y) \cap w(y,x) = \phi$ and if $z \in w(x,y)$, then $w(z,y) \subset w(x,y)$.

It remains to show that if $x \notin \overline{K}$, then $x \notin \overline{\cup\{w(y,x) : y \in K\}}$. To this end, let N denote an integer such that $x \notin \overline{\sigma\left(\overline{K}, N\right)}$. For each $i \in \omega$, let $K_i = \{x \in K : n(y,x) = i\}$ and $W_i = \cup\{w(y,x) : y \in K_i\}$. By construction, if $i \geq N$, $W_i \subset \sigma\left(\overline{K}, N\right)$; and so, $x \notin \overline{\cup\{W_i : i \geq N\}}$. If $i \leq N$ and $y \in K_i$, then $w(x,y) \cap W_i = \phi$. □

Question 6. Would monotone normality plus strongly monotonically T_2 give strong monotone normality?

It is not known whether M_1-spaces are hereditary. In fact, if it could be shown that even closed subspaces of M_1-spaces were M_1-spaces, then this would answer the $M_3 \Rightarrow M_1$ question in the affirmative. We prove here that strong monotone normality and nested normality are also hereditary.

Theorem 21. *If X is strongly monotonically normal (nestedly normal) and T is a subspace of X, then T is strongly monotonically normal (nestedly normal).*

Proof: Let G be a strong monotone normality operator on X, satisfying condition (*b.*) of Theorem 5. We shall construct a strong monotone normality operator G' on T which also satisfies condition (*b.*) of Theorem 5. Let H and K be mututally separated sets in T. Then H and K are mutually separated in X. Define $G'(H,K) = G(H,K) \cap T$. The result follows easily. □

The proof for nested normality is essentially the same with G satisfying condition (*c.*) of Theorem 7.

It follows directly from the previous theorem, together with Theorem 6, that generalized order spaces (GO-spaces), another important class of topological spaces, are strongly monotonically normal.

Corollary 22. *Any generalized order space is strongly monotonically normal.*

The following theorem on adjunction spaces is an analog of those proved by Borges [1] for stratifiable spaces, by Miwa [22] for monotonically normal spaces, and by Moody and Roscoe [26] for acyclically monotonically normal spaces. We extend them to the strongly monotonically normal and nestedly normal cases. The approach is closer to that for the stratifiable and monotonically normal spaces than it is to Moody and Roscoe's approach in the acyclic monotone normality case. It essentially follows the proof given in [22], with some modifications.

Definition 23. Let X and Y be topological spaces, let F be a closed subspace of X, and let $f : F \to Y$ be a continuous function. Let $X \cup Y$ denote the topological disjoint union of X and Y, and let Z be the quotient space which we get from identifying an element x belonging to F with $f(x)$ belonging to Y. Then $Z = X \cup_f Y$ is called the *adjunction space* of X and Y.

Theorem 24. *Let X and Y be strongly monotonically normal (nestedly normal) spaces, F a closed subspace of X, and $f : F \to Y$ be a continuous map. Then the adjunction space $Z = X \cup_f Y$ is strongly monotonically normal (nestedly normal).*

Proof: Let $h : X \to Z$ and $k : Y \to Z$ be the natural projections. If $A \subset Z$, we will let $A_X = h^{-1}(A)$ and $A_Y = k^{-1}(A)$. Also, let $A_F = A_X \cap F = h^{-1}(A) \cap F = f^{-1}(A_Y) = f^{-1}\left(k^{-1}(A)\right)$ and let $A_{X\backslash F} = A_X \backslash F$. Then we have that $A_X = A_F \cup A_{X\backslash F}$.

Let G_X and G_Y be strong monotone normality operators for X and Y, respectively, which satisfy condition $(b.)$ of Theorem 5. Let A and B be separated subsets of Z; then A_Y and B_Y are separated subsets of Y, and A_X and B_X are separated subsets of X. Hence, $A_{X\backslash F}$ and $B_{X\backslash F}$ are also separated subsets of X.

Now we define $A_1 = A_{X\backslash F} \cup f^{-1}(G_Y(A_Y, B_Y))$ and $B_1 = B_{X\backslash F} \cup \left(F\backslash f^{-1}(cl_Y G_Y(A_Y, B_Y))\right)$. It is not hard to see that A_1 and B_1 are separated in X.

Since A_1 and B_1 are separated in X, there exists a $G_X(A_1, B_1)$ which is open in X, such that $A_1 \subset G_X(A_1, B_1) \subset cl_X G_X(A_1, B_1) \subset X\backslash B_1$. Hence, there is a V open in X such that $V \cap F = f^{-1}(G_Y(A_Y, B_Y))$, so we let $U_A = (G_X(A_1, B_1) \cap V) \cup (G_X(A_1, B_1)\backslash F)$. Then $U_A \backslash F = G_X(A_1, B_1)\backslash F$ and $U_A \cap F = f^{-1}(G_Y(A_Y, B_Y))$. Note that $U_A \subset G_X(A_1, B_1)$ and define $G(A, B) = h(U_A\backslash F) \cup k(G_Y(A_Y, B_Y))$, an open set in Z. We must show that G is a strong monotone normality operator on Z.

Suppose that A and B are separated in Z. Let z be an element of A. If $z \in k(Y)$, we have that $z = k\left(k^{-1}(z)\right) \in k(G_Y(A_Y, B_Y)) \subset G(A, B)$. If $z \notin k(Y)$, we have that $z = h\left(h^{-1}(z)\right) \in h\left(A_{X\backslash F}\right) \subset h(U_A\backslash F) \subset G(A, B)$. Thus, we have that $A \subset G(A, B) \subset \overline{G(A, B)}$. Now $B \subset G(B, A) = h(U_B\backslash F) \cup k(G_Y(B_Y, A_Y))$, with $(U_A\backslash F) \cap (U_B\backslash F) = \phi$ and $G_Y(A_Y, B_Y) \cap G_Y(B_Y, A_Y) = \phi$. Therefore, $G(B, A)$ is an open neighborhood of B disjoint from $G(A, B)$ and $A \subset G(A, B) \subset \overline{G(A, B)} \subset X\backslash B$. This gives us conditions $(i.)$ and $(iii.)$ of $(b.)$ in Theorem 5.

Now let A' and B' be separated sets in Z, with $A \subset A'$ and $B \supset B'$. Then we have that $k(G_Y(A_Y, B_Y)) \subset k(G_Y(A'_Y, B'_Y))$, that $A_X\backslash F \subset A'_X\backslash F$, and that $B_X\backslash F \supset B'_X\backslash F$. We also have that $F\backslash f^{-1}(cl_Y G_Y(A'_Y, B'_Y)) \subset F\backslash f^{-1}(cl_Y G_Y(A_Y, B_Y))$, so we have $A_1 \subset A'_1$ and $B_1 \supset B'_1$. Therefore, we have that $G_X(A_1, B_1) \subset G_X(A'_1, B'_1)$. Hence, $U_A\backslash F = G_X(A_1, B_1)\backslash F \subset G_X(A'_1, B'_1)\backslash F = U_{A'}\backslash F$. Thus, it follows that $G(A, B) = (h(U_A\backslash F) \cup k(G_Y(A_Y, B_Y))) \subset (h(U_{A'}\backslash F) \cup k(G_Y(A'_Y, B'_Y))) = G(A', B')$.

Finally, suppose $A' \subset G(A,B) = h(U_A \backslash F) \cup\ k(G_Y(A_Y, B_Y))$. Now, $A'_{X\backslash F} =$

$$h^{-1}\left(A'\right)\backslash F \subset U_A\backslash F = G_X\left(A_1, B_1\right)\backslash F \subset G_X\left(A_1, B_1\right)$$

and we have

$$f^{-1}\left(G_Y\left(A'_Y, B_Y\right)\right) \subset f^{-1}\left(G_Y\left(A_Y, B_Y\right)\right) \subset A_1 \subset G_X\left(A_1, B_1\right).$$

Thus, $A'_1 = A'_{X\backslash F} \cup f^{-1}\left(G_Y\left(A'_Y, B_Y\right)\right) \subset G_X\left(A_1, B_1\right)$. Also, it follows that

$$B'_1 = B_{X\backslash F} \quad \cup \quad (F\backslash f^{-1}(cl_Y G_Y(A'_Y, B_Y)))$$
$$\supset B_{X\backslash F}\ \cup\ \left(F\backslash f^{-1}\left(cl_Y G_Y\left(A_Y, B_Y\right)\right)\right) = B_1,$$

so by the preceding paragraph, $G_X\left(A'_1, B'_1\right) \subset G_X\left(A'_1, B_1\right) \subset G_X\left(A_1, B_1\right)$. Then $U_{A'}\backslash F = G_X\left(A'_1, B'_1\right)\backslash F \subset G_X\left(A_1, B_1\right)\backslash F = U_A\backslash F$. So $G\left(A', B\right) = h(U_{A'}\backslash F) \cup k(G_Y(A'_Y, B_Y)) \subset h\left(U_A\backslash F\right) \cup k(G_Y(A_Y, B_Y)) = G(A, B)$.

Therefore, G satisfies (*b.*) of Theorem 5 and is, therefore, a strong monotone normality operator on $Z = X \cup_f Y$. □

The proof for nested normality is essentially the same with G satisfying the conditions of (*c.*) in Theorem 7.

Two corollaries on stratifiable and monotonically normal spaces being absolute (neighborhood) extensors if and only if they are absolute (neighborhood) retracts carry over to strongly monotonically normal (nestedly normal) spaces.

Definition 25. By an *absolute retract* for the class $\mathcal{C}$ of topological spaces, denoted $AR\left(\mathcal{C}\right)$, we mean a topological space Y belonging to the class $\mathcal{C}$ such that every homeomorphic image of Y as a closed subspace of a space Z belonging to the class $\mathcal{C}$ is necessarily a retract of Z. By an *absolute neighborhood retract* for the class $\mathcal{C}$, denoted $ANR\left(\mathcal{C}\right)$, we mean a space Y belonging to $\mathcal{C}$ such that every homeomorphic image of Y as a closed subspace of a space Z belonging to $\mathcal{C}$ is necessarily a neighborhood retract of Z.

Definition 26. A topological space Y is called an *absolute extensor* for the class $\mathcal{C}$ of topological spaces, denoted $AE\left(\mathcal{C}\right)$, if whenever X is a topological space belonging to the class $\mathcal{C}$ and F is a closed set in X, then every continuous map f from F into Y can be extended to a continuous map from X into Y. A topological space Y is called an *absolute neighborhood extensor* for the class $\mathcal{C}$, denoted $ANE\left(\mathcal{C}\right)$, if whenever X is a topological space belonging to the class $\mathcal{C}$ and F is a closed set in X, then every continuous map f from F into Y can be extended to a continuous map from some open neighborhood of F into Y.

Corollary 27. *Let Y be a strongly monotonically normal space. Then Y is an $AR(SMN)$ if and only if Y is an $AE(SMN)$.*

Proof: Suppose that Y is an $AR(SMN)$. Let X be a strongly monotonically space and A be a closed subset of X, with $f : A \to Y$ a continuous map. Then by Theorem 24, the adjunction space is strongly monotonically normal. Y is closed in $X \cup_f Y$, and $h^{-1}(Y) = A$ is closed in X ($h : X \to X \cup_f Y$ is the natural projection). Since Y is an absolute retract, there is a $g : X \cup_f Y \to Y$ that keeps Y pointwise fixed. Then $g \circ h : X \to Y$ is a continuous extension of f. Thus, Y is an $AE(SMN)$.

Now let Y be an $AE(SMN)$ and Y be closed in X, which is strongly monotonically normal. Consider $i : Y \to Y$. Since Y is an absolute extensor, there is an $h : X \to Y$ that extends i. Then h is onto and $h(y) = y$ for all $y \in Y$. Thus, Y is an $AR(SMN)$. □

Corollary 28. *Let Y be a strongly monotonically normal space. Then Y is an $ANR(SMN)$ if and only if Y is an $ANE(SMN)$.*

Proof: Essentially the same as for the previous corollary. □

The following extension theorem for nestedly normal spaces (and hence for strongly monotonically normal spaces) holds.

Theorem 29. *Let X be a nestedly normal space. Let $\mathcal{C}_H$ denote the set of continuous functions f such that the domain of f, dom(f), is a closed subset of X and the range of f is $[0,1]$. Then there is a function $\phi : \mathcal{C}_H \to \mathcal{C}(X,[0,1])$ such that:*

i.) if $f \in \mathcal{C}_H$, then $\phi(f)(x) = f(x)$ for all $x \in$ dom(f);

ii.) if $f, g \in \mathcal{C}_H$, with dom$(g) \subseteq$ dom(f), and $\phi(g)(x) \le f(x)$ for all $x \in$ dom(f), then $\phi(g)(x) \le \phi(f)(x)$ for all $x \in X$.

Proof: Let G be a nested normality operator on X satisfying condition *(c.)* of Theorem 7. Let $\mathcal{P} = \{\frac{p}{2^q} : 0 \le p \le 2^q$ with p, q nonnegative integers$\}$, the dyadic rationals. Define $H_f(r) = f^{-1}([0,r))$ and $K_f(r) = f^{-1}((r,1])$, mutually separated sets in X. Now we inductively construct open sets $\{D_f(r) : r \in \mathcal{P}\}$ as follows: Let $D_f(0) = \varnothing$ and $D_f(1) = X$. Then for $p = 1, 2, \ldots, 2^N$, let

$\widetilde{H}_f\left(\frac{2p-1}{2^{N+1}}\right) = H_f\left(\frac{2p-1}{2^{N+1}}\right) \cup \overline{D_f\left(\frac{2p-2}{2^{N+1}}\right)} = H_f\left(\frac{2p-1}{2^{N+1}}\right) \cup \overline{D_f\left(\frac{p-1}{2^N}\right)}$ and $\widetilde{K}_f\left(\frac{2p-1}{2^{N+1}}\right) = K_f\left(\frac{2p-1}{2^{N+1}}\right) \cup X\backslash D_f\left(\frac{2p}{2^{N+1}}\right) = K_f\left(\frac{2p-1}{2^{N+1}}\right) \cup X\backslash D_f\left(\frac{p}{2^N}\right)$. It is clear that $\widetilde{H}_f\left(\frac{2p-1}{2^{N+1}}\right)$ and $\widetilde{K}_f\left(\frac{2p-1}{2^{N+1}}\right)$ are mutually separated. Now, let $D_f\left(\frac{2p-1}{2^{N+1}}\right) = G\left(\widetilde{H}_f\left(\frac{2p-1}{2^{N+1}}\right), \widetilde{K}_f\left(\frac{2p-1}{2^{N+1}}\right)\right)$. By the construction, we have that if $r_1, r_2 \in \mathcal{P}$ with $r_1 \leq r_2$, then $\overline{D_f(r_1)} \subseteq D_f(r_2)$. For $x \in X$, define $\phi(f)(x) = glb\{r \in \mathcal{P} : x \in D_f(r)\}$. It is straightforward to see that $\phi(f)(x) = f(x)$ for all $x \in dom(f)$ and that $\phi(f)$ is continuous.

Next, suppose that $f, g \in \mathcal{C}_H$, with $dom(g) \subseteq dom(f)$, and $\phi(g)(x) \leq f(x)$ for all $x \in dom(f)$. We need to show that $\phi(g)(x) \leq \phi(f)(x)$ for all $x \in X$, which will hold if we can show that $D_f(r) \subseteq D_g(r)$ for all $r \in \mathcal{P}$. Note that $D_g(0) = D_f(0) = \phi$ and that $D_g(1) = D_f(1) = X$. Suppose that $D_f(r) \subseteq D_g(r)$ for $r = \frac{p}{2^N}$, $p = 0, ..., 2^N$. Since $g(x) \leq f(x)$ for all $x \in dom(g) \subseteq dom(f)$, we have that $\widetilde{K}_g\left(\frac{2p-1}{2^{N+1}}\right) = K_g\left(\frac{2p-1}{2^{N+1}}\right) \cup X\backslash D_g\left(\frac{p}{2^N}\right) \subseteq K_f\left(\frac{2p-1}{2^{N+1}}\right) \cup X\backslash D_f\left(\frac{p}{2^N}\right) = \widetilde{K}_f\left(\frac{2p-1}{2^{N+1}}\right)$. Thus, by property ii.) of G, $D_f\left(\frac{2p-1}{2^{N+1}}\right) = G\left(\widetilde{H}_f\left(\frac{2p-1}{2^{N+1}}\right), \widetilde{K}_f\left(\frac{2p-1}{2^{N+1}}\right)\right) \subseteq$
$G\left(\widetilde{H}_f\left(\frac{2p-1}{2^{N+1}}\right), \widetilde{K}_g\left(\frac{2p-1}{2^{N+1}}\right)\right)$. Also, note that if $x \in H_f\left(\frac{2p-1}{2^{N+1}}\right)$, then $\phi(g)(x) \leq f(x) \leq \frac{2p-1}{2^{N+1}}$. Therefore, $x \in D_g\left(\frac{2p-1}{2^{N+1}}\right)$. By assumption, $\overline{D_f\left(\frac{p-1}{2^N}\right)} \subseteq \overline{D_g\left(\frac{p-1}{2^N}\right)}$. We now have that $\widetilde{H}_f\left(\frac{2p-1}{2^{N+1}}\right) \subseteq G\left(\widetilde{H}_g\left(\frac{2p-1}{2^{N+1}}\right), \widetilde{K}_g\left(\frac{2p-1}{2^{N+1}}\right)\right)$. By property iii.) of G, $D_f\left(\frac{2p-1}{2^{N+1}}\right) \subseteq G\left(\widetilde{H}_g\left(\frac{2p-1}{2^{N+1}}\right), \widetilde{K}_g\left(\frac{2p-1}{2^{N+1}}\right)\right) = D_g\left(\frac{p-1}{2^N}\right)$. □

Question 7. Does acyclic monotone normality imply strong monotone normality?

Question 8. What could be added to strong monotone normality to give acyclic monotone normality?

Question 9. Would stratifiable plus strong monotone normality be equivalent to (imply) M_1? If so, since stratifiability and strong monotone normality are hereditary, that would answer the $M_3 \Rightarrow M_1$ question.

References

[1] C. J. R. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.

[2] C. J. R. Borges, *Elastic spaces are monotonically normal*, Notices Amer. Math. Soc. **18** (1971), 840, Abstract #71T-G158.

[3] C. J. R. Borges, *Four generalizations of stratifiable spaces*, General Topology and Its Relation to Modern Analysis and Algebra. III, (Proc. Third Prague Topology Sympos., 1971), Academia, Prague, 73–76.

[4] C. J. R. Borges, *A study of monotonically normal spaces*, Proc. Amer. Math. Soc. **38** (1973), 211–214.

[5] R. E. Buck, *Monotone separation and basis properties*, Ph.D. Thesis, University of Pittsburgh, 1993.

[6] R. E. Buck, *Some weaker monotone separation and basis properties*, Topology Appl. **69** (1996), 1–12.

[7] J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.

[8] L. B. Fuller, *Trees and proto-metrizable spaces*, Pacific J. Math. **104** (1983), 55–75.

[9] P. Gartside and P. J. Moody, *A note on proto-metrizable spaces*, Top. Appl. **52** (1993), 1–9.

[10] G. Gruenhage, *Stratifiable spaces are M_2*, Top. Proc. **1** (1976), 221–226.

[11] G. Gruenhage, *On the $M_3 \Rightarrow M_1$ question*, Top. Proc. **5** (1980), 77–104.

[12] R. W. Heath, *Extension Properties of Generalized Metric Spaces*, Lecture Notes - The University of North Carolina at Greensboro (1975).

[13] R. W. Heath and R. E. Hodel, *Characterizations of σ-spaces*, Fund. Math. **77** (1973), 271–275.

[14] R. W. Heath and H. J. K. Junilla, *Stratifiable spaces as subspaces and continuous images of M_1-spaces*, Proc. Amer. Math. Soc. **83** (1981), 146–148.

[15] R. W. Heath and D. J. Lutzer, *A note on monotone normality*, Notices Amer. Math. Soc. **18** (1971), 783, Abstract #687-54-1.

[16] R. W. Heath and D. J. Lutzer, *A characterization of monotone normality*, Notices Amer. Math. Soc. **18** (1971), 1066, Abstract #689-G8.

[17] R. W. Heath, D. J. Lutzer, and P. L. Zenor, *Monotonically normal spaces*, Trans. Amer. Math. Soc. **178** (1973), 481–493.

[18] R. W. Heath and P. L. Zenor, *Stronger forms of monotone normality I*, Abstract, Summer Conference on General Topology and Applications in Honor of Mary Ellen Rudin and Her Work, University of Wisconsin, Madison, 1991.

[19] R. W. Heath and P. L. Zenor, *Stronger forms of monotone normality II*, Abstract, Summer Conference on General Topology and Applications in Honor of Mary Ellen Rudin and Her Work, University of Wisconsin, Madison, 1991.

[20] H. J. K. Junilla, *Neighbornets*, Pacific J. Math. **765** (1978), 83–108.
[21] D. J. Lutzer, *On generalized ordered spaces*, Dissertations Math. **89** (1971).
[22] T. Miwa, *Adjunction spaces of monotonically normal spaces and spaces dominated by monotonically normal subsets*, Proc. Amer. Math. Soc. **87** (1983), 357–362.
[23] P. J. Moody, *Acyclic monotonically normal spaces*, Ph.D. Thesis, Oxford, 1988.
[24] P. J. Moody, *Monotone normality and topological groups*, preprint.
[25] P. J. Moody, G. M. Reed, A. W. Roscoe, and P. J. Collins, *A lattice of conditions on topological spaces II*, Fund. Math. **138** (1991), 69–81.
[26] P. J. Moody and A. W. Roscoe, *Acyclic monotone normality*, Top. Appl. **47** (1992), 53–67.
[27] P. J. Nyikos, *Some surprising base properties in topology*, Studies in Topology, Academic Press, 427–450.
[28] M. E. Rudin, *A cyclic monotonically normal space which is not* K_0, Proc. Amer. Math. Soc. **119** (1993), 303–307.
[29] M. E. Rudin, *Compact, separable, linearly ordered spaces*, Top. Appl. **82** (1998), 397–419.
[30] M. E. Rudin, *Zero dimensionality and monotone normality*, Top. Appl. **85** (1998), 319–333.
[31] M. E. Rudin, *Nikiel's conjecture*, Top. Appl. **116** (2001), 305–331.
[32] F. G. Slaughter, *The closed image of a metrizable space is* M_1, Proc. Amer. Math. Soc. **37** (1973), 309–314.
[33] P. Zenor, *Monotonically normal spaces*, Notices Amer. Math. Soc. **17** (1970), 1034, Abstract #679-G2.

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