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CARDINAL INVARIANTS RELATED TO STAR COVERING PROPERTIES

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Abstract. We investigate cardinal invariants which are related to star covering properties of a topological space. We compare them in the class of Tychonoff spaces and figure out how big the gaps between them are. In particular, we show that in most cases a cardinal invariant has no upper bounds when the cardinal invariants smaller than it are $\aleph_0$.

1. Introduction

Although the study of star covering properties of a topological space could be dated to 1970s or even earlier, a systematic investigation on them was done by van Douwen et al. [8] in the 1990s. Since then, this area has attracted many topologists. One of the most influential study in this area is Matveev's survey article [14]. The aim of the present paper is to study star covering properties from the point of view of cardinal functions. Let X be a topological space, and let $\mathcal{P}(X)$ denote the collection of all subsets of X. $[X]^{<\kappa}$ ($[X]^{\leq\kappa}$) is the collection of all subsets of X with cardinality $< \kappa$ ($\leq \kappa$). For $\mathcal{U} \subseteq \mathcal{P}(X)$ and $B \subseteq X$, let $\mathrm{st}(B,\mathcal{U}) = \mathrm{st}^1(B,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap B \neq \emptyset\}$; $\mathrm{st}^{n+1}(B,\mathcal{U}) = \mathrm{st}(\mathrm{st}^n(B,\mathcal{U}),\mathcal{U})$ for each $n \in \mathbb{N}$. As usual, we write $\mathrm{st}^n(x,\mathcal{U})$ for $\mathrm{st}^n(\{x\},\mathcal{U})$.

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Definition 1.1. [8, 14] A space X is called *n-starcompact* (*$n\frac{1}{2}$-starcompact*) if for every open cover $\mathcal{U}$ of X, there exists a finite $A \subseteq X$ ($\mathcal{V} \subseteq \mathcal{U}$) such that $\mathrm{st}^n(A,\mathcal{U}) = X$ ($\mathrm{st}^n(\bigcup \mathcal{V},\mathcal{U}) = X$).

Clearly, n-starcompact $\rightarrow n\frac{1}{2}$-starcompact $\rightarrow (n+1)$-starcompact. We notice that n-starcompact spaces are also called *n-pseudocompact* in [1]. In [8], n-starcompact spaces are called *strongly n-starcompact*, while $n\frac{1}{2}$-starcompact spaces are named *n-starcompact*. For convenience, 1-starcompact spaces are simply called *starcompact*. Every countably compact space is starcompact [8].

Definition 1.2. [3, 13] A space X is *absolutely countably compact* if for every open cover $\mathcal{U}$ of X and every dense subset $D \subseteq X$, there exists a finite $F \subseteq D$ such that $\mathrm{st}(F,\mathcal{U}) = X$.

It is natural to extend properties in definitions 1.1 and 1.2 by introducing n-star-Lindelöf number $st_n\text{-}l(X)$, $n\frac{1}{2}$-star-Lindelöf number $st_{n\frac{1}{2}}\text{-}l(X)$, and absolute star-Liundeöf number $a\text{-}st\text{-}l(X)$ [3] as follows:

$st_n\text{-}l(X) = \aleph_0 \cdot \min\{\kappa :$ for every open cover $\mathcal{U}$ of X,
there exists an $A \in [X]^{\leq\kappa}$ such that $\mathrm{st}^n(A,\mathcal{U}) = X\}$;

$st_{n\frac{1}{2}}\text{-}l(X) = \aleph_0 \cdot \min\{\kappa :$ for every open cover $\mathcal{U}$ of X,
there exists a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $\mathrm{st}^n(\bigcup \mathcal{V},\mathcal{U}) = X\}$;

$a\text{-}st\text{-}l(X) = \aleph_0 \cdot \min\{\kappa :$ for every open cover $\mathcal{U}$ of X, every
dense $D \subseteq X$, there is an $A \in [D]^{\leq\kappa}$ such that $\mathrm{st}(A,\mathcal{U}) = X\}$.

A space X is said to be *n-star-Lindelöf* (*$n\frac{1}{2}$-star-Lindelöf*, *absolutely star-Lindeöf*, respectively) if $st_n\text{-}l(X) = \aleph_0$ ($st_{n\frac{1}{2}}\text{-}l(X) = \aleph_0$, $a\text{-}st\text{-}l(X) = \aleph_0$, respectively). We write $st\text{-}l(X)$ instead of $st_1\text{-}l(X)$. Another relevant cardinal function is the Aquaro number. Recall that the *Aquaro number* $Aqu(X)$ of a T_1 space X is the smallest infinite cardinal κ such that for each open cover $\mathcal{U}$ of X there is a closed and discrete $A \in [X]^{\leq\kappa}$ with $\mathrm{st}(A,\mathcal{U}) = X$. If $Aqu(X) = \aleph_0$, then X is called *discretely star-Lindelöf* [17].

In section 2, we shall investigate general relationships among cardinal invariants defined above, as well as cellularity, discrete cellularity, extent, etc. A new cardinal invariant called $\mathcal{L}$-star-Lindelöf number is introduced and studied in section 3. The last section is dedicated to investigate those cardinal invariants between discrete cellularity and cellularity. Specifically, we shall show the following fact: In the class of Tychonoff spaces, a particular cardinal invariant does not have upper bounds under the restriction that certain smaller cardinal invariants are countable. These improve some recent results of Matveev and Tree.

Throughout this paper, $D(\kappa)$ stands for the discrete space of cardinality κ. The cofinality of κ is denoted by $\mathrm{cf}(\kappa)$. Ordinals are always given the usual order topology. For undefined terms and symbols, refer to [9], [11] and [14].

2. Preliminary results

For a space X, $e(X)$, $c(X)$ and $l(X)$ shall stand for the extent, the cellularity, and the Lindelöf number of X, respectively. The *discrete cellularity* of X is defined as $dc(X) = \aleph_0 \cdot \sup\{\ |\mathcal{V}| : \mathcal{V}$ is a discrete open family in $X\}$ [2]. In addition, if every pairwise disjoint subfamily of a family $\mathcal{F} \subseteq \mathcal{P}(X)$ is countable, then $\mathcal{F}$ is said to be a *ccc-family* in X.

Proposition 2.1. *Let X be a space. Then the following (1)-(4) hold:*

(1) $st\text{-}l(X) \le a\text{-}st\text{-}l(X) \le l(X)$;

(2) $st_2\text{-}l(X) \le st_{1\frac{1}{2}}\text{-}l(X) \le st\text{-}l(X)$;

(3) $dc(X) \le c(X)$;

(4) $st_{n\frac{1}{2}}\text{-}l(X) \le st_n\text{-}l(X) \le st_{2\frac{1}{2}}\text{-}l(X) \le \min\{st_2\text{-}l(X), dc(X)\}$ for all $n \ge 3$.

If X is a T_1 space, then the following (5) and (6) hold:

(5) $st\text{-}l(X) \le Aqu(X) \le e(X) \le l(X)$;

(6) $dc(X) \le l(X)$.

Furthermore, if X is a regular space, then

(7) $st_{n\frac{1}{2}}\text{-}l(X) = st_n\text{-}l(X) = st_{2\frac{1}{2}}\text{-}l(X) = dc(X)$ for all $n \ge 3$.

Proof: Most of proofs are either straightforward or similar to those in [8] and [14]. For the sake of completeness, we shall include a simple proof of the inequality $st_{2\frac{1}{2}}\text{-}l(X) \le dc(X)$ here. Let κ be an infinite cardinal. Suppose $st_{2\frac{1}{2}}\text{-}l(X) > \kappa$. Then there exists an open cover $\mathcal{U}$ of X such that for any subfamily $\mathcal{V} \subseteq \mathcal{U}$ with $|\mathcal{V}| \le \kappa$, $\mathrm{st}^2(\bigcup \mathcal{V}, \mathcal{U}) \ne X$. By using transfinite induction, we can select a sequence $\{U_\alpha : \alpha < \kappa^+\} \subseteq \mathcal{U}$ such that $U_\alpha \cap \mathrm{st}((\bigcup_{\beta<\alpha} U_\beta), \mathcal{U}) = \emptyset$ for all $\alpha < \kappa^+$, where κ^+ is the smallest cardinal greater than κ. It can be checked readily that $\{U_\alpha : \alpha < \kappa^+\}$ is discrete, which implies $dc(X) > \kappa$. Since κ is arbitrary, then $st_{2\frac{1}{2}}\text{-}l(X) \leq dc(X)$. □

The *para-Lindelöf number* of a space X is defined as the smallest infinite cardinal κ such that each open cover of X has a locally κ open refinement [2]. Clearly, X is para-Lindelöf iff $pl(X) = \aleph_0$. Let $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{n\frac{1}{2} : n \in \mathbb{N}\}$.

Theorem 2.2. *For any space X and $n \in \widetilde{\mathbb{N}}$, $pl(X) \cdot st_n\text{-}l(X) = l(X)$.*

Proof: Since $st_{n\frac{1}{2}}\text{-}l(X) \leq st_n\text{-}l(X) \leq l(X)$ and $pl(X) \leq l(X)$, it suffices to prove $l(X) \leq pl(X) \cdot st_{n\frac{1}{2}}\text{-}l(X)$ for any $n \in \mathbb{N}$. To do this, let $pl(X) = \alpha$ and $st_{n\frac{1}{2}}\text{-}l(X) = \beta$. We shall show that $l(X) \leq \alpha \cdot \beta$. Let $\mathcal{U}$ be any open cover of X, then there exists an open refinement $\mathcal{V}_1$ of $\mathcal{U}$ such that every point $x \in X$ has an open neighbourhood O_x with $|\{V \in \mathcal{V}_1 : V \cap O_x \neq \emptyset\}| \leq \alpha$. So, we can choose an open refinement $\mathcal{W}_1$ of $\mathcal{V}_1$ such that $|\{V \in \mathcal{V}_1 : W \cap V \neq \emptyset\}| \leq \alpha$ for each $W \in \mathcal{W}_1$. By applying $pl(X) = \alpha$ to $\mathcal{W}_1$, there exists an open refinement $\mathcal{V}_2$ of $\mathcal{W}_1$ such that every point $x \in X$ has an open neighbourhood G_x with $|\{V \in \mathcal{V}_2 : V \cap G_x \neq \emptyset\}| \leq \alpha$. Again, we can select an open refinement $\mathcal{W}_2$ of $\mathcal{V}_2$ such that $|\{V \in \mathcal{V}_2 : W \cap V \neq \emptyset\}| \leq \alpha$ for each $W \in \mathcal{W}_2$. Continuing the above process in n many steps, we obtain two finite sequences $\{\mathcal{V}_k : 1 \leq k \leq n\}$ and $\{\mathcal{W}_k : 1 \leq k \leq n\}$ of open covers of X such that $\mathcal{W}_k \preceq \mathcal{V}_k \preceq \mathcal{W}_{k-1} \preceq \mathcal{V}_{k-1}$ for $2 \leq k \leq n$ and $|\{V \in \mathcal{V}_k : V \cap W \neq \emptyset\}| \leq \alpha$ for $W \in \mathcal{W}_k$ and $1 \leq \kappa \leq n$. Since $st_{n\frac{1}{2}}\text{-}l(X) = \beta$, there exists a $\mathcal{C}_n \in [\mathcal{W}_n]^{\leq\beta}$ such that $\mathrm{st}^n(\bigcup \mathcal{C}_n, \mathcal{W}_n) = X$. Thus, we have $|\{V \in \mathcal{V}_n : V \cap (\bigcup \mathcal{C}_n) \neq \emptyset\}| \leq \alpha \cdot \beta$, which can be combined with $X = \mathrm{st}^n(\bigcup \mathcal{C}_n, \mathcal{W}_n)$ to obtain a $\mathcal{C}_{n\text{-}1} \in [\mathcal{V}_n]^{\leq \alpha\cdot\beta}$ such that $\mathrm{st}^{n\text{-}1}(\bigcup \mathcal{C}_{n\text{-}1}, \mathcal{V}_n) = X$. On the other hand, each member of $\mathcal{C}_{n\text{-}1}$ is contained in some member of $\mathcal{W}_{n\text{-}1}$, so it intersects at most α many elements of $\mathcal{V}_{n\text{-}1}$. Inductively, we could define $\mathcal{C}_i = \{V \in \mathcal{V}_i : V \cap (\bigcup \mathcal{C}_{i+1}) \neq \emptyset\}$ for $0 \leq i < n$. Then $|\mathcal{C}_i| \leq \alpha \cdot \beta$ for $0 \leq i < n$ and $\mathcal{C}_0$ is a subcover of $\mathcal{V}_1$. Therefore, $l(X) \leq \alpha \cdot \beta$. □

Corollary 2.3. *A space X is Lindelöf if and only if it is para-Lindelöf and n-star-Lindelöf for some $n \in \widetilde{\mathbb{N}}$.*

Remark 2.4. Blair [2] has shown for any space X, $pl(X) \cdot dc(X) = l(X)$. Furthermore, Bonanzinga and Matveev [4] proved that a para-Lindelöf and $1\frac{1}{2}$-star-Lindelöf space is star-Lindelöf. It is clear that Theorem 2.2 improves both of these two results.

A family $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be *n-linked* (*centered*) if every $\mathcal{A} \in [\mathcal{F}]^{\leq n}$ ($\mathcal{A} \in [\mathcal{F}]^{<\omega}$) has nonempty intersection. The *n-linked Lindelöf number* $l_n l(X)$ (*centered Lindelöf number* $ct\text{-}l(X)$) of a space X is defined to be the smallest infinite cardinal κ such that every open cover of X has a subcover representable as the union of at most κ many n-linked (centered) subfamilies [5]. We simply write $l_2 l(X)$ as $ll(X)$. The *ccc-Lindelöf number* $ccc\text{-}l(X)$ of a space X is defined to be the smallest infinite cardinal κ such that every open cover $\mathcal{U}$ of X has a subcover $\mathcal{V}$ such that the cardinality of each pairwise disjoint subfamily of $\mathcal{V}$ is at most κ [5]. If $l_n l(X) = \aleph_0$ ($ct\text{-}l(X) = \aleph_0$, $ccc\text{-}l(X) = \aleph_0$, respectively), then X is called *n-linked Lindelöf* (*centered Lindelöf, ccc-Lindelöf*, respectively).

Proposition 2.5. *The following statements hold for any space X:*
(1) $ccc\text{-}l(X) \leq c(X)$;
(2) $ccc\text{-}l(X) \leq l_n l(X) \leq l_{n+1} l(X) \leq ct\text{-}l(X) \leq st\text{-}l(X)$ *for any* $n \geq 2$;
(3) $st_{1\frac{1}{2}}\text{-}l(X) \leq ccc\text{-}l(X)$.

Recall that a space X is (*strongly*) *collectionwise Hausdorff*, abbreviated as (strongly) CWH, if for every closed discrete set A in X, the points can be separated by a (discrete) disjoint collection of open sets.

Proposition 2.6. *Let X be a T_1 space.*
(1) If X is CWH, then $ccc\text{-}l(X) = e(X)$.
(2) If X is strongly CWH, then $st_2\text{-}l(X) = e(X)$.

Proof: Since (1) and (2) are similar, we only need to show (2). It suffices to show $e(X) \leq st_2\text{-}l(X)$. For any closed discrete subset $A = \{x_\alpha : \alpha < \kappa\}$ of X with cardinality κ. Since X is strongly CWH, then there exists a discrete family $\{U_\alpha : \alpha < \kappa\}$ of open subsets such that $U_\alpha \cap A = \{x_\alpha\}$ for each $\alpha < \kappa$. For any $x \notin A$, there exists an open neighbourhood $U(x)$ of x such that $U(x)$ intersects at most one element of $\{U_\alpha : \alpha < \kappa\}$. Consider the open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{U(x) \setminus A : x \notin A\}$ of X. Let $F \subseteq X$ with

$|F| < \kappa$. We can choose an $\alpha < \kappa$ such that $U_\alpha \cap (U(x) \setminus A) = \emptyset$ for all $x \in F \setminus A$, which implies $U_\alpha \cap \mathrm{st}(F,\mathcal{U}) = \emptyset$. Now, since U_α is the only element of $\mathcal{U}$ containing x_α, then $x_\alpha \notin \mathrm{st}^2(F,\mathcal{U})$. Therefore, $st_2\text{-}l(X) \geq |A|$. Since A is arbitrary, it follows that $e(X) \leq st_2\text{-}l(X)$. □

Remark 2.7. First, for regular strongly CWH spaces (thus, for collectionwise normal spaces), all cardinal functions mentioned in this section, except the cellularity, coincide. Second, it is noticed in [6, Remark 5.2] that the extent of a normal ccc-star-Lindelöf space can be arbitrarily large. Thus, the (strong) collectionwise Hausdorffness in Proposition 2.6 cannot be replaced by non-collectionwise separation properties, such as normality. Finally, we shall show in section 4 (see Remark 4.3) that the word "strongly" in Proposition 2.6 (2) cannot be dropped.

3. $\mathcal{L}$-star-Lindelöf number

Recall that a space X is said to be *$\mathcal{L}$-starcompact* if for every open cover $\mathcal{U}$ of X there exists a Lindelöf subspace $L \subseteq X$ such that $\mathrm{st}(L,\mathcal{U}) = X$ [10], [12]. By definition, we have the following implications:

$$\text{star-Lindelöf} \rightarrow \mathcal{L}\text{-starcompact} \rightarrow 1\tfrac{1}{2}\text{-star-Lindelöf}.$$

It is not difficult to find examples of Tychonoff spaces to show that none of the above implications is reversible. In this section, we shall study a new cardinal invariant and show that the gaps between the above three notions could be arbitrarily large.

Definition 3.1. The *$\mathcal{L}$-star-Lindelöf number* of a space X, denoted by $\mathcal{L}\text{-}st\text{-}l(X)$, is defined as the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of X there exists a subspace $L \subseteq X$ with $l(L) \leq \kappa$ and $\mathrm{st}(L,\mathcal{U}) = X$.

By definition, $st_{1\frac{1}{2}}\text{-}l(X) \leq \mathcal{L}\text{-}st\text{-}l(X) \leq st\text{-}l(X)$ for any space X. Obviously, a space X is $\mathcal{L}$-starcompact iff $\mathcal{L}\text{-}st\text{-}l(X) = \aleph_0$.

Theorem 3.2. *For each cardinal $\kappa \geq \aleph_0$, there exists an $\mathcal{L}$-starcompact Tychonoff space X with $st\text{-}l(X) \geq \kappa$.*

Proof: Let $X = (\beta(D(\kappa)) \times (\omega+1)) \setminus ((\beta(D(\kappa)) \setminus D(\kappa)) \times \{\omega\})$ be the subspace of $\beta(D(\kappa)) \times (\omega+1)$, where $\beta(D(\kappa))$ is the Stone-Čech compactification of $D(\kappa)$. Then X is $\mathcal{L}$-starcompact, since

$\beta(D(\kappa)) \times \omega$ is a dense Lindelöf subspace of X. Consider the open cover

$$\mathcal{U} = \{\{d\} \times (\omega+1) : d \in D(\kappa)\} \cup \{\beta(D(\kappa)) \times \omega\}$$

of X. For every subset F of X with $|F| < \kappa$, there exists a point $d^* \in D(\kappa)$ such that $F \cap (\{d^*\} \times (\omega+1)) = \emptyset$. Since $U = \{d^*\} \times (\omega+1)$ is the only element of $\mathcal{U}$ containing $\langle d^*, \omega\rangle$, then $\langle d^*, \omega\rangle \notin \mathrm{st}(F, \mathcal{U})$. This shows that $st\text{-}l(X) \geq \kappa$. □

In fact, the conclusion of Theorem 3.2 can be improved slightly (see the remark after Corollary 4.2). Next, we shall show that the $\mathcal{L}$-star-Lindelöf number of a $1\frac{1}{2}$-star-Lindelöf Tychonoff space could be arbitrarily large. To achieve this, we need some preparation. For each $\alpha < \kappa$, let $z_\alpha \in {}^\kappa 2$ be the point such that $z_\alpha(\alpha) = 1$ and $z_\alpha(\beta) = 0$ for all $\beta \in \kappa \setminus \{\alpha\}$. Let $Z = \{z_\alpha : \alpha < \kappa\}$. Define a subspace Y of ${}^\kappa 2 \times (\omega+1)$ as

$$Y = ({}^\kappa 2 \times (\omega+1)) \setminus (({}^\kappa 2 \setminus Z) \times \{\omega\}).$$

Then $Z \times \{\omega\} \subseteq Y$ is closed discrete with $|Z \times \{\omega\}| = \kappa$. Hence, $e(Y) = \kappa$.

Lemma 3.3. [15] *Assume that there exists a family $\{U_\alpha : \alpha < \kappa\}$ of open subsets in ${}^\kappa 2$ such that $z_\alpha \in U_\alpha$ for each $\alpha < \kappa$. Then there exists a countable $S \subseteq {}^\kappa 2$ such that $S \cap U_\alpha \neq \emptyset$ for all $\alpha < \kappa$ and $(Cl_{{}^\kappa 2} S) \cap Z = \emptyset$, where $Cl_{{}^\kappa 2} S$ denotes the closure of S in ${}^\kappa 2$.*

Theorem 3.4. *For every cardinal $\kappa \geq \aleph_0$, there exists a $1\frac{1}{2}$-star-Lindelöf Tychonoff space X with $\mathcal{L}$-st-$l(X) \geq \kappa$.*

Proof: We may assume $\mathrm{cf}(\kappa) > \omega$ (otherwise, select a cardinal $k' > k$ with $\mathrm{cf}(\kappa') > \omega$). Let Y be the space given in the above. Now, define X as

$$X = (Y \times \kappa) \setminus (Z \times \{\omega\} \times (\kappa \setminus \{0\})).$$

We topologize X as follows: ${}^\kappa 2 \times \omega \times \kappa$ has the usual product topology and is an open subspace of X. A basic neighborhood of any point in X with $\langle z_\alpha, \omega, 0\rangle \in Z \times \{\omega\} \times \{0\}$ takes the form

$$G_{U,n,\beta}(\langle z_\alpha, \omega, 0\rangle) = ((U \cap Z) \times \{\omega\} \times \{0\}) \cup (U \times [n, \omega) \times [\beta, \kappa)),$$

where U is a neighborhood of z_α in ${}^\kappa 2$, $n < \omega$ and $\beta < \kappa$. For each $n < \omega$, let $X_n = {}^\kappa 2 \times \{n\} \times \kappa$ and $X_\omega = Z \times \{\omega\} \times \{0\}$.

Then $X = X_\omega \cup \bigcup_{n<\omega} X_n$, and X equipped with this topology is Tychonoff.

To show $st_{1\frac{1}{2}}\text{-}l(X) = \omega$, let $\mathcal{U}$ be an open cover of X. By refining $\mathcal{U}$ or taking a subcover of $\mathcal{U}$, we may assume that $\mathcal{U} = \mathcal{V}_\omega \cup \bigcup_{n<\omega} \mathcal{V}_n$, where $\mathcal{V}_\omega$ and $\mathcal{V}_n$ $(n < \omega)$ are defined as follows: $\mathcal{V}_\omega = \{V_\alpha : \alpha < \kappa\}$, where each V_α takes the form $V_\alpha = G_{U_\alpha, n_\alpha, \beta_\alpha}(\langle z_\alpha, \omega, 0\rangle)$ for some open neighborhood U_α of z_α in ${}^\kappa 2$, some $n_\alpha \in \omega$ and some $\beta_\alpha < \kappa$; $\mathcal{V}_n = \{V(x) : x \in X_n\}$, where each $V(x)$ is an open neighborhood of x in X_n. For each $n \in \omega$, since X_n is countably compact, we can find a finite subfamily $\mathcal{V}'_n \subseteq \mathcal{V}_n$ such that $X_n \subseteq \mathrm{st}(\bigcup \mathcal{V}'_n, \mathcal{U})$. If we put $\mathcal{W} = \bigcup\{\mathcal{V}'_n : n \in \omega\}$, then $\mathcal{W}$ is countable and ${}^\kappa 2 \times \omega \times \kappa \subseteq \mathrm{st}(\bigcup \mathcal{W}, \mathcal{U})$. By Lemma 3.3, there exists an $S \in [{}^\kappa 2]^{\leq\omega}$ such that $S \cap U_\alpha \neq \emptyset$ for every $\alpha < \kappa$. Define $T = S \times \omega$. For each $\alpha < \kappa$, there exists a point $f(\alpha) \in T$ such that $\{f(\alpha)\} \times [\beta_\alpha, \kappa) \subseteq V_\alpha$. Let $P = \{f(\alpha) : \alpha < \kappa\}$. Now, $f : \kappa \to P$ is a surjective mapping with $|P| \leq \omega$. For each $p \in P$, choose an $\alpha(p) \in f^{-1}(p)$. Let $\mathcal{W}' = \{V_{\alpha(p)} \in \mathcal{V}_\omega : p \in P\}$. Then $\mathcal{W}' \in [\mathcal{V}_\omega]^{\leq\omega}$. Since $\langle z_\alpha, \omega, 0\rangle \in V_\alpha$ and $\{f(\alpha)\} \times [\beta_\alpha, \kappa) \subseteq V_\alpha \cap V_{\alpha(f(\alpha))}$, we obtain $X_\omega \subseteq \mathrm{st}(\bigcup \mathcal{W}', \mathcal{U})$. Consequently, if we put $\mathcal{G} = \mathcal{W} \cup \mathcal{W}'$, then $\mathcal{G} \in [\mathcal{U}]^{\leq\omega}$ and $X = \mathrm{st}(\bigcup \mathcal{G}, \mathcal{U})$. Hence, $st_{1\frac{1}{2}}\text{-}l(X) = \omega$.

Next, we show $\mathcal{L}\text{-}st\text{-}l(X) \geq \kappa$. Since $Z \times \{\omega\}$ is closed and discrete in Y, then there exists an open set U_α in ${}^\kappa 2$ and a $n_\alpha < \omega$ such that $(U_\alpha \times [n_\alpha, \omega]) \cap (Z \times \{\omega\}) = \{\langle z_\alpha, \omega\rangle\}$ for every $\alpha < \kappa$. Let us consider the open cover $\mathcal{U}$ of X defined by

$$\mathcal{U} = \{G_{U_\alpha, n_\alpha, \alpha}(\langle z_\alpha, \omega, 0\rangle) : \alpha < \kappa\} \cup \{{}^\kappa 2 \times \omega \times [0, \alpha) : \alpha < \kappa\}.$$

For every subspace $F \subseteq X$ with $l(F) < \kappa$, since $|F \cap (Z \times \{\omega\} \times \{0\})| < \kappa$, we can pick some $\beta' < \kappa$ such that $F \cap \{\langle z_\alpha, \omega, 0\rangle : \alpha > \beta'\} = \emptyset$. For each $n < \omega$, there exists a $\beta_n < \kappa$ such that $F \cap ({}^\kappa 2 \times \{n\} \times [\beta_n, \kappa)) = \emptyset$. Hence, we can pick $\beta'' < \kappa$ such that $\beta'' > \beta_n$ for all $n \in \omega$. Then, $({}^\kappa 2 \times \omega \times [\beta'', \kappa)) \cap F = \emptyset$. Pick an ordinal $\beta > \max\{\beta', \beta''\}$, then $G_{U_\beta, n_\beta, \beta}(\langle z_\beta, \omega, 0\rangle) \cap F = \emptyset$. Since $G_{U_\beta, n_\beta, \beta}(\langle z_\beta, \omega, 0\rangle)$ is the only element of $\mathcal{U}$ containing $\langle z_\beta, \omega, 0\rangle$, $\langle z_\beta, \omega, 0\rangle \notin \mathrm{st}(F, \mathcal{U})$. This implies $\mathcal{L}\text{-}st\text{-}l(X) \geq \kappa$. $\square$

4. Cardinal invariants between discrete cellularity and cellularity

In this section, we shall distinguish those cardinal invariants which are between discrete cellularity and cellularity in the class of Tychonoff spaces. Let X be a Tychonoff space. From the discussion in section 2,

$$dc(X)(= st_{2\frac{1}{2}}\text{-}l(X)) \leq st_2\text{-}l(X) \leq st_{1\frac{1}{2}}\text{-}l(X) \leq ccc\text{-}l(X) \leq c(X).$$

Now, for any infinite cardinal κ with $\mathrm{cf}(\kappa) > \omega$, the *Noble plank of* X, which is denoted by $\mathbf{N}_\kappa[X]$, is constructed by

$$\mathbf{N}_\kappa[X] = (\beta X \times (\kappa + 1)) \setminus ((\beta X \setminus X) \times \{\kappa\}),$$

where βX stands for the Stone-Čech compactification of X. It is known that $\mathbf{N}_\kappa[X]$ is a pseudocompact space which contains $X \times \{\kappa\}$ as a closed subspace [19], [22].

Theorem 4.1. *Let κ and τ be two infinite cardinals.*

(1) If κ is a regular cardinal, then $\mathbf{N}_\kappa[D(\kappa)]$ is a 2-starcompact space with $st_{1\frac{1}{2}}\text{-}l(\mathbf{N}_\kappa[D(\kappa)]) = \kappa$.

(2) If τ is a cardinal with $\mathrm{cf}(\tau) > \kappa$, then $\mathbf{N}_\tau[D(\kappa)]$ is a $1\frac{1}{2}$-starcompact space with $ccc\text{-}l(\mathbf{N}_\tau[D(\kappa)]) = \kappa$.

Proof: (1) Since $\beta(D(\kappa)) \times \kappa$ is a dense countably compact subspace of $\mathbf{N}_\kappa[D(\kappa)]$, then $\mathbf{N}_\kappa[D(\kappa)]$ is 2-starcompact. Enumerate $D(\kappa)$ as $D(\kappa) = \{x_\alpha : \alpha < \kappa\}$. For every $\alpha < \kappa$, define open subsets U_α and V_α as $U_\alpha = \{x_\alpha\} \times (\alpha, \kappa]$ and $V_\alpha = \beta(D(\kappa)) \times [0, \alpha)$ respectively. Then $\mathcal{U} = \{U_\alpha : \alpha < \kappa\} \cup \{V_\alpha : \alpha < \kappa\}$ is an open cover of $\mathbf{N}_\kappa[D(\kappa)]$. Let $\mathcal{V}$ be a subcollection of $\mathcal{U}$ with $|\mathcal{V}| < \kappa$. Since κ is regular, there exists an $\alpha_* < \kappa$ such that $\alpha_* > \sup\{\alpha < \kappa : U_\alpha \in \mathcal{V}, V_\alpha \in \mathcal{V}\}$. Then $U_{\alpha_*} \cap (\bigcup \mathcal{V}) = \emptyset$. As U_{α_*} is the only element of $\mathcal{U}$ containing $\langle x_{\alpha_*}, \kappa\rangle$, $\langle x_{\alpha_*}, \kappa\rangle \notin \mathrm{st}(\bigcup \mathcal{V}, \mathcal{U})$. Thus, $st_{1\frac{1}{2}}\text{-}l(\mathbf{N}_\kappa[D(\kappa)]) \geq \kappa$. Since $\beta(D(\kappa)) \times \kappa$ is countably compact, $st_{1\frac{1}{2}}\text{-}l(\mathbf{N}_\kappa[D(\kappa)])$ is at most κ. Therefore, $st_{1\frac{1}{2}}\text{-}l(\mathbf{N}_\kappa[D(\kappa)]) = \kappa$.

(2) Since $\mathrm{cf}(\tau) > l(D(\kappa))$, by Theorem 36 (2) of [14] (cf. [14, p. 31]), the space $\mathbf{N}_\tau[D(\kappa)]$ is $1\frac{1}{2}$-starcompact. Let $U_x = \{x\} \times (\tau + 1)$ for each $x \in D(\kappa)$, and consider the open cover $\mathcal{U} = \{U_x : x \in D(\kappa)\} \cup \{\beta(D(\kappa)) \times \tau\}$ of $\mathbf{N}_\tau[D(\kappa)]$. Note that $\mathcal{U}$ is irreducible, so it has no proper subcover. Since $\{U_x : x \in D(\kappa)\}$ is a pairwise disjoint

subfamily of $\mathcal{U}$ with cardinality of κ, $ccc\text{-}l(\mathbf{N}_\tau[D(\kappa)]) \geq \kappa$. As in (1), $ccc\text{-}l(\mathbf{N}_\tau[D(\kappa)])$ is at most κ. Hence, $ccc\text{-}l(\mathbf{N}_\tau[D(\kappa)]) = \kappa$. $\square$

Corollary 4.2. *Let κ be an infinite cardinal. Then there exist*

(1) a 2-starcompact Tychonoff space X with $st_{1\frac{1}{2}}\text{-}l(X) \geq \kappa$;

(2) a $1\frac{1}{2}$-starcompact Tychonoff space Y with $ccc\text{-}l(Y) \geq \kappa$.

Remark 4.3. In fact, by the same argument as that in Theorem 4.1 (2), we can show $ccc\text{-}l(X) = \kappa$ for the space X defined in Theorem 3.2. Furthermore, we notice that if $\mathrm{cf}(\tau) > \kappa$ then $\mathbf{N}_\tau[D(\kappa)]$ is CWH, but not strongly CWH. By combining this observation and Theorem 4.1(2), we conclude that either $ccc\text{-}l(X)$ in Proposition 2.6 (1) cannot be replaced by the smaller cardinal function $\mathrm{st}_{1\frac{1}{2}}\text{-}l(X)$, or the strong collectionwise Hausdorffness in Proposition 2.6(2) cannot be weakened as collectionwise Hausdorffness.

It is fairly easy to find a ccc-Lindelöf Tychonoff (even normal) space whose cellularity is equal to any given cardinal, for example, any uncountable cardinal κ with the order topology. Recently, Matveev [18] also gave an example of Tychonoff spaces with properties described in Corollary 4.2 (1). In [21], Tree gave a pseudocompact space Y^+ which is not 2-star-Lindelöf. We observe that $st_2\text{-}l(Y^+)$ is precisely $2^{\aleph_0}$. In addition, Matveev [18] also pointed out that the difference between $dc(X)$ and $st_2\text{-}l(X)$ is at least one exponential step for Tychonoff spaces. Thus, the following natural question arises.

Question 4.4. How big can the difference be between $dc(X)$ and $st_2{-}l(X)$ for a Tychonoff space X?

We shall answer this question in our next theorem by showing that the difference between discrete cellularity and 2-star-Lindelöf number is arbitrarily large in the class of Tychonoff spaces. Clearly, this also improves the main result of Tree in [21]. The key idea is to use a well-known construction of Shakhmatov in [20]. Roughly speaking, Shakhmatov started with any zero-dimensional and non-pseudocompact T_1 space X_0 with a point-countable base and without isolated points, then constructed an increasing chain $(X_\alpha : \alpha < \delta)$ of zero-dimensional T_1 space with point-countable bases and without isolated points, where δ ($\mathrm{cf}(\delta) > \omega$) is large enough such that each sequence of locally finite open sets in X_β ($\beta < \delta$)

has a cluster point in X_α for some $\beta < \alpha < \delta$. Then $X = \bigcup_{\alpha<\delta} X_\alpha$ is a pseudocompact zero-dimensional and T_1 space with a point-countable base which contains X_0 as a closed subspace.

Theorem 4.5. *For every infinite cardinal κ, there exists a pseudocompact Tychonoff space $\mathbf{S}[X]$ (hence, $dc(\mathbf{S}[X]) = \aleph_0$) with st_2-l $(\mathbf{S}[X]) \geq \kappa$.*

Proof: Without loss of generality, we may assume that κ is regular. Let Z be any zero-dimensional and non-pseudocompact T_1 space with a point-countable base and without isolated points (For instance, Z can be the space of rationals, or the Pixley-Roy space over reals). Let X_0 be the free sum of κ many copies of Z, that is, $X_0 = \bigoplus_{\lambda<\kappa} Z_\lambda$. Using the method in [20], we can define an increasing chain $(X_\alpha : \alpha < \delta)$ of zero-dimensional T_1 spaces with point-countable bases. For every $\alpha < \delta$, we assign a triple $(B_\alpha, \mathcal{B}_\alpha, \theta_\alpha)$ to X_α, where B_α and $\mathcal{B}_\alpha$ are sets such that $\mathcal{B}_\alpha = B_\alpha \times \mathbb{N}$, and $\theta_\alpha : \mathcal{B}_\alpha \to \mathcal{P}(X_\alpha) \setminus \{\emptyset\}$ is a mapping such that $\theta_\alpha(\mathcal{B}_\alpha)$ is a point-countable base consisting of clopen sets for the space X_α. Furthermore, as it is done in [20], the family $\{(B_\alpha, \mathcal{B}_\alpha, \theta_\alpha) : \alpha < \delta\}$ can be chosen to satisfy the following additional properties (a)-(f) for any $\beta < \alpha < \delta$:

(a) $B_\beta \subseteq B_\alpha$ and $\theta_\alpha(b, n) \cap X_\beta = \emptyset$ whenever $(b, n) \in \mathcal{B}_\alpha \setminus \mathcal{B}_\beta$,

(b) $\theta_\alpha(b, n) \cap X_\beta = \theta_\beta(b, n)$ whenever $(b, n) \in \mathcal{B}_\beta$,

(c) for each $x \in X_\alpha \setminus X_\beta$, there is $(b, n) \in \mathcal{B}_\alpha \setminus \mathcal{B}_\beta$ with $x \in \theta_\alpha(b, n)$,

(d) for each $x \in X_\beta$, there is $(b, n) \in \mathcal{B}_\beta$ with $x \in \theta_{\beta+1}(b, n) \subseteq X_\beta$,

(e) $\theta_\beta(a, m) \cap \theta_\beta(b, n) = \emptyset$ implies $\theta_\alpha(a, m) \cap \theta_\alpha(b, n) = \emptyset$,

(f) for any family $\{(b_n, i_n) : n \in \mathbb{N}\} \subseteq \mathcal{B}_\beta$, there exists an ordinal γ such that $\beta < \gamma < \delta$ and some $z^* \in X_\gamma$ such that if $(a, j) \in \mathcal{B}_\gamma$ and $z^* \in \theta_\gamma(a, j)$, then $\{n \in \mathbb{N} : \theta_\gamma(a, j) \cap \theta_\gamma(b_n, i_n) \neq \emptyset\}$ must be infinite.

Let $\mathbf{S}[X] = \bigcup_{\alpha<\delta} X_\alpha$, $B = \bigcup_{\alpha<\delta} B_\alpha$, and $\mathcal{B} = B \times \mathbb{N}$. We define a mapping $\theta : \mathcal{B} \to \mathcal{P}(\mathbf{S}[X]) \setminus \{\emptyset\}$ such that $\theta(b, n) = \bigcup\{\theta_\alpha(b, n) : \alpha < \delta, (b, n) \in \mathcal{B}_\alpha\}$ for all $(b, n) \in \mathcal{B}$. As it is observed in [20], $\theta(\mathcal{B})$ is a point-countable base consisting of clopen sets for a pseudocompact T_1 topology on $\mathbf{S}[X]$. Next, we shall show that st_2-$l(\mathbf{S}[X]) \geq \kappa$. To do this, let $Y_{\alpha+1} = X_{\alpha+1} \setminus X_\alpha$ for each $\alpha < \delta$, and $Y_0 = X_0$. If $x \in \mathbf{S}[X]$, then there exists a unique $\alpha < \delta$ such that $x \in Y_\alpha$. For each $\alpha < \delta$ and $x \in Y_{\alpha+1}$, by (b)-(d), we can pick a pair $(b(x), n(x)) \in \mathcal{B}_{\alpha+1}$ such that

(g) $x \in \theta_{\alpha+1}(b(x), n(x)) \subseteq Y_{\alpha+1}$, and $\theta_{\alpha+2}(b(x), n(x)) \subseteq X_{\alpha+1}$.

For each $x \in X_0$, pick a pair $(b(x), n(x)) \in \mathcal{B}_0$ with $x \in \theta_1(b(x), n(x)) \subseteq X_0$. It follows from (e) and (g) that $\theta(b(x), n(x)) \cap \theta(b(y), n(y)) = \emptyset$ whenever $x \in Y_\alpha, y \in Y_\beta$ for $\alpha \neq \beta$. Now, $\mathcal{U} = \{\theta(b(x), n(x)) : x \in \mathbf{S}[X]\}$ is an open cover of $\mathbf{S}[X]$. For an arbitrary $A \subseteq \mathbf{S}[X]$ with $|A| < \kappa$, define

$$\mathcal{V} = \{\theta(b(x), n(x)) : \theta(b(x), n(x)) \cap A \neq \emptyset \text{ and } x \in \mathbf{S}[X]\},$$
$$\alpha_0 = \sup\{\alpha < \delta : \theta(b(x), n(x)) \in \mathcal{V} \text{ for some } x \in Y_\alpha\}.$$

Let $\Delta = \{\lambda < \kappa : (\bigcup \mathcal{V}) \cap Z_\lambda \neq \emptyset\}$. Since $\mathcal{V}$ is a point-countable family of subsets of $\mathbf{S}[X]$, then $|\mathcal{V}| = |A| < \kappa$. By the regularity of κ, $\alpha_0 < \kappa$ and $|\Delta| < \kappa$. Hence, by (f), we can select a locally finite sequence $(U_n : n < \omega)$ of nonempty open subsets in X_0 such that $U_n \subseteq \bigcup_{\lambda \in \kappa \setminus \Delta} Z_\lambda$ for all $n < \omega$, and $(U_n : n < \omega)$ has a cluster point $z^* \in Y_\alpha$ for some $\alpha \in \delta \setminus (\alpha_0 + 1)$. It can be easily checked that $z^* \notin \mathrm{st}^2(\bigcup \mathcal{V}, \mathcal{U})$, which implies $\mathrm{st}^2(\bigcup \mathcal{V}, \mathcal{U}) \neq \mathbf{S}[X]$. Therefore, $st_2\text{-}l(\mathbf{S}[X]) \geq \kappa$. □

We conclude the paper with some additional remarks. In [6], Cao and Song have considered some cardinal invariants between star-Lindelöf number and Lindelöf number for Tychonoff spaces, and have shown the gaps among them could be arbitrarily big. They also observed that the linked Lindelöf number of a normal *ccc*-Lindelöf space could be arbitrarily large. It is still not clear how big the gaps are among those cardinal invariants between linked Lindelöf number and centered Lindelöf number for Tychonoff spaces (even in larger classes of spaces).

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