Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 26, 2001-2002
Pages 97-125

NONSEPARATING SUBCONTINUA AND MAPPINGS

JANUSZ J. CHARATONIK

Abstract. Various kinds of nonseparating subcontinua of a given continuum, as end continua, extremal, terminal, absolutely terminal continua, and several related concepts are studied in the paper. A special attention is paid to their mapping properties with respect to a variety of classes of mappings: open, monotone, atomic, confluent, and many other applicable ones.

1. Introduction

One of outstanding early results in continuum theory, exploited as a useful tool in proofs of numerous theorems, says that the concept of an end point of a curve is an invariant under open mappings. The result has been extended in several directions. In a number of papers one can find studies of nonseparating points and/or nonseparating subcontinua, but with an emphasis on structural properties rather than to mapping ones, see e.g. [2], [28], [32], [34], [41], [47], [50], [54], [56], [57], [58], [64], [65].

Further, in the literature of the subject, the same name is often used for different objects; for example the term "end point" has various meanings in [3], [40], [50], etc. The same concerns the name "terminal continuum" in some papers. Thus it has become necessary to systematize these concepts, and to study some of their

2000 *Mathematics Subject Classification.* 54C10, 54E40, 54F15, 54F50, 54F55.

Key words and phrases. confluent, continuum, end, mapping, monotone, open, terminal.

mapping properties. This is just the main goal of the present paper. Special attention is paid to the following problem.

(1.1) Let $\mathfrak{C}$ be a family of nonseparating subcontinua of a given continuum, and let $\mathfrak{M}$ be a class of mappings between continua. For what $\mathfrak{C}$ and $\mathfrak{M}$ implication

$$A \in \mathfrak{C} \text{ and } f \in \mathfrak{M} \implies f(A) \in \mathfrak{C}$$

is true?

The paper consists of five chapters. After Introduction and Preliminaries, we consider properties of end points and of end continua in the third chapter. End points in the sense of Menger-Urysohn and in the classical sense are discussed first. Next, we study properties of ends in the sense of Bing, absolute end points, and ends in the sense of Bennett and Fugate. We close this chapter with E-continua in the sense defined by Miller. The fourth chapter is devoted to terminal continua. We start with terminal continua in the sense of Bennett and Fugate; next, we consider extremal continua as defined by Owens, followed by terminal continua in the sense of Gordh, and finally, terminal continua in the sense of Wallace. In the last chapter, a direction of a further study in the area is indicated.

2. Preliminaries

The term *space* means a topological Hausdorff space, and a *mapping* is a continuous function.

Given a subset A of a space X, we denote by card A, cl A, int A and bd A its cardinality, closure, interior and boundary, respectively. In a case when X is metric, the symbol diam A denotes the diameter of A.

A *continuum* means a compact connected space. A point p of a space X *disconnects* X *between some points* a *and* b *of* X provided that a and b belong to different components of the set $X \setminus \{p\}$. A point p of a space X *disconnects* X provided that there are two points of X such that p disconnects X between them. A continuum A is called an *arc* provided that there are exactly two points of A each of which does not disconnect A. These points are called the *end points* of the arc. A continuum X is said to be *arcwise connected* provided that for every two points of X there is an arc containing these points and contained in X. The union of two arcs having just their end points in common is called a *simple closed curve*.

A concept of an *order* of a point p in a space X (in the sense of Menger-Urysohn), written $\operatorname{ord}(p, X)$, is defined as follows. Let $\mathfrak{n}$ be a cardinal number. We write:

- $\operatorname{ord}(p, X) \leq \mathfrak{n}$ provided that for every open neighborhood U of p there is an open neighborhood V of p such that $V \subset U$ and $\operatorname{card} \operatorname{bd} V \leq \mathfrak{n}$;
- $\operatorname{ord}(p, X) = \mathfrak{n}$ provided that $\operatorname{ord}(p, X) \leq \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\operatorname{ord}(p, X) \leq \mathfrak{m}$ does not hold;
- $\operatorname{ord}(p, X) = \omega$ provided that for every open neighborhood U of p there are open neighborhoods V of p such that $V \subset U$ with finite boundaries $\operatorname{bd} V$ and the numbers $\operatorname{card} \operatorname{bd} V$ are not bounded by any positive integer n.

Thus for any continuum X we have

$$\operatorname{ord}(p, X) \in \{1, 2, \ldots, n, \ldots, \omega, \aleph_0, 2^{\aleph_0}\}$$

(convention: $\omega < \aleph_0$); see [40, §51, I, p. 274]. A point $p \in X$ such that $\operatorname{ord}(p, X) = 1$ is called an *end point* of X (in the sense of Menger-Urysohn); compare [66, Chapter IV, 1, p. 64].

Replacing in the above definition the point p by a subcontinuum P of the space X we get the concept of the (Menger-Urysohn) *order of a subcontinuum* P in the space X, writing $\operatorname{ord}(P, X)$.

By a *simple* $\mathfrak{n}$*-od* with the center p we mean the union of $\mathfrak{n}$ arcs every two of which have p as the only common point which is an end point of each of the arcs. Let a space X be given. A point $p \in X$ is said to be a *point of order at least* $\mathfrak{n}$ *in the classical sense* provided that p is the center of an $\mathfrak{n}$-od contained in X. We say that p is a *point of order* $\mathfrak{n}$ *in the the classical sense* provided that $\mathfrak{n}$ is the minimum cardinality for which the above condition is satisfied; see [4, p. 229]. In particular, if $\mathfrak{n} = 1$, then p is called an *end point of* X *(in the classical sense)*; see e.g., [41, §2, p. 301], [54] and [56]; if $\mathfrak{n} \geq 3$, then p is called a *ramification point of* X *(in the classical sense)*; see e.g., [4, p. 229] and [55].

A subcontinuum I of a continuum X is said to be *irreducible about a subset* $S \subset X$ provided that no proper subcontinuum of X contains S. A continuum X is said to be *irreducible* provided that there are two points a and b in X such that X is irreducible about

$\{a, b\}$. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two of its subcontinua is connected. A *dendrite* means a locally connected metric continuum containing no simple closed curve. A hereditarily unicoherent and arcwise connected metric continuum is called a *dendroid*. A dendroid X is said to be *smooth* provided that there exists a point $v \in X$ such that for each point $x \in X$ and each sequence of points x_n tending to x the sequence of arcs vx_n in X tends to the arc vx.

Given a space X and a point $p \in X$, the set C of all points of X that can be joined with p by a closed, connected proper subset of X is called a *composant* of p in X. By a *composant of the space* we mean a composant of some point in the space, see [40, §48, VI, p. 208] or [52, Exercise 5.20, p. 83].

A continuum X is said to be *acyclic* provided that each mapping from X into the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ (here $\mathbb{C}$ stands for the complex plane) is homotopic to a constant mapping; i.e., for all mappings $f : X \to \mathbb{S}^1$ and $c : X \to \{p\} \subset \mathbb{S}^1$, there exists a mapping $h : X \times [0, 1] \to \mathbb{S}^1$ such that for each point $x \in X$ we have $h(x, 0) = f(x)$ and $h(x, 1) = c(x)$. A *tree* means a one-dimensional acyclic connected polyhedron, i.e., a dendrite with finitely many end points. A continuum X is said to be *tree-like (arc-like)* provided that every open cover of X can be refined by a finite open cover whose nerve is a tree (an arc); equivalently, for the metric case, if for each $\varepsilon > 0$ there is a tree (an arc) T and a surjective mapping $f : X \to T$ such that f is an *ε-mapping*, (i.e., $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in T$). Let us mention that a metric continuum X is tree-like (arc-like) if and only if it is the inverse limit of an inverse sequence of trees (of arcs) with surjective bonding mappings; see [52, p. 24]; for the original definition using tree-chains see Bing's paper [3, p. 653].

Definitions of various concepts of end, terminal, extremal, etc. subcontinua (or points) are given at their proper places, in sections where they are discussed. Needed definitions of classes of mappings are collected below.

A surjective mapping $f : X \to Y$ between topological spaces is said to be:

– *open* provided that the images of open sets under f are open;

– *interior at a point* $p \in X$ provided that for every open set U

$p \in U$ implies $f(p) \in \operatorname{int} f(U)$; see [66, p. 149]; thus f is open if and only if it is interior at each point of its domain;
– a *local homeomorphism* provided that for each point $x \in X$ there exists an open neighborhood of x such that $f(U)$ is an open neighborhood of $f(x)$ and that $f|U : U \to f(U)$ is a homeomorphism; thus any local homeomorphism is open;
– *monotone* provided that for each point $y \in Y$, the set $f^{-1}(y)$ is connected;
– *quasi-monotone* provided that for each subcontinuum Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ has finitely many components, each of which is mapped onto Q under f;
– *feebly monotone* provided that if A and B are proper subcontinua of Y such that $Y = A \cup B$, then their inverse images $f^{-1}(A)$ and $f^{-1}(B)$ are connected;
– *atomic* provided that for each subcontinuum K of X such that the set $f(K)$ is nondegenerate condition $K = f^{-1}(f(K))$ holds, see [1]; any atomic mapping is known to be (hereditarily) monotone, see [31, Theorem 1, p. 49] and [44, 4.14, p. 17];
– *light* provided that for each point $y \in Y$ the set $f^{-1}(y)$ has one-point components (note that if the point-inverses are compact, this condition is equivalent to the property that they are zero-dimensional);
– *confluent* provided that for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto Q under f; see [5];
– *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;
– *weakly confluent* provided that for each subcontinuum Q of Y some component of $f^{-1}(Q)$ is mapped onto Q under f.

Given a class $\mathfrak{M}$ of mappings between continua, a mapping $f : X \to Y$ is said to be *hereditarily* $\mathfrak{M}$ provided that for each subcontinuum $K \subset X$ the partial mapping $f|K : K \to f(K) \subset Y$ is in $\mathfrak{M}$. The reader is referred to [44, Table II, p. 28] to see interrelations between most of the classes of mappings mentioned above.

A space X is said to be *homogeneous with respect to a class* $\mathfrak{M}$ *of mappings* provided that for every two points p and q of X there exists a surjective mapping $f : X \to X$ such that $f \in \mathfrak{M}$ and $f(p) = q$. If $\mathfrak{M}$ is the class of homeomorphisms, we get the well-known notion of a *homogeneous* space.

If a closed subset C of a continuum X is given, then X/C is the quotient space obtained by shrinking C to a point. Thus, if C is a continuum, the quotient mapping $q : X \to X/C$ is monotone. See [66, Chapter 7, p. 122] for the details. If $f : X \to Y$ is a surjection between continua, K is a subcontinuum of X, and $q : X \to X/K$ and $r : Y \to Y/f(K)$ are the quotient mappings, then f determines the *induced mapping* $f_* : X/K \to Y/f(K)$ defined by $f_*(q(x)) = r(f(x))$ for each $x \in X$ (see e.g. [30, Theorem 7.7, p. 17]).

3. End continua

3A. Around Whyburn's theorem. A classical Whyburn's result says that order of a point in a locally compact space cannot be increased under an open mapping [66, Corollary 7.31, p. 147]. The result can be extended by using the concept of the order of a subcontinuum in place of the order of a point. The proof of the extension is the same one as of the original result, namely a consequence of [66, (7.3), p. 147]. Thus, we have the following theorem.

Theorem 3.1. *The* Menger-Urysohn *order of a continuum in a locally compact space never increases under an open mapping.*

Corollary 3.2. *If K is a continuum contained in a locally compact space X such that* $\operatorname{ord}(K, X) = 1$*, and if a mapping $f : X \to Y$ is open, then* $\operatorname{ord}(f(K), f(X)) = 1$.

Recall that the conclusion of the above theorem is not true if the concept of the order is understood in the classical sense. This can be seen from [21, Example 2.1, p. 3728] where an open mapping $f : X \to T$ is defined on a smooth dendroid X onto a simple triod T such that there is a point $p \in X$ which is of order two in the classical sense, while the order of $f(p)$ in T is three. Further, in [21, Example 2.2, p. 3729], an open mapping $f : X \to A$ of a plane dendroid X onto an arc A is defined such that an end point (in the classical sense) of X is mapped onto an interior point of A. The dendroid X in this example is not smooth. The following question is asked in [21, Question 2.3, p. 3730], where the set of all end points in the classical sense of a continuum X is denoted by $E(X)$.

Question 3.3. Let $f : X \to f(X)$ be an open mapping defined on a *smooth* dendroid X. Does then the inclusion $f(E(X)) \subset E(f(X))$ hold true?

Whyburn's result is still true if we assume that the domain space is a hereditarily unicoherent continuum and the mapping under consideration is light, even for a larger class of mappings than open ones. Namely, we have the following theorem, see [21, Theorem 3.1, p. 3731].

Theorem 3.4. *Let a continuum X be hereditarily unicoherent, and let a mapping $f : X \to Y = f(X)$ be confluent and light. Then, for each point $p \in X$, the order in the classical sense of $f(p)$ in Y is not greater than the order in the classical sense of p in X.*

Remark 3.5. Examples 2.1 and 2.2 of [21, p. 3728 and 3729], mentioned above, show that lightness of the mapping is essential in Theorem 3.4. Confluence of f is indispensable and cannot be weakened to semi-confluence, [21, Remark 3.4, p. 3731]. Hereditary unicoherence of X is essential as well, see [21, Example 3.5, p. 3731].

Note that the set of all end points in the classical sense in a continuum X is, by its definition, a boundary subset of X. Thus, Theorem 3.4 leads to the following result.

Corollary 3.6. *Let a hereditarily unicoherent continuum X contain an end point p in the classical sense, and let a mapping $f : X \to Y = f(X)$ be confluent and light. Then $f(p)$ is an end point in the classical sense of Y.*

Since each dendroid X contains (at least two) end points in the classical sense, Corollary 3.6 gives an easy argument for the following result, which is both new and interesting.

Corollary 3.7. *No dendroid is homogeneous with respect to the class of light confluent mappings.*

Remark 3.8. One can ask if lightness of the mappings can be omitted in Corollary 3.7. A partial positive answer to the above question is known, under an additional assumption that the set of all ramification points (in the classical sense) of the considered dendroid is finite, see [39, Proposition 2.2, p. 59]. The same conclusion holds for the class of open mappings, see [8, Theorem, p. 409]. But in general the answer is negative: there exist dendrites which are homogeneous with respect to the class of monotone mappings, see [39, Example 2.4, p. 59]. For more results and problems in this direction see [11, Section 7, pp. 185–186], [10], [16], [24], [17], [18], [22], [19, Section 2, pp. 80–82], and references therein.

3B. Ends in the sense of Bing. Besides the concepts of an end point in the sense of Menger-Urysohn and in the classical sense, which apply to any continua, recall the following, introduced by Bing, which applies to arc-like continua only (see [3, Condition (C), p. 660]).

Definition 3.9. A point p of a continuum X is called an *end point in the sense of* Bing provided that the continuum X is arc-like and for each chain $\mathcal{C}$ covering X there is a chain $\mathcal{D}$ which refines $\mathcal{C}$, which also covers X, and such that only the first link of $\mathcal{D}$ contains p.

This definition of an end point $p \in X$ is equivalent (for arc-like continua X) to the following two, which are mutually equivalent in the realm of all continua, see [3, theorems 12 and 13, p. 661]:

(3.10.A) each nondegenerate subcontinuum of X containing the point p is irreducible from p to some other point;

(3.10.B) for every two subcontinua K and L of X, if $p \in K \cap L$ then either $K \subset L$ or $L \subset K$.

Remark 3.11. A condition close to (3.10.A) was considered earlier by Miller. Namely in [50, p. 190], a point p of a (metric) continuum X is defined to be a *terminal* point of X provided that every irreducible subcontinuum of X which contains p is irreducible between p and some other point.

Since each subcontinuum of an arc-like continuum is arc-like, thus irreducible, we see by (3.10.A) that

(3.11.1) *if a continuum X is arc-like, then a point of X is terminal if and only if it is an end point in the sense of* Bing.

The concept of an end point of a continuum X in the sense of condition (3.10.B) above (that can be applied to an arbitrary continuum X) can easily be extended to a concept of an end continuum of X as follows.

Definition 3.12. A subcontinuum E of a continuum X is called an *end continuum in the sense of* Bing (abbreviated B-*end continuum*) provided that for every two subcontinua K and L of X the condition $E \subset K \cap L$ implies that either $K \subset L$ or $L \subset K$. If E is a singleton $\{p\}$, then p will be named B-*end point*.

Remark 3.13. Note that X is a B-end continuum of itself. Further, a subcontinuum E of X is a B-end continuum of X if and only if E is a B-end point of the decomposition space X/E of the monotone upper semicontinuous decomposition of X whose only nondegenerate element is E, see [28, p. 385].

The next theorem says that the notion of a B-end continuum is invariant under confluent mappings.

Theorem 3.14. *Let a mapping $f : X \to Y$ between continua X and Y be confluent. If $E \subset X$ is a* B-*end continuum of X, then $f(E)$ is a* B-*end continuum of Y.*

Proof: Let K and L be subcontinua of Y with $f(E) \subset K \cap L$, and let A and B be components of $f^{-1}(K)$ and $f^{-1}(L)$, respectively. Since E is a B-end continuum of X, we have either $A \subset B$ or $B \subset A$. Further, $f(A) = K$ and $f(B) = L$ by confluence of f. Thus, either $K \subset L$ or $L \subset K$, as needed. □

As a consequence we get the following result (see [6, Lemma 1, p. 288] and [7, Lemma 1, p. 270]).

Corollary 3.15. *If f is a confluent mapping from a continuum X onto $f(X)$, then the image of each* B-*end point of X is a* B-*end point of $f(X)$.*

Corollary 3.16. *If f is a confluent mapping between arc-like continua X and Y then the concept of an end point in the sense of* Bing *is preserved under f.*

Since the pseudo-arc is characterized as an arc-like continuum, each point of which is an end point in the sense of Bing, see [3, Theorem 16, p. 662], we get the following consequence of Corollary 3.16.

Corollary 3.17. *If $f : X \to Y$ is a confluent mapping from the pseudo-arc X onto an arc-like continuum Y, then Y is the pseudo-arc.*

We do not know if the arc-likeness of Y is essential in corollaries 3.16 and 3.17. This question is related to the following famous problem of Lelek (see [42, Problem 4, p. 94]).

Question 3.18. Let a curve X be the image of an arc-like curve under a confluent mappings. Is X arc-like?

A partial answer to this question for the metric case was given by McLean who proved that such a continuum must be tree-like, [49, Theorem 2.1, p. 468, and Corollary 2.2, p. 472]. An extension of this result to Hausdorff continua and semi-confluent mappings is given in [35, Corollary 4.6, p. 353].

Since open mappings of compact spaces are confluent, [66, Theorem 7.5, p. 148], and since they preserve arc-likeness of continua, [63, Theorem 1.0, p. 259], we get the next two corollaries which are due to Rosenholtz, see [63, Corollary 1.2 and Theorem 1.3, p. 260].

Corollary 3.19. *The concept of an end point in the sense of* Bing *is preserved under open mappings.*

Corollary 3.20. *A nondegenerate image of the pseudo-arc under an open mapping is the pseudo-arc.*

Remark 3.21. Corollary 3.15 is an important step in the proof of the following result (see [27, Corollary 3.6, p. 33]).

(3.21.1) *Each confluently homogeneous nondegenerate arc-like continuum is the pseudo-arc.*

Remark 3.22. Confluence of the mapping f in Theorem 3.14 and corollaries 3.15 and 3.16 is essential and cannot be relaxed to semi-confluence, even if the continua X and Y are arcs. Namely, let $f : [-1, 2] \to [0, 2]$ be defined by $f(t) = |t|$ (see [44, Example 3.12, p. 14]). Then f is semi-confluent, and the end point -1 of the domain is mapped to the midpoint 1 of the range.

3C. Absolute end points. A more restrictive concept than that of an end point in the sense of Bing has been introduced by Rosenholtz in [64], where an equivalence of seven certain conditions was shown in [64, Theorem 1.0, p. 1308], again for arc-like continua. Limits of application of the notion have been extended to arbitrary continua in [15] by taking one of these conditions as the definition. It runs as follows.

Definition 3.23. A point p of a continuum X is called an *absolute end point of* X provided that $X \setminus \{p\}$ is a composant of X.

However, since

(3.24) *if a continuum* X *contains an absolute end point* p*, then* X *is irreducible between* p *and some other point of* X*, and* X *is decomposable,*

(see [15, Lemma 3.1, p. 22, and Proposition 3.10, p. 24]), the results on absolute end points concern irreducible decomposable continua only. Further,

(3.25) *each continuum has, at most, two absolute end points.*

The obtained structural results can be summarized as follows (see [64, Theorem 1.0, p. 1308]; [15, Proposition 3.2, p. 22; Proposition 3.5 and Corollary 3.7, p. 23]).

3.26. Theorem. *Consider the following conditions for a point* p *of a continuum* X*.*

(3.26.1) $X \setminus \{p\}$ *is a composant of* X *(i.e.,* p *is an absolute end point of* X*);*
(3.26.2) *if* X *is irreducible between points* x *and* y*, then either* x *or* y *is* p*;*
(3.26.3) X *is irreducible between* p *and some other point, and* X *is connected im kleinen at* p*;*
(3.26.4) X *is irreducible between* p *and some other point, and* X *is locally connected at* p*;*
(3.26.5) X *is irreducible between* p *and some other point, and* X *is aposyndetic at* p *with respect to any other point of* X*;*
(3.26.6) X *is irreducible between* p *and some other point, and* X *is semi-locally connected at* p*;*
(3.26.7) X *is locally connected at* p*, and* p *does not separate* X*;*
(3.26.8) X *is locally connected at* p*, and whenever* $C_1, \dots, C_n$ *is an* ε*-chain in the metric continuum* X *with* p *belonging to a connected link* C_k*, then either* (a) $k = 1$, (b) $k = n$, (c) $k = 2$*, and* $C_1 \subset \operatorname{cl} C_2$ *or* (d) $C_n \subset \operatorname{cl} C_{n-1}$*;*

(3.26.9) *for each positive number* ε *there is a positive number* δ *such that, if* $C_1, \dots, C_n$ *is an* δ*-chain in the metric continuum* X *with* p *belonging to* C_k*, then either* $\bigcup\{C_j : j \in \{1, \dots, k\}\}$ *or* $\bigcup\{C_j : j \in \{k, \dots, n\}\}$ *is contained in an* ε*-neighborhood of* p.

Then

- *if the continuum* X *is irreducible, then conditions* (3.26.1)-(3.26.6) *are equivalent, and each of them implies* (3.26.7);
- *if the continuum* X *is irreducible and hereditarily unicoherent at* p*, then* (3.26.7) *implies each of conditions* (3.26.1)-(3.26.6), thus (3.26.1)-(3.26.7) *are equivalent*;
- *if the metric continuum* X *is arc-like (thus irreducible and hereditarily unicoherent at each of its points), then all conditions* (3.26.1)-(3.26.9) *are equivalent*.

In general, an absolute end point need not be a B-end point, see [15, Example 4.4, p. 25]. However, under some additional assumptions, the implication holds. Namely we have the following result, see [64, Remark, p. 1310] and [58, Proposition, p. 62].

Theorem 3.27. *If a continuum* X *is either arc-like or irreducible and atriodic, then*

(3.27.1) *each absolute end point of* X *is a* B*-end point of* X.

Pyrih posed in [58, p. 62] the following problem.

Problem 3.28. Characterize (irreducible) continua X for which implication (3.27.1) is true.

The concept of an absolute end point has been localized as follows, [26, p. 106].

Definition 3.29. A point p of a continuum X is called a *local absolute end point* of X provided that there is a subcontinuum K of X such that $p \in \operatorname{int} K$ and p is an absolute end point of K.

Below we collect known structural properties of local absolute end points, see [26, theorems 3.3 and 3.4, p. 106–107].

Theorem 3.30. *The following conditions on a continuum* X *and a point* $p \in X$ *are equivalent:*

(3.30.1) p *is a local absolute end point of* X;

(3.30.2) *there is a subcontinuum K of X irreducible between p and some other point of X, such that K is locally connected at p and $p \in \operatorname{int} K$;*
(3.30.3) *for each nondegenerate subcontinuum K of X if $p \in K$ then $p \in \operatorname{int} K$;*
(3.30.4) *p is a local absolute end point of each nondegenerate subcontinuum of X containing p.*

Theorem 3.31. *A local absolute end point of a continuum X is an absolute end point of X if and only if X is irreducible.*

The known mapping properties of absolute end points and local absolute end points are connected with a concept of partially confluent mappings, which generalizes the one of confluent mappings. A surjective mapping $f : X \to Y$ between continua X and Y is said to be *partially confluent at a point* $p \in X$ provided that for each nondegenerate subcontinuum Q of Y such that $f(p) \in Q$ the component of $f^{-1}(Q)$ containing the point p is nondegenerate. This concept should not be confused with a concept of a partially confluent mapping considered in [53] and defined in a different way. Below we summarize the most important mapping properties as in [26, Section 4, pp. 107-111].

Theorem 3.32. *Let a surjective mapping $f : X \to Y$ between continua X and Y be both interior at a point $p \in X$ and partially confluent at p. If p is a local absolute end point of X, then $f(p)$ is a local absolute end point of Y.*

Theorem 3.33. *Let p be a local absolute end point of a continuum X and let a mapping $f : X \to Y$ be a surjection. If*
(3.33.1) *f is both confluent and interior at p, or*
(3.33.2) *f is open,*
then the point $f(p)$ is a local absolute end point of Y.

Each of the following conditions implies that $f(p)$ is an absolute end point of Y:
(3.33.3) *f is open and X is arc-like*;
(3.33.4) *f is both confluent and interior at p, and X and Y are irreducible*;
(3.33.5) *f is open and X and Y are irreducible*;
(3.33.6) *f is both monotone and interior at p, and X is irreducible*;
(3.33.7) *f is atomic and X is irreducible.*

The reader is referred to [26, Section 4, pp. 109-111] for a discussion of essentiality of the assumptions made in the quoted results.

3D. Ends in the sense of Bennett and Fugate. This section concerns the following concept (see [2, Definition 1.9, p. 8]).

Definition 3.34. A proper subcontinuum E of a continuum X is said to be an *end continuum of X in the sense of* Bennett *and* Fugate (abbreviated BF-*end continuum)* provided that X is not the union of two proper subcontinua each intersecting E.

The following characterization of BF-end continua is due to Rosen [62, p. 118]; see also [2, Theorem 1.16, p. 10].

Theorem 3.35. *A subcontinuum K is a* BF-*end continuum of a continuum X if and only if there is a point $p \in X \setminus K$ such that X is irreducible between p and any point of K.*

Therefore, the existence of a BF-end continuum in a continuum X implies irreducibility of X.

An essential part of the following result has been proved in [20, Corollary 5.6, p. 22]. For a part of the conclusion, see also [33, Theorem 3, p. 222]; (compare also [44, (8.1), p. 71] and [12, Theorem 7, p. 278] for a more precise formulation) and see also [44, Corollary 4.45, p. 26].

Theorem 3.36. *If a continuum X is irreducible between points a and b, and a surjective mapping $f : X \to Y$ satisfies one of the following conditions*:

(3.36.1) *f is quasi-monotone*,
(3.36.2) *f is feebly monotone*,
(3.36.3) *f is confluent with point-inverses having finitely many components*,
(3.36.4) *f is hereditarily confluent*,
(3.36.5) *f is a local homeomorphism*,

then the continuum Y is irreducible between $f(a)$ and $f(b)$.

The next theorem summarizes several results dispersed in the literature. In particular, compare [13, Theorem 16, p. 73, and corollaries 17 and 18, p. 74] and [20, Theorem 7.15, p. 27].

Theorem 3.37. *Let X and Y be continua, and let a surjective mapping satisfy one of the conditions* (3.36.1)-(3.36.5). *If $E \subset X$ is a* BF*-end continuum of X, then $f(E)$ either equals the whole Y or it is a* BF*-end continuum of Y.*

Proof: Assume $f(E) \neq Y$. By Theorem 3.35 there is a point $p \in X \setminus E$ such that X is irreducible between p and any point of E. By Theorem 3.36 the continuum Y is irreducible between $f(p)$ and any point of $f(E)$. Thus $f(p) \in Y \setminus f(E)$, and applying Theorem 3.36 once more we get the conclusion. □

The reader is referred to [13, Remark 20, p. 75] to see that the conclusion of Theorem 3.37 does not hold for essentially larger classes of mappings.

Another concept which is related to the one of a BF-end continuum is the notion of an E-continuum. Its idea is taken from [50, Lemma A, p. 183 and Definition, p. 184]. We quote here its definition in a simpler form, after [2, Definition 1.24, p. 15].

Definition 3.38. A subcontinuum K of a continuum X is called an *E-continuum of X* provided that $X \setminus K$ is a composant of X.

Thus, if an E-continuum of X is a singleton $\{p\}$, then p is an absolute end point of X (see Section 3C above). The concepts of an E-continuum and BF-end continuum are related by the following result [2, Theorem 1.25, p. 15].

Theorem 3.39. *A subcontinuum K of a continuum X is an E-continuum of X if and only if K is a maximal* BF*-end continuum of X (i.e., K is not properly contained in any* BF*-end continuum of X).*

A continuum X is said to be *locally connected at a subcontinuum K of X* provided that for each open set U containing K there is an open connected V such that $K \subset V \subset U$. Notice that this is a weaker condition than the one saying that X is locally connected at each point of K. The following result [2, Theorem 1.30, p. 16] gives another characterization of E-continua.

Theorem 3.40. *The E-continua of a continuum X are exactly those* BF*-end continua of X at which X is locally connected.*

Theorem 3.41. *Let $f : X \to Y$ be a local homeomorphism between continua X and Y. If K is an E-continuum of X, then $f(K)$ either equals the whole Y or it is an E-continuum of Y.*

Proof. By Theorem 3.37, used with (3.36.4), the continuum $f(K)$ is either Y or a BF-end continuum of Y. Assume $f(K) \neq Y$. Let $f(K) \subset U$, where U is an open subset of Y. Thus, $K \subset f^{-1}(f(K)) \subset f^{-1}(U)$ and $f^{-1}(U)$ is open in X. Hence, by Theorem 3.40, the continuum X is locally connected at K; i.e., there is a connected open set V' in X such that $K \subset V' \subset f^{-1}(U)$. Since f, being a local homeomorphism, is open, $V = f(V')$ is an open connected set such that $K \subset V \subset U$, as needed. □

4. Terminal continua

In the topological literature, or in continuum theory (to be more precise), the term "terminal," when applied either to subcontinua of a given continuum or to points, has several quite different meanings. Thus, if this concept is considered, the reader should always be informed which definition of "terminality" is used. The term was already defined with respect to points after [50, p. 190], and recalled above in Remark 3.11. Below we will discuss some other, and more frequently used, variants of this concept related to continua.

4A. Terminal continua in the sense of Bennett and Fugate. We start with a definition and a nice characterization (see [2, Definition 1.1, p. 7, and Corollary 1.14, p. 9], where the term "terminal" was used for the defined concept).

Definition 4.1. A subcontinuum K of a continuum X is said to be *terminal in the sense of* Bennett *and* Fugate (more succinctly BF-*terminal*) subcontinuum of X, provided that K is a proper subset of X and that if whenever A and B are proper subcontinua of X whose union is X and such that $A \cap K \neq \emptyset \neq B \cap K$, then either $X = A \cup K$ or $X = B \cup K$.

Theorem 4.2. *A subcontinuum K of a continuum X is a* BF-*terminal subcontinuum of X if and only if the continuum X/K is irreducible from K to some point.*

For various structural properties of BF-terminal continua see [2]. In particular, each BF-terminal continuum is non-separating, [2, Theorem 1.3, p. 7]. A relation between BF-terminal continua and BF-end continua (Definition 3.34) is described in the next result, [2, Theorem 1.16, p. 10].

Theorem 4.3. *A subcontinuum K of a continuum X is a* BF*-end subcontinuum of X if and only if K is a* BF*-terminal continuum with empty interior.*

The following theorem describes the main mapping property of BF-terminal continua, see [13, Theorem 5, p. 71] and [20, Theorem 7.3, p. 25].

Theorem 4.4. *Let a proper subcontinuum K of a continuum X be given, and let a surjective mapping $f : X \to Y$ be such that*

(4.4.1) *the induced mapping $f_* : X/K \to Y/f(K)$ is either feebly monotone or quasi-monotone.*

Then

(4.4.2) *if K is a* BF*-terminal continuum of X, then $f(K)$ either is the whole Y or is a* BF*-terminal continuum of Y.*

It is known that for each subcontinuum K of a continuum X and for each quasi-monotone (feebly monotone) surjection $f : X \to Y$ the induced mapping f_* is quasi-monotone (feebly monotone, respectively), [13, Lemma 6, p. 71] and [20, Proposition 7.5]. Further, any mapping satisfying (3.36.3)-(3.36.5) is quasi-monotone, see [33, theorems 5 and 7, p. 223 and 224] and [44, Corollary 4.45, p. 26]. Therefore, Theorem 4.4 implies a corollary.

Corollary 4.5. *Let a proper subcontinuum K of a continuum X be given, and let a surjective mapping $f : X \to Y$ satisfy one of the conditions* (3.36.1)-(3.36.5). *Then implication* (4.4.2) *holds.*

Recall that open retractions do not preserve BF-terminality of subcontinua, even for arc-like continua, as it is shown in [9, Example 15, p. 379].

Now we pass to another concept.

Definition 4.6. A subcontinuum K of a continuum X is said to be *absolutely* BF*-terminal* provided that K is BF-terminal in each subcontinuum L of X which properly contains K.

Note that absolutely BF-terminal continua are named "absolutely terminal" in [2, Definition 4.1, p. 36], and "terminal" in

Fugate's paper [32, p. 461] and in Nadler's book [51, 1.54, p. 107]. See [2, p. 35] for a discussion on relations to some other concepts for which the name "terminal" (or a similar one) is used by other authors.

Obviously each absolutely BF-terminal continuum is BF-terminal but not conversely. Various structural properties of absolutely BF-terminal continua are discussed in [2, Chapter 4, p. 34]. The next result can easily be deduced from [32, p. 461] and [51, Lemma 1.55, p. 107].

Theorem 4.7. *The following conditions are equivalent for a proper subcontinuum K of a continuum X:*

(4.7.1) *K is an absolutely* BF*-terminal continuum of X;*
(4.7.2) *for every two proper subcontinua A and B of X, if $A \cap K \neq \emptyset \neq B \cap K$, then either $A \subset B \cup K$ or $B \subset A \cup K$;*
(4.7.3) *for every two proper subcontinua A and B of X, if $K \subset A \cap B$, then either $A \subset B$ or $B \subset A$.*

The subsequent result concerns a mapping invariance of absolutely BF-continua as shown in [9, Theorem 5, p. 378].

Theorem 4.8. *Let a subcontinuum K of a continuum X be given, and let a mapping $f : X \to Y$ satisfy the condition*

(4.8.1) *for each subcontinuum Q of Y containing $f(K)$ there exists a component C of $f^{-1}(Q)$ such that $K \subset C$ and $f(C) = Q$.*

Then

(4.8.2) *if K is an absolutely* BF*-terminal continuum of X, then $f(K)$ either is the whole Y or is an absolutely* BF*-terminal continuum of Y.*

Remarks 4.9. (a) The assumption (4.8.1) holds if the mapping f is confluent with respect to each subcontinuum Q of Y which contains $f(K)$, in particular if f is confluent. Therefore, for such mappings, implication (4.8.2) holds.

(b) The assumption (4.8.1) of Theorem 4.8 cannot be relaxed to weak confluence of f with respect to Q. See [9, Example 11, p. 379].

(c) It follows from [9, Example 18, p. 380] that if f is a quasi-monotone retraction defined on an arc-like continuum X, then the implication (4.8.2) need not be true.

4B. Extremal continua. The notion of a BF-terminal continuum has been used by Owens in [57, p. 264] to define another class of nonseparating subcontinua of a given continuum, namely the class of extremal continua. The definition runs as follows.

Definition 4.10. A proper subcontinuum S of a continuum X is called an *extremal subcontinuum of* X provided that, for each irreducible subcontinuum I in X such that $I \cap S \neq \emptyset \neq I \setminus S$, the continuum S is a BF-terminal subcontinuum of the union $I \cup S$.

In [57] various structural properties of these continua are shown. In particular, they are nonseparating ones in a very strong sense: if S is an extremal subcontinuum of a continuum X, and Y is an arbitrary subcontinuum of X, then $Y \setminus S$ is connected, see [57, Theorem 2.2, p. 265]. In the next theorem, characterizations of extremal continua are collected, see [57, Theorem 3.3, p. 268 and Corollary 3.4, p. 269].

Theorem 4.11. *Let S be a proper subcontinuum of a continuum X. The following conditions are equivalent*:

(4.11.1) *S is an extremal subcontinuum of X;*
(4.11.2) *S is a point of irreducibility of each irreducible subcontinuum of the quotient continuum X/S that contains S;*
(4.11.3) *each irreducible subcontinuum of X which meets S has a point of irreducibility in S.*

If S is an extremal subcontinuum of a continuum X and if the quotient space X/S is arcwise connected, then S is an end point of X/S in the classical sense, [57, Corollary 3.7, p. 270]; the opposite implication does not hold, [57, Example 3.8, p. 270].

Mapping properties of extremal continua were studied in [12] and in [14]. Below we summarize and simplify the results obtained there. The main one says that the concept is invariant under mappings which preserve some irreducible continua in both directions. Precisely, we have the following result [12, Theorem 2, p. 276].

Theorem 4.12. *Let S be an extremal subcontinuum of a continuum X. If a surjective mapping $f : X \to Y$ satisfies the two conditions:*

(4.12.1) *for each subcontinuum I of X which is irreducible from a point $p \in S$ to some point of X, its image $f(I)$ is irreducible from $f(p)$ to some point of Y,*

(4.12.2) *for each irreducible subcontinuum J of Y such that $J \cap f(S) \neq \emptyset$, its inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X,*

then the image $f(S)$ is an extremal subcontinuum of Y.

The theorem says when the image $f(S)$ of a particular extremal subcontinuum S of the domain is an extremal subcontinuum of the range. In [12] conditions are formulated under which the invariance holds globally. In particular, we have the following corollary [12, Corollary 5, p. 277].

Corollary 4.13. *Let a surjective mapping $f : X \to Y$ between continua X and Y satisfy the two conditions*:

(4.13.1) *for each subcontinuum I of X which is irreducible from a point $p \in X$ to some point of X, its image $f(I)$ is irreducible from $f(p)$ to some point of Y,*

(4.13.2) *for each irreducible subcontinuum J of Y, its inverse image $f^{-1}(J)$ is an irreducible subcontinuum of X.*

Then

(4.13.3) *the images of extremal subcontinua of the domain are extremal subcontinua of the range.*

It should be underlined that each of the conditions (4.12.1) and (4.13.1) is stronger than simple preservation of irreducible continua under f. It demands that not only irreducible continua but also points of irreducibility have to be preserved under the mapping. A natural question arises: What known classes of mappings satisfy such conditions. Recall that Theorem 3.36 above concerns this. Thus, if the mapping $f : X \to Y$ satisfies one of the conditions (3.36.1)-(3.36.5) and condition (4.13.2), then (4.13.3) holds. However, mappings for which (4.13.1) holds need not satisfy (4.13.2), which has to be separately assumed. Moreover, the reader can verify, simply by constructing suitable examples, that no one of the conditions (3.36.1)-(3.36.5) implies (4.13.2). On the other hand, (see [12, Corollary 14, p. 280])

(4.14) *if a surjective mapping $f : X \to Y$ between continua X and Y satisfy condition* (4.13.2)*, then it is monotone* and monotone mappings preserve points of irreducibility, see e.g. [40, §48, I, Theorem 3, p. 192]. This leads to the following result.

Proposition 4.15. *If a surjective mapping* $f : X \to Y$ *between continua* X *and* Y *satisfies the two conditions:* (4.13.2) *and*

(4.15.1) *for each irreducible subcontinuum* I *of* X *the partial mapping* $f|I : I \to f(I)$ *is monotone*,

then (4.13.3) *holds.*

Hence, one can omit (4.13.1) in Corollary 4.13 provided that f is hereditarily monotone, see [12, Corollary 12, p. 279]. This is the case when X is hereditarily unicoherent, in particular when X is arc-like and f is monotone [12, Corollaries 17 and 18, p. 281].

Corollary 4.16. *If the continuum* X *is hereditarily unicoherent, then* (4.13.2) *implies* (4.13.3).

Corollary 4.17. *Condition* (4.13.3) *holds if* f *is a monotone mapping of an arc-like continuum* X*.*

Finally, let us recall that both (4.13.1) and (4.13.2) hold for atomic mappings, see [44, (4.14), p. 17] and [14, Theorem 2, p. 132], and therefore the following result is true, see [14, Corollary 5, p. 133].

Corollary 4.18. *If a surjective mapping between continua is atomic, then for each extremal subcontinuum of the domain its image is an extremal subcontinuum of the range.*

Neither the inverse implication to that of Corollary 4.18 holds, nor can it be extended to hereditarily monotone mappings [14, p. 133].

4C. Terminal continua in the sense of Gordh. To avoid confusion or misunderstanding in the terminology, we have to use another name for the concept of a terminal continuum as defined by Gordh in [34]. Since he restricts his considerations to subcontinua of hereditarily unicoherent continua only, we rename this concept, following [9, Section 3, p. 380], as HU-terminal.

Definition 4.19. A subcontinuum K of a hereditarily unicoherent continuum X is called an *HU-terminal continuum of* X, provided that K is contained in an irreducible subcontinuum of X, and for each irreducible subcontinuum I of X containing K, there is a point $x \in X$ such that I is irreducible about the union $K \cup \{x\}$.

Note that if a continuum X is irreducible (and hereditarily unicoherent), then it is an HU-terminal subcontinuum of itself. The next theorem (see [34, Theorem 3.1, p. 463]) describes interrelations between HU-terminal continua and absolutely BF-terminal continua (Definition 4.6). To formulate it we have to recall some definitions. A continuum T is called a *triod* provided there exists a proper subcontinuum Q of T such that $T \setminus Q$ is the union of three mutually disjoint sets. A continuum which contains no triod is said to be *atriodic*. It is well known that each arc-like continuum is hereditarily unicoherent, irreducible and atriodic.

Theorem 4.20. *Let a hereditarily unicoherent continuum X be atriodic. Then a proper subcontinuum of X is HU-terminal if and only if it is absolutely* BF*-terminal.*

The above equivalence (proved in [34, Theorem 3.1, p. 463]) need not be true if X is not atriodic, see [34, Example 1, p. 463] and [2, Example 4.3, p. 35].

Theorems 4.20 and 4.8 imply that, in the realm of atriodic hereditarily unicoherent continua, mappings satisfying condition (4.8.1) for some HU-terminal subcontinuum K of X (in particular confluent mappings, see Remark 4.9 (a)) preserve its HU-terminality. However, if one is looking for a suitable class of mappings which preserve HU-terminality in the realm of all hereditarily unicoherent continua, then any condition expressed in terms of confluence is rather inadequate for such invariance. A much stronger condition is atomicity, and it works here. The next theorem answers [9, Question 22, p. 382].

Theorem 4.21. *Let $f : X \to Y$ be a surjective atomic mapping defined on a hereditarily unicoherent continuum X. If K is an HU-terminal subcontinuum of X, then its image $f(K)$ is an HU-terminal subcontinuum of Y.*

Proof: Observe first that since each atomic mapping is hereditarily monotone, [44, (4.14), p. 17], thus hereditarily confluent, and since hereditarily confluent mappings preserve hereditary unicoherence of continua, [44, (7.5), p. 59], the continuum Y is hereditarily unicoherent, and therefore the concept of HU-terminality can be applied to $f(K)$. If I_0 is an irreducible subcontinuum of X containing K, then $f(K) \subset f(I_0)$, and $f(I_0)$ is in turn an irreducible subcontinuum of Y since monotone mappings preserve irreducibility [40, §48, I, Theorem 3, p. 192]. Further, if J is an irreducible subcontinuum of Y containing $f(K)$, then its preimage $f^{-1}(J)$ is an irreducible subcontinuum of X according to [14, Theorem 2, p. 132], which obviously contains K. Thus, by the definition of HU-terminality, there exists a point $x \in X$ such that the continuum $f^{-1}(J)$ is irreducible about the union $K \cup \{x\}$ (then K is called the set of irreducibility of $f^{-1}(J)$, see [44, p. 8]). Since f is hereditarily monotone, the partial mapping $f|f^{-1}(J) : f^{-1}(J) \to f(f^{-1}(J))$ is monotone, so quasi-monotone. Quasi-monotone mappings preserve sets of irreducibility, [44, (8.1), p. 71], i.e., $f(f^{-1}(J))$ is irreducible about $f(K) \cup \{f(x)\}$. Finally, since $f(f^{-1}(J)) = J$ by atomicity of f, the proof is complete. □

An example is known (see [9, p. 381]) which shows that atomicity of f is an essential assumption in Theorem 4.21, and the result cannot be extended to hereditarily monotone mappings.

4D. Terminal continua in the sense of Wallace. Now we will discuss a quite different concept of terminality than that in the sense of Bennett and Fugate (Definition 4.1), as well as in the sense of Gordh (Definition 4.19), a concept which was used by Maćkowiak and Tymchatyn in [48], and widely exploited in continuum theory by various authors. The difference relies on the property of separation of the whole continuum by the considered subcontinuum: all special subcontinua discussed above as end continua, E-continua or terminal ones in various senses were nonseparating. Terminal continua in the sense we intend to discuss below do not have this property. So, discussion about this subject has been joined to the present paper because of terminology rather than because of properties.

The idea of this notion comes from Wallace, who introduced so called C-sets and studied their properties in semigroups, [65, p. 639]. Recall that a proper subset Q of a continuum X is called a *C-set of* X provided that Q is a subset of each subcontinuum of X which intersects both Q and its complement $X \setminus Q$. Wallace proved that C-sets are connected and have empty interior, [65, Lemma 1, p. 639]. C-sets need not be closed: for example, a composant of a solenoid X is a C-set of X. If closedness is assumed, we get the concept of a terminal continuum as defined in [48, p. 17].

Definition 4.22. A subcontinuum Q of a continuum X is called *terminal in the sense of* Wallace (abbreviated W-*terminal*) provided that for each subcontinuum K of X the condition $K \cap Q \neq \emptyset$ implies $K \subset Q$ or $Q \subset K$.

Observe that, according to the above definition, the whole continuum X is a W-terminal subcontinuum of itself, and that each singleton is W-terminal.

A mapping $f : X \to Y$ between continua X and Y is said to be *confluent with respect to a subcontinuum* Q *of* Y provided that for each component C of $f^{-1}(Q)$ we have $f(C) = Q$. Ingram characterized W-terminal continua in [36, Theorem 1, p. 84] as follows:

Theorem 4.23. *A proper subcontinuum Q of a continuum X is* W-*terminal in X if and only if each mapping from a continuum onto X is confluent with respect to Q.*

Since a continuum X is hereditarily indecomposable if and only if each mapping from a continuum onto X is confluent (see [29, Theorem 4, p. 243] and [43, 5.7, p. 111]); equivalently, each confluent mapping from a continuum onto X is hereditarily confluent [44, (6.11), p. 53]. It follows that a continuum is hereditarily indecomposable if and only if each of its subcontinua is W-terminal, see [36, Remark, p. 85], which was used as the key argument to show that a continuum X is hereditarily indecomposable if and only if each monotone mapping from X is atomic [31, Theorem 4, p. 51]. The previously mentioned characterization of hereditarily indecomposable continua via W-terminality resembles another one saying that they are characterized as such continua for which each of their points is a B-end point (Definition 3.12), see [27, Fact 2.3, p. 31]. Replacing B-end points by B-end continua, we get a similar characterization with the same easy proof. Finally, using Theorem 4.2 and the invariance of hereditary indecomposability with respect to monotone mappings (see [44, (8.10), p. 72]), the same can be obtained with BF-terminal continua in place of B-end continua. Summarizing, we have the following result.

Theorem 4.24. *The following conditions are equivalent for a continuum X*:

(4.24.1) X *is hereditarily indecomposable*;
(4.24.2) *each mapping from a continuum onto X is confluent*;
(4.24.3) *each confluent mapping from a continuum onto X is hereditarily confluent*;
(4.24.4) *each monotone mapping from X onto a continuum is atomic*;
(4.24.5) *each point of X is a* B-*end point of X*;
(4.24.6) *each subcontinuum of X is a* B-*end continuum of X*;
(4.24.7) *each subcontinuum of X is* BF-*terminal*;
(4.24.8) *each subcontinuum of X is* W-*terminal.*

The following invariance of W-terminality has recently been shown in [25, Proposition 3].

Proposition 4.25. *If a mapping $f : X \to Y$ between continua X and Y is confluent, and if Q is a* W-*terminal subcontinuum of X, then $f(Q)$ is a* W-*terminal subcontinuum of Y.*

Proof. Let S be a subcontinuum of Y such that $S \cap f(Q) \neq \emptyset$. Then $Q \cap f^{-1}(S) \neq \emptyset$, so there exists a component C of $f^{-1}(S)$ such that $C \cap Q \neq \emptyset$. Since Q is W-terminal in X, we have either $Q \subset C$, which implies $f(Q) \subset f(C) = S$, or $C \subset Q$, which implies $S = f(C) \subset f(Q)$, by the confluence of f. The proof is complete. □

Finally, recall that some properties of W-terminal continua were used in [45] and [46] to investigate various properties and to construct interesting examples of continua by condensation of singularities. Further, the hyperspace $T(X)$ consisting of all W-terminal subcontinua of a given (metric) continuum X, and understood as a subspace of the hyperspace $C(X)$ equipped with the Hausdorff metric, was studied in [47] and applied there to investigate homogeneity of continua. Also the use of W-terminal continua in the aposyndetic decomposition theorem of Jones, see [37, Theorem 1, p. 736] and [38, Theorem, p. 51], is known to be essential in the study of homogeneous continua (compare [59, Section 1, p. 451], [60, Section 2, p. 216] and [61, Section 2, p. 344]).

5. A final remark

We close the paper with the following remark. The main stream of study of mapping properties of (nonseparating) subcontinua has its source in problem (1.1) mentioned in the beginning of the paper. Besides, an opposite implication can also be considered. Precisely, one can consider the following problem.

(5.1) Let $\mathfrak{C}$ be a family of nonseparating subcontinua of a given continuum, and let $\mathfrak{M}$ be a class of mappings between continua. For what $\mathfrak{C}$ and $\mathfrak{M}$ implication

$$f(A) \in \mathfrak{C} \text{ and } f \in \mathfrak{M} \implies A \in \mathfrak{C}$$

is true?

Only a few results related to this problem are known. Recall some of them.

If $E(X)$ denotes the set of all end points (in the classical sense) of X, then the inclusion $E(Y) \subset f(E(X))$ holds if (a) X is a fan (i.e., a dendroid with exactly one ramification point) and f is confluent, [23, Theorem 4.1, (10), p. 14] or if X is a dendrite and f is monotone, [24, Proposition 4.20, p. 11].

In [45, Proposition 4, p. 576] a characterization of atomic mappings via W-terminal continua is given. Namely a mapping $f : X \to Y$ between continua X and Y is atomic if and only if the inverse image of any W-terminal subcontinuum of Y is a W-terminal subcontinuum of X. For other families $\mathfrak{C}$ of subcontinua of a given continuum, in particular for families of nonseparating subcontinua, and for various classes $\mathfrak{M}$ of mappings, a systematic investigation of problem (5.1) above is a subject for further study.

References

[1] R. D. Anderson, *Atomic decompositions of continua*, Duke Math. J., **24** (1956), 507–514.

[2] D. E. Bennett and J. B. Fugate, *Continua and their non-separating subcontinua*, Dissertationes Math. (Rozprawy Mat.), **149** (1977), 1–46.

[3] R. H. Bing, *Snake-like continua*, Duke Math. J., **18** (1951), 653–663.

[4] J. J. Charatonik, *On ramification points in the classical sense*, Fund. Math., **51** (1962), 229–252.

[5] J. J. Charatonik, *Confluent mappings and unicoherence of continua*, Fund. Math., **56** (1964), 213–220.

[6] J. J. Charatonik, *Some generalizations of homogeneity of spaces*, Topology, (Moscow 1979), Trudy Mat. Inst. Steklov, **154** (1983): 267–274 (Russian); English transl. in Proc. Steklov Institute, (1984), 287–294.

[7] J. J. Charatonik, *Generalized homogeneity and some characterizations of solenoids*, Bull. Polish Acad. Sci. Math., **31** (1983), 171–174.

[8] J. J. Charatonik, *Generalized homogeneity of continua and a question of H. Kato*, Bull. Polish Sci. Math., **36** (1988), 409–411.

[9] J. J. Charatonik, *Absolutely terminal continua and confluent mappings*, Comment. Math. Univ. Carolin., **32** (1991), 377–382.

[10] J. J. Charatonik, *On generalized homogeneity of locally connected plane continua,* Comment. Math. Univ. Carolin., **32** (1991), 769–774.

[11] J. J. Charatonik, *Monotone mappings of universal dendrites*, Topology Appl., **38** (1991), 163–187.

[12] J. J. Charatonik, *Mapping invariance of extremal continua*, Topology Appl., **43** (1992), 275–282.

[13] J. J. Charatonik, *Terminal continua and quasi-monotone mappings*, Topology Appl., **47** (1992), 69–77.

[14] J. J. Charatonik, *Atomic mappings and extremal continua*, Extracta Math., **7** (1992), 131–134.

[15] J. J. Charatonik, *Absolute end points of irreducible continua*, Math. Bohemica, **118** (1993), 19–28.

[16] J. J. Charatonik, *On generalized homogeneity of curves of a constant order*, Rocky Mountain J. Math., **24** (1994), 491–504.

[17] J. J. Charatonik, *Problems on generalized homogeneity of some continua*, Questions Answers Gen. Topology, **14** (1996), 9–13.

[18] J. J. Charatonik, *An uncountable family of monotone homogeneous dendrites*, Glasnik Mat., **31 (51)** (1996), 289–294.

[19] J. J. Charatonik, *Several problems in continuum theory*, Aportaciones Mat. Comun., **20** (1997), 77–86.

[20] J. J. Charatonik, *On feebly monotone and related classes of mappings*, Topology Appl., **105** (2000), 15–29.

[21] J. J. Charatonik and W. J. Charatonik, *Open mappings increasing order*, Proc. Amer. Math. Soc., **125** (1997), 3725–3733.

[22] J. J. Charatonik and W. J. Charatonik, *Monotone homogeneity of dendrites*, Comment. Math. Univ. Carolin., **38** (1997), 361–370.

[23] J. J. Charatonik, W. J. Charatonik and S. Miklos, *Confluent mappings of fans*, Dissertationes Math. (Rozprawy Mat.), **301** (1990), 1–86.

[24] J. J. Charatonik, W. J. Charatonik and J. R. Prajs, *Mapping hierarchy of dendrites*, Dissertationes Math. (Rozprawy Mat.), **333** (1994), 1–52.

[25] J. J. Charatonik, W. J. Charatonik and P. Pyrih, *On open mappings of compactifications of the ray onto an arc*, Bull. Polish Acad. Sci. Math. **50** (2002), to appear.

[26] J. J. Charatonik, P. Krupski and J. R. Prajs, *Absolute end points and their mapping properties*, Math. Pannonica, **4** (1993), 103–111.

[27] J. J. Charatonik and T. Maćkowiak, *Confluent and related mappings on arc-like continua — an application to homogeneity*, Topology Appl., **23** (1986), 29–39.

[28] J. J. Charatonik and T. Maćkowiak, *Continua with a dense set of end points*, Rocky Mountain J. Math., **17** (1987), 385–391.

[29] H. Cook, *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math., **60** (1967), 241–249.

[30] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[31] A. Emeryk and Z. Horbanowicz, *On atomic mappings*, Colloq. Math., **27** (1973), 49–55.

[32] J. B. Fugate, *Decomposable chainable continua*, Trans. Amer. Math. Soc., **123** (1966), 460–468.

[33] J. B. Fugate and L. Mohler, *Quasi-monotone and confluent images of irreducible continua*, Colloq. Math., **28** (1973), 221–224.

[34] G. R. Gordh, Jr., *Terminal subcontinua of hereditarily unicoherent continua*, Pacific J. Math., **47** (1973), 457–464.

[35] J. Grispolakis and E. D. Tymchatyn, *Semi-confluent mappings and acyclicity*, Houston J. Math., **4** (1978), 343–357.

[36] W. T. Ingram, *C-sets and mappings of continua*, Topology Proc., **7** (1982), 83–90.

[37] F. B. Jones, *On a certain type of homogeneous plane continuum*, Proc. Amer. Math. Soc., **6** (1955), 735–740.

[38] F. B. Jones, *The aposyndetic decomposition of homogeneous continua*, Topology Proc., **8** (1983), 51–54.

[39] H. Kato, *Generalized homogeneity of continua and a question of J. J. Charatonik*, Houston J. Math., **13** (1987), 51–63.

[40] K. Kuratowski, *Topology*, vol. 2 Academic Press and PWN, New York, London and Warszawa, 1968.

[41] A. Lelek, *On plane dendroids and their end points in the classical sense*, Fund. Math., **49** (1961), 301–319.

[42] A. Lelek, *Some problems concerning curves*, Colloq. Math., **23** (1971), 93–98.

[43] A. Lelek and D. R. Read, *Compositions of confluent mappings and some other classes of functions*, Colloq. Math., **29** (1974), 101–112.

[44] T. Maćkowiak, *Continuous mappings on continua*, Dissertationes Math. (Rozprawy Mat.), **158** (1979), 1–91.

[45] T. Maćkowiak, *The condensations of singularities in an arc-like continua*, Houston J. Math., **11** (1985), 535–558.

[46] T. Maćkowiak, *Singular arc-like continua*, Dissertationes Math. (Rozprawy Mat.), **257** (1986), 1–35.

[47] T. Maćkowiak, *Terminal continua and homogeneity*, Fund. Math., **127** (1987), 177–186.

[48] T. Maćkowiak and E. D. Tymchatyn, *Continuous mappings on continua II*, Dissertationes Math. (Rozprawy Mat.), **225** (1984), 1–57.

[49] T. B. McLean, *Confluent images of tree-like curves are tree-like*, Duke Math. J., **39** (1972), 465–473.

[50] H. C. Miller, *On unicoherent continua*, Trans. Amer. Math. Soc., **69** (1950), 179–194.
[51] S. B. Nadler, Jr., *Hyperspaces of Sets*, M. Dekker, New York, 1978.
[52] S. B. Nadler, Jr., *Continuum Theory: An Introduction*, M. Dekker, New York, Basel and Hong Kong, 1992.
[53] V. C. Nall, *Partially confluent maps and n-ods*, Houston J. Math., **15** (1989), 409–415.
[54] J. Nikiel, *A characterization of dendroids with uncountably many end-points in the classical sense*, Houston J. Math., **9** (1983), 421–432.
[55] J. Nikiel, *On dendroids and their ramification points in the classical sense*, Fund. Math., **123** (1984), 39–46.
[56] J. Nikiel, *On dendroids and their end-points and ramification points in the classical sense*, Fund. Math., **124** (1984), 99–108.
[57] M. A. Owens, *Extremal continua: a class of non-separating subcontinua*, Topology Appl., **23** (1986), 262–270.
[58] P. Pyrih, *Each absolute end point of an atriodic irreducible continuum is an end point*, Questions Answers Gen. Topology, **17** (1999), 61–62.
[59] J. T. Rogers, Jr., *Completely regular mappings and homogeneous, aposyndetic continua*, Canad. J. Math., **33** (1981), 450–453.
[60] J. T. Rogers, Jr., *Homogeneous continua*, Topology Proc., **8** (1983), 213–233.
[61] J. T. Rogers, Jr., *Classifying homogeneous continua*, Topology Appl., **44** (1991), 341–352.
[62] R. H. Rosen, *On tree-like continua and irreducibility*, Duke Math. J., **26** (1959), 113–122.
[63] I. Rosenholtz, *Open maps of chainable continua*, Proc. Amer. Math. Soc., **42** (1974), 258–264.
[64] I. Rosenholtz, *Absolute end points of chainable continua*, Proc. Amer. Math. Soc., **103** (1988), 1305–1314.
[65] A. D. Wallace, *The position of C-sets in semigroups*, Proc. Amer. Math. Soc., **6** (1955), 639–642.
[66] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ. 28, Providence, 1942. reprinted with corrections 1971.

Mathematical Institute, University of Wrocław, Wrocław, Poland

Instituto de Matemáticas, Universidad Nacional Autónoma de México, México, D. F., México
E-mail address: jjc@math.unam.mx