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**NON-REALCOMPACT PRODUCTS WITH A
METRIC FACTOR**

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ABSTRACT. Given a metric space X we characterize those Y so that the realcompactification of $X \times Y$ is just the product of X with the realcompactification of Y . Examples are constructed to illustrate that the properties involved do depend on the metric spaces

1. INTRODUCTION

We are studying the general question of when $v(X \times Y)$ is the same as $vX \times vY$ where vX denotes the realcompactification of the space X (see below for definitions). The situation for $\beta(X \times Y)$ has a very nice resolution and is well understood by Glicksberg's Theorem [6]. However, Hušek [7] has shown that there is no topological property of $X \times Y$ which is equivalent to the equality $v(X \times Y) = vX \times vY$ (where equality of extensions of any space X is understood to mean via a homeomorphism which is the identity on X). In this paper we are studying a restricted question in which the factor X is assumed to be metric. Comfort [2] has shown that if X is locally compact and realcompact, then $v(X \times Y) = X \times vY$ for all spaces Y .

The results in this paper are improvements of results that appeared in [3]. The examples included in this paper are much simpler than the construction in [3] and the theorem is a generalization

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from the space \mathbb{Q} to all metric spaces. We have attempted to make the paper fairly self-contained.

2. PRELIMINARIES

For a space X , $C(X)$ denotes the set of all real-valued continuous functions defined on X , while $C^*(X)$ denotes the set of members of $C(X)$ which are bounded. We say that a subspace S of X is C -embedded (C^* -embedded) in X if every function in $C(S)$ ($C^*(S)$) can be extended to a function in $C(X)$ ($C^*(X)$). A subset of X which is the preimage, by a member of $C(X)$, of a closed subset of the reals is a zero-set of X , and cozero sets are the preimages of open subsets of the reals. We will say that the *type* of an open set G in a space X is the least cardinal κ such that G is equal to the union of at κ cozero subsets of X . Recall that sets A and B are said to be *completely separated* in X if there is a function $f \in C(X)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

A space X is said to be realcompact if X can be embedded as a closed subspace of a power of the real line. Note that this property is therefore closed hereditary. The Hewitt realcompactification, νX , of a space X is the smallest subspace of βX , containing X that is realcompact, where βX is the Stone-Cech compactification of X . It is known that νX is the largest subspace of βX in which X is C -embedded. It also follows immediately that a point $p \in \beta X$ is a member of νX precisely if for each $f \in C^*(\beta X)$, $f(p) = f(x)$ for some $x \in X$.

Locally finite families of subsets of a space play a very important role in our results. A family of subsets of a space X is called locally finite if every point of X has a neighborhood which meets only finitely many members of the family. Metric spaces can be characterized as those regular spaces X which possess a basis for the topology which can be written as a countable union of locally finite subcollections (see [4, 4.4.7]).

The density of a space X is the smallest cardinality of a dense subset. A space is separable if its density is countable. A space is locally separable, or locally of density at most κ , if every point has a neighborhood which is separable, or of density at most κ .

Some other well known results will be required.

Proposition 2.1. [5, 1.15] *Two sets are completely separated if (and only if) they are contained in disjoint zero sets.*

Proposition 2.2. [4, 4.1.15] *If X is a metric space and κ is a cardinal which is less than the density of X , then X contains a closed discrete set of cardinality κ .*

We will just sketch the proof of this well-known result since it is so integral to most of our constructions. The usual proof [5, 5.12] in the case $\kappa = \omega_1$ generalizes trivially.

Proposition 2.3. *Let κ be an uncountable regular cardinal. Every real-valued continuous function on the ordinal space κ is constant on a final segment of κ .*

Proof: The basic fact upon which this result depends is that the closures of any two cofinal subsets of the ordinal space κ will meet. To see this, suppose that A and B are cofinal in κ . Inductively, choose a sequence of points a_n, b_n in A, B , respectively, so that $a_n < b_n < a_{n+1}$ for each n . By the assumption on κ , the sequence $\{a_n : n \in \omega\}$ will have a least upper bound in κ and this will be a common limit point of A and B . A similar argument shows easily that κ is a countably compact space.

Fix any f in $C(\kappa)$ and consider the subspace $f(\{\alpha : \alpha < \kappa\})$ of the reals. The intersection of the descending chain of closed sets, $\{\overline{f((\alpha, \kappa))} : \alpha < \kappa\}$ is not empty because κ has uncountable cofinality. Indeed, if for each integer M , there were some α_M such that $[-M, M] \cap \overline{f((\alpha_M, \kappa))}$ is empty, then setting α greater than α_M for each M would mean that $\overline{f((\alpha, \kappa))}$ were empty. Therefore, there must be an integer M so that each of the closed sets $\overline{f((\alpha, \kappa))}$ meets the compact set $[-M, M]$.

Now, let r be any point in this intersection. It follows then that for each $n > 0$, $f^{-1}([r + 1/n, \infty) \cup (-\infty, r - 1/n])$ and $f^{-1}([r - 1/(2n), r + 1/(2n)])$ are disjoint closed subsets of κ with the latter being unbounded. Therefore, there is some $\alpha_n < \kappa$ bounding the former, and by the uncountable cofinality of κ , a single α such that $f^{-1}([r + 1/n, \infty) \cup (-\infty, r - 1/n]) \subset [0, \alpha]$ for each n . This shows that $f(\beta) = r$ for each $\beta > \alpha$. \square

It will be useful to note that it now follows that if κ is an uncountable regular cardinal, the ordinal space κ has the property

that $v\kappa$ is just the one-point compactification; equivalently, the space $\kappa + 1 = [0, \kappa]$. By contrast, if X is a metric space whose cardinality is not *measurable*, then X is realcompact (see [4]). This assertion is virtually the same as the definition of a measurable cardinal; a reader unfamiliar with the concept will likely be satisfied with the assurance that such cardinals are extremely large and cannot even be proven to exist using the usual axioms of set-theory.

3. THE THEOREMS

In [3, Theorem 5.5], a characterization is found of those Y such that $v(\mathbb{Q} \times Y) = \mathbb{Q} \times vY$. The result is exactly as in Theorem 3.1 below for the case $X = \mathbb{Q}$. As mentioned above we extend that result by considering all metric spaces X . More precisely we do not concern ourselves with non-realcompact metric spaces X as these involve large cardinals and do not seem immediately of interest. A reader interested in those spaces can note that our results will actually characterize those Y with the property that $X \times Y$ is C^* -embedded in $X \times vY$ because of the following result.

Proposition 3.1. *If X is a realcompact space then $v(X \times Y) = X \times vY$ if and only if $X \times Y$ is C^* -embedded in $X \times vY$.*

Proof: First of all, if $v(X \times Y) = X \times vY$ then by the definition of $v(X \times Y)$, it even follows that $X \times Y$ is C -embedded in $X \times vY$.

Conversely, assume that $X \times Y$ is C^* -embedded in $X \times vY$. It follows that, up to homeomorphism, $X \times vY$ is a subspace of $\beta(X \times Y)$. To see that it is $v(X \times Y)$ it suffices to check that it is realcompact and that no smaller subspace containing $X \times Y$ is realcompact. First of all, it is realcompact because it is the product of the realcompact spaces X and vY (the product of a closed subset of a power of \mathbb{R} with another is again a closed subset of yet a larger power of \mathbb{R}). Additionally, if $p \in vY \setminus Y$ and $x \in X$, then $X \times vY \setminus \{(x, p)\}$ is not realcompact since it has a closed subspace, namely $\{x\} \times (vY \setminus \{p\})$, which is not realcompact as it is homeomorphic to $vY \setminus \{p\}$ which is not realcompact. \square

First, we consider the locally separable case.

Theorem 3.1. *Let X be a locally separable non-locally compact metric space, then, for all Y ,*

$$v(X \times Y) = X \times vY$$

iff each locally finite countable family of cozero-set subsets of Y is also locally finite in vY .

Proof: First assume that each locally finite countable family of cozero-set subsets of Y is also locally finite in vY . Fix any function $f \in C^*(X \times Y)$. We prove that it extends continuously to $X \times vY$.

Let $(x, p) \in X \times vY$. Since Y is C -embedded in vY , $f \upharpoonright \{x\} \times Y$ clearly extends to $\{x\} \times vY$, let r denote the appropriate value for $f(x, p)$. We will prove that this value will make f continuous on $X \times (Y \cup \{p\})$. It suffices to prove that for any $\epsilon > 0$, (x, p) is not in the closure of $f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))$.

Since f is continuous on $\{x\} \times Y \cup \{p\}$, there is a cozero set neighborhood W of p such that $f(\{x\} \times W) \subset (r - \epsilon/2, r + \epsilon/2)$. Therefore, $X \times W$ is a neighborhood of (x, p) . Let W_Y denote the cozero set $W \cap Y$.

By hypothesis, we can find a countable set, say S , which is dense in a neighborhood of x . Note, then that if

$$\begin{aligned} (x, p) \in \text{cl}[f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))] \quad \text{and} \\ (x, p) \in \text{cl}[(X \times W) \cap (S \times Y) \cap f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))] \quad \text{then} \\ (x, p) \in \text{cl}[(S \times W_Y) \cap f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))] \end{aligned}$$

So it will suffice, as we now do, to prove that

$$(x, p) \notin \text{cl}[(S \times W_Y) \cap f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))] .$$

For each $s \in S$, let C_s denote the cozero-set subset of Y such that

$$(1) \{s\} \times C_s = (\{s\} \times Y) \cap [(S \times W) \cap f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))] .$$

Also, for each $n \in \omega$ and $s \in S$, let

$$(2) U_n = \bigcup \{C_s : s \in S \text{ and } d(s, x) < 1/n\} .$$

Since S is countable, it follows that U_n is a cozero set (i.e., it has type ω). If $y \notin \overline{W}$, then $Y \setminus \overline{W}$ is a neighborhood of y which, by (1)

and (2), misses all of the U_n 's. If $y \in \overline{W}$, then, by the continuity of f at (x, y) , we can find an integer n and an open set W_1 , such that $y \in W_1 \subset W$ and, with $B(x; 1/n)$ denoting the usual ball, we have

$$(3) f(B(x; 1/n) \times W_1) \subset (r - \frac{2\epsilon}{3}, r + \frac{2\epsilon}{3}) .$$

Now take any $k > n$ and any $s \in S$ such that $d(s, x) < 1/k$. Since $s \in B(x; 1/k) \subset B(x; 1/n)$, the following inclusions hold:

$$f(\{s\} \times C_s) \subset \mathbb{R} \setminus (r - \epsilon, r + \epsilon) \text{ by (1) and (2)}$$

and

$$f(\{s\} \times W_1) \subset (r - \frac{2\epsilon}{3}, r + \frac{2\epsilon}{3}) \text{ by (3)} .$$

This shows that $C_s \cap W_1$ is empty for each $s \in B(x; 1/k)$, which proves that $W_1 \cap U_k$ is empty for all $k > n$. So, $\{U_n : n \in \omega\}$ is locally finite in Y , and by the hypothesis on Y , is also locally finite in vY .

Since $\{U_n : n \in \omega\}$ is locally finite at p , there is a neighborhood $W_2 \subset W$ of p such that $W_2 \cap U_n$ is empty for all $n > k$ for some fixed k . So, $B(x; \frac{1}{k+1}) \times W_2$ is a neighborhood of (x, p) and for all $s \in B(x; \frac{1}{k+1}) \cap S$, we have

$$(4) (B(x; \frac{1}{k+1}) \times W_2) \cap (\{s\} \times C_s) = \emptyset$$

since $C_s \subset U_{k+1}$ and $W_2 \cap U_{k+1} = \emptyset$. We claim that

$$(B(x; \frac{1}{k+1}) \times W_2) \cap [(S \times W_Y) \cap f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))] = \emptyset .$$

Assume otherwise, and let s be a member of $S \cap B(x; \frac{1}{k+1})$ such that

$$(\{s\} \times W_2) \cap [(\{s\} \times W_Y) \cap f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))] \neq \emptyset .$$

Using (1), this implies $(\{s\} \times W_2) \cap (\{s\} \times C_s) \neq \emptyset$, which contradicts (4). Therefore, (x, p) is not in the closure of $(S \times W_Y) \cap f^{-1}(\mathbb{R} \setminus (r - \epsilon, r + \epsilon))$.

For the converse direction, assume that there is a countable family $\{U_n : n \in \omega\}$ of cozero sets of Y which are locally finite in Y but not at some $p \in vY$. Fix $x \in X$ such that x does not have a

compact neighborhood. Hence for each $\epsilon > 0$, $\text{cl}[B(x; \epsilon)]$ contains an infinite closed discrete set.

Inductively select a sequence of closed discrete sets as follows. Let $\epsilon_1 = 1$ and let $\{s(1, m) : m \in \omega\} \subset B(x; \epsilon_1)$ be a closed discrete subset of X which does not contain x . Let $\epsilon_2 > 0$ be chosen so that $B(x; 2\epsilon_2)$ is disjoint from $\{s(1, m) : m \in \omega\}$. Continue choosing closed discrete sets $\{s(n, m) : m \in \omega\} \subset B(x, \epsilon_n)$ and values $\epsilon_{n+1} > 0$ so that $B(x; 2\epsilon_{n+1}) \cap \{s(n, m) : m \in \omega\} = \emptyset$ for all $n \geq 1$.

For each n, m , choose $0 < \delta_{n,m} < \frac{1}{n+m}$ small enough so that $B(s(n, m); \delta_{n,m}) \subset B(x; \epsilon_n) \setminus B(x; \epsilon_{n+1})$ and $d(s(n, m), s(n', m')) > 2\delta_{n,m}$ for all $(n, m) \neq (n', m')$. The family $\{B(s(n, m); \delta_{n,m}) : 0 \neq n, m \in \omega\}$ is clearly not locally finite at x .

Claim: $\{B(s(n, m); \delta_{n,m}) : 0 \neq n, m \in \omega\}$ is pairwise disjoint and is locally finite at each point of X other than x .

Let $x' \in X$ be distinct from x and let n be large enough so that $x' \notin B(x; 2\epsilon_n)$. Then $B(x'; \epsilon_n)$ misses $B(s(n', m'); \delta_{n',m'})$ for all $n' \geq n$. It suffices to prove then that $\{B(s(k, m); \delta_{k,m}) : m \in \omega\}$ is locally finite for any given $k \geq 1$. Recall that $\{s(k, m) : m \in \omega\}$ is closed and discrete. Fix $\epsilon' > 0$ so that $B(x'; \epsilon') \cap \{s(k, m) : m \in \omega\}$ has at most one point. Now consider $B(x'; \epsilon'/2)$. Let $m_0 > 0$ be such that $1/m_0 < \epsilon'/2$ and let $m > m_0$. Then $\delta_{n,m} < \frac{1}{k+m} \leq 1/m < \frac{1}{m_0}$ so $\delta_{k,m} < \frac{\epsilon'}{2}$. Since $d(x', s(k, m)) > \epsilon'/2 + \delta_{k,m}$, we have

$$B(x'; \epsilon'/2) \cap B(s(k, m); \delta_{k,m}) = \emptyset$$

which proves the Claim.

For each n , fix an increasing sequence $\{Z(n, m) : m \in \omega\}$ of zero sets in Y whose union is U_n . Now $B(s(n, m); \delta_{n,m}) \times U_n$ is a cozero set since it is a product of cozero sets and similarly, $\{s(n, m)\} \times Z(n, m)$ is a zero set. The disjoint zero sets $\{s(n, m)\} \times Z(n, m)$ and $X \times Y \setminus [B(s(n, m); \delta_{n,m}) \times U_n]$ will be completely separated. So, we can pick a continuous function

$$f_{n,m} : X \times Y \rightarrow [0, 1]$$

such that

$$f_{n,m}(X \times Y \setminus [B(s(n, m); \delta_{n,m}) \times U_n]) = 0$$

and

$$f_{n,m}(\{s(n,m)\} \times Z(n,m)) = 1 .$$

Define the function $f = \sum_{n,m} f_{n,m}$. We will show that f is continuous on $X \times Y$. There are two cases.

Case 1: Let $x' \in X \setminus \{x\}$. The family $\{B(s(n,m); \delta_{n,m}) : n, m \in \omega\}$ is locally finite at x' , so in a neighborhood of (x', y) we just get a finite number of nonzero terms in the summation $\sum_{n,m} f_{n,m}$. Hence, f is continuous at all points of $\{x'\} \times Y$.

Case 2: To prove the continuity of f at all points on $\{x\} \times Y$ we fix a point $y \in Y$. We are given that $\{U_n : n \in \omega\}$ is locally finite at y so there exists a neighborhood W of y and $m_0 \in \omega$ such that $W \cap U_n = \emptyset$ for all $n > m_0$. We fix $k > m_0$ large enough so that $B(s(n,m); \delta_{n,m})$ and $B(x; \epsilon_k)$ are disjoint for all $n \leq m_0$. It follows that

$$(B(s(n,m); \delta_{n,m}) \times U_k) \cap (B(x; \epsilon_k) \times W) = \emptyset$$

by considering $n \leq m_0$ and $n > m_0$ separately. Since the support of $f_{k,m}$ is contained in $B(s(k,m); \delta_{k,m}) \times U_k$, $B(x; \epsilon_k) \times W$ is a neighborhood of (x, y) which misses all supports of $f_{n,m}$. Hence f is identically 0 on a neighborhood of (x, y) , and note that f is identically 0 on $\{x\} \times Y$. Thus, (x, p) , being a closure point of $\{x\} \times Y$, is in the closure of $f^{-1}(0)$.

Since the family $\{U_n : n \in \omega\}$ is not locally finite at p and for all n, m , $Z(n, m) \subset U_n$, (x, p) is also a closure point of the zero set $f^{-1}(1)$ which contains $\bigcup_{n,m} \{s(n,m)\} \times Z(n,m)$. Therefore, the function f does not extend to (x, p) and, consequently, does not extend to $X \times vY$. \square

The next theorem handles the locally non-separable case.

Theorem 3.2. *Let X be any realcompact metric space. Let κ be the smallest cardinal such that each point has a neighborhood of density $< \kappa$. Then for all Y ,*

$$v(X \times Y) = X \times vY$$

iff for each $\lambda < \kappa$ and each locally finite family $\{U_n : n \in \omega\}$ of open sets of type λ in Y , the family is also locally finite in vY .

Proof: Let $(x, p) \in X \times vY$ and $f \in C^*(X \times Y)$. We assume that for each $\lambda < \kappa$ and each locally finite countable family $\{U_n : n \in \omega\}$

of open subsets of Y of type λ is also locally finite in vY and prove that f extends continuously to $X \times vY$.

The first part of the proof is basically the same as the proof of Theorem 3.1 by choosing a set S which is of cardinality $\lambda < \kappa$ and is dense in a neighborhood of x .

For the converse, we may assume that $\kappa > \omega_1$ since we have already proven Theorem 3.1. We assume that a countable family of open sets $\{U_n : n \in \omega\}$ is locally finite in Y but not at $p \in vY$, where each U_n is of type λ and $\lambda < \kappa$. We can write

$$U_n = \bigcup_{\alpha < \lambda} C_\alpha^n$$

where C_α^n are cozero sets in Y . Also, each C_α^n can be expressed as $\bigcup_{m \in \omega} Z_{\alpha,m}^n$ where each $Z_{\alpha,m}^n$ is a zero set in Y . So we have

$$U_n = \bigcup_{\alpha < \lambda} \bigcup_{m \in \omega} Z_{\alpha,m}^n.$$

Note that for each n, α, m , $Z_{\alpha,m}^n$ is completely separated from $Y \setminus U_m$.

Fix $x \in X$ such that x has a neighborhood of density not less than $\max\{\lambda, \omega_1\}$. By Proposition 2.2, the closure of each $B(x; \epsilon)$ contains a closed (in X) discrete subset of cardinality λ .

We start an inductive construction again by setting $\epsilon_1 = 1$. Let $\{s(1, (\alpha, m)) : (\alpha, m) \in \lambda \times \omega\} \subset B(x; \epsilon_1) \setminus \{x\}$ be closed and discrete in X . Since $x \notin \text{cl} [\{s(1, (\alpha, m)) : (\alpha, m) \in \lambda \times \omega\}]$ we pick $0 < \epsilon_2 \leq \frac{1}{2}$ such that $d(x, s(1, (\alpha, m))) > 2\epsilon_2$, for all $(\alpha, m) \in \lambda \times \omega$. Repeat this induction for $n = 2, 3, \dots$ so that $\{s(n, (\alpha, m)) : (\alpha, m) \in \lambda \times \omega\}$ is a closed discrete subset of $B(x; \epsilon_n) \setminus B(x; 2\epsilon_{n+1})$ where $0 < \epsilon_n \leq \frac{1}{n}$.

Again, for each n and (α, m) choose $0 < \delta_{n,(\alpha,m)} < \frac{1}{n+1}$ small enough so that

$$B(s(n, (\alpha, m)); \delta_{n,(\alpha,m)}) \subset B(x; \epsilon_n) \setminus B(x; \epsilon_{n+1})$$

and

$$d(s(n, (\alpha, m)), s(n', (\alpha', m'))) > 2\delta_{n,(\alpha,m)},$$

for all $(n', (\alpha', m')) \neq (n, (\alpha, m))$. It is easily checked that the family

$$\{B(s(n, (\alpha, m)); \delta_{n,(\alpha,m)}) : n, m \in \omega, \alpha < \lambda\}$$

is not locally finite at x .

We claim that $\{B(s(n, (\alpha, m))); \delta_{n, (\alpha, m)}) : n, m \in \omega, \alpha < \lambda\}$ is pairwise disjoint and locally finite at each point of X other than x .

The proof of this claim is similar to the analogous claim in Theorem 3.1.

Now $B(s(n, (\alpha, m))); \delta_{n, (\alpha, m)}) \times C_\alpha^n$ is a cozero set which contains the zero set $\{s(n, (\alpha, m))\} \times Z_{\alpha, m}^n$. So, we can pick a continuous function

$$f_{n, (\alpha, m)} : X \times Y \rightarrow [0, 1]$$

such that

$$f_{n, (\alpha, m)}(X \times Y \setminus [B(s(n, (\alpha, m))); \delta_{n, (\alpha, m)}) \times C_\alpha^n] = 0$$

and

$$f_{n, (\alpha, m)}(\{s(n, (\alpha, m))\} \times Z_{\alpha, m}^n) = 1 .$$

Define the function $f = \sum_{n, (\alpha, m)} f_{n, (\alpha, m)}$. We can prove that f is continuous on $X \times Y$. The proof is also a routine modification of the arguments in Theorem 3.1.

The point (x, p) is in the closure of $f^{-1}(0)$. Also, (x, p) is a closure point of the set $\bigcup_{n, (\alpha, m)} \{s(n, (\alpha, m))\} \times Z_{\alpha, m}^n$. Since $\bigcup_{n, (\alpha, m)} \{s(n, (\alpha, m))\} \times Z_{\alpha, m}^n$ is contained in $f^{-1}(1)$, (x, p) is then a closure point of the zero set $f^{-1}(1)$. So the function f does not extend to $X \times vY$. \square

4. EXAMPLES

Before presenting the examples, we recall a few technical results which will allow us to more easily construct vY for some Y . In the first of these, we do not know if the assumption that the basis elements be clopen is necessary (requiring $X \setminus U$ is C^* -embedded is sufficient, of course), and we also do not have a good reference, although it is probably well-known.

Lemma 4.1. *Assume that a space X has a local basis of clopen sets, U , at the point x such that $X \setminus U$ is realcompact, then X is realcompact.*

Proof: Assume that $p \in \beta X \setminus X$; we prove that $p \notin vX$. Fix any U in the clopen basis for x so that p is not in the closure of U . Now consider the space $\{p\} \cup (X \setminus U)$. Since $X \setminus U$ is realcompact, $X \setminus \{p\}$ is not C -embedded in $\{p\} \cup (X \setminus U)$, so we may choose a function $f \in C(X \setminus U)$ which does not extend continuously to $\{p\}$.

Clearly the function g extending f to all of X by setting $g(y) = 0$ for all $y \in U$ is in $C(X)$ but will not extend continuously to p , which shows that $p \notin vX$. \square

Proposition 4.2. [5, 12G] *If κ is not measurable then the topological sum of a family of κ many realcompact spaces is realcompact.*

Proposition 4.3. [5, 8.15] *A first-countable realcompact space is hereditarily realcompact.*

We have already seen that if κ is an uncountable regular cardinal, then the ordinal space κ has the property that $v\kappa$ is the one-point compactification and that every continuous real-valued function is eventually constant. Our examples will be based on the classic example known as the Tychonoff plank (subspace of the product)

$$\mathbb{T} = ((\omega_1 + 1) \times (\omega + 1)) \setminus \{(\omega_1, \omega)\}$$

which has a pair of distinguished closed sets, the top edge $\omega_1 \times \{\omega\}$ and the right edge $\{\omega_1\} \times \omega$.

It is also the case that $v\mathbb{T}$ is its one-point compactification, and each of these spaces has the property that locally finite families of cozero sets are finite. Such spaces are said to be *pseudocompact* and figure prominently in Glicksberg's theorem. As we saw in the proof of Theorem 3.1, such families can be used to construct continuous functions, even unbounded functions.

Example 1. *Let X_κ denote any realcompact metric space in which each point has a neighborhood of density no less than the regular uncountable cardinal κ . We show there is a space Y for which $v(\mathbb{Q} \times Y) = \mathbb{Q} \times vY$ but $v(X_\kappa \times Y) \neq X_\kappa \times vY$.*

Proof: Consider first, the product space $[(\omega + 1) \times L_\kappa]$. Here, we let L_κ denote the space obtained by taking two copies (**bottom** and **top**) of the set $\kappa + 1$, i.e., $\{b_\alpha : \alpha \leq \kappa\}$ and $\{t_\alpha : \alpha \leq \kappa\}$, identifying $\{b_\kappa, t_\kappa\}$, making every point isolated except the collapsed point, and neighborhoods of this point are the complements of sets of size less than κ . We will use t_κ to refer to the collapsed point.

Let Y denote the subspace

$$[(\omega + 1) \times L_\kappa] \setminus [\{\omega\} \times (\{b_\alpha : \alpha < \kappa\} \cup \{t_\kappa\})].$$

For convenience, let $t(n, \alpha)$ denote the point (n, t_α) and similarly for $t(\omega, \alpha), b(n, \alpha)$.

The neighborhood base for $t(n, \alpha)$, $\alpha < \kappa$, is $\{t(n, \alpha)\}$ and the neighborhood base for $t(\omega, \alpha)$ is

$$\{ B(t(n, \omega); m) = \{t(n, \alpha) : m < n \leq \omega\} : m \in \omega \}.$$

Also, the neighborhood base for $b(n, \alpha)$ is $\{b(n, \alpha)\}$, while the neighborhood base for $t(n, \kappa)$ is $\{t(n, \alpha), b(n, \alpha) : \beta < \alpha < \kappa\} \cup \{t(n, \kappa)\}$. Observe that if $f \in C(Y)$ and $f(t(n, \kappa)) = r$, then there exists $\beta < \kappa$ such that $f(t(n, \alpha)) = f(b(n, \alpha))$, for all $\beta < \alpha < \kappa$. Also, if $\{Z_\alpha : \alpha < \lambda\}$ with $\lambda < \kappa$ are zero sets of Y and $t(n, \kappa) \notin \bigcup_{\alpha < \lambda} Z_\alpha$, then there exists $\beta < \kappa$ such that $t(n, \gamma), b(n, \gamma) \notin \bigcup_{\alpha < \lambda} Z_\alpha$ for all $\beta < \gamma < \kappa$.

Now we show that $vY \setminus Y$ is just a single point which we will call p .

Claim 1. *Let p have the same neighborhood base as the point $t(\omega, \kappa)$ (if we had included it), then $p \in vY$.*

Proof: Let f be any member of $C(Y)$. We prove that there exists a real r such that for all $\epsilon > 0$, there exists $j \in \omega$ and $\beta < \kappa$ such that $f(t(k, \alpha)) = f(b(k, \alpha)) = f(t(k, \kappa)) \in (r - \epsilon, r + \epsilon)$ for all $k > j$ and $\alpha > \beta$. Thus, $f(t(\omega, \alpha)) = r$ for all $\alpha > \beta$.

For each k , let $r_k = f(t(k, \kappa))$ and fix β_k so that $f(t(k, \alpha)) = r_k = f(b(k, \alpha))$ for all $\alpha > \beta_k$. Let $\beta = \sup_k \beta_k$. Let $r = f(t(\omega, \beta + 1))$ and $\epsilon > 0$ be given. Fix $\alpha_0 > \beta + 1$; there exists j_{α_0} such that $f(B(t(\omega, \alpha_0); j_{\alpha_0})) \subset (r - \epsilon/2, r + \epsilon/2)$. Now check the conclusion: let $\alpha > \beta$ and $k > j = j_{\alpha_0}$; it follows that

$$f(t(k, \alpha)) = f(b(k, \alpha)) = r_k = f(t(k, \alpha_0));$$

hence, $f(t(k, \alpha)), f(b(k, \alpha)) \in (r - \epsilon/2, r + \epsilon/2)$ and, by continuity, $f(t(\omega, \alpha)) \in [r - \epsilon/2, r + \epsilon/2]$. Therefore, f extends continuously to p completing the proof that $p \in vY$.

Furthermore, the neighborhood filter for p looks like

$$\begin{aligned} & \{ \{t(k, \alpha) : j < k \leq \omega, \beta < \alpha \leq \kappa\} \cup \\ & \{b(k, \alpha) : j < k < \omega, \beta < \alpha \leq \kappa\} : j \in \omega, \beta < \kappa \} \end{aligned}$$

which is the same as would be for the point $t(\omega, \kappa)$. \square

To see that $vY = Y \cup \{p\}$, we have to check that $Y \cup \{p\}$ is realcompact. Now that we know the neighborhood basis for p , this follows easily from lemmas 4.1 and 4.2.

Now we show that $v(\mathbb{Q} \times Y) = \mathbb{Q} \times vY$ by using Theorem 3.1. Suppose that $\{W_n : n \in \omega\}$ is a locally finite family of open sets in Y such that each W_n is of type $\lambda < \kappa$. We must prove that $\{W_n : n \in \omega\}$ is also locally finite in vY . Note that for all k and n , if $b(k, \kappa) \in \overline{W_n}$, then $b(k, \kappa) \in W_n$. There are two cases.

Case 1: There exists j such that for all $k > j$ and for all $n > j$, $b(k, \kappa) \notin W_n$.

Since each W_n , $n > j$, has type $\lambda < \kappa$, W_n is a union of $\leq \lambda$ zero sets. Set $W_n = \bigcup_{\alpha < \lambda} Z(n, \alpha)$. Fix any $k > j$; hence, $b(k, \kappa) \notin \bigcup_{n > j} W_n$.

Also, we can now observe by Claim 2 that p has a neighborhood avoiding $\bigcup_{n > j} W_n$.

Claim 2. *There exists β_k such that for all $\alpha > \beta_k$, $b(k, \alpha), t(k, \alpha) \notin \bigcup_{n > j} W_n$.*

For each $n > j$ and $\alpha < \lambda$, $b(k, \kappa) \notin Z(n, \alpha)$. Hence, there exists $\gamma_{n, \alpha}$ such that for all $\delta > \gamma_{n, \alpha}$, $b(k, \delta)t(k, \delta) \notin Z(n, \alpha)$. The family $\{\gamma_{n, \alpha} : n > j, \alpha < \lambda\}$ has cardinality less than κ , so it has a supremum, β_k which works.

Similarly, let $\beta = \sup_{k > j} \beta_k$, and we observe that for all $k > j$, $t(k, \alpha), b(k, \alpha) \notin \bigcup_{n > j} W_n$. Fix any $\alpha > \beta$. We have

$$\{t(m, \alpha) : j < m < \omega\} \cap \bigcup_{n > j} W_n = \emptyset.$$

Since $\bigcup_{n > j} W_n$ is open, it follows that $t(\omega, \alpha) \notin \bigcup_{n > j} W_n$.

Now we handle the other case:

Case 2: For each j , there are $k > j$ and $n > j$ such that $b(k, \kappa) \in W_n$. In this case we show that $\{W_n : n \in \omega\}$ is not locally finite in Y . We recursively choose members of increasing sequences: $k_1, n_1 < k_2, n_2 < \dots$ such that $b(k_j, \kappa) \in W_{n_j}$. For each $j \in \omega$, there exists β_j such that $g(k_j, \alpha) \in W_{n_j}$ for all $\alpha > \beta_j$. Let $\beta = \sup_{j \in \omega} \{\beta_j\}$ and let $\alpha > \beta$. It follows that $g(k_j, \alpha) \in W_{n_j}$ for each j ; hence, $t(\omega, \alpha)$ does not have a neighborhood meeting only finitely many of the $\{W_{n_j} : n \in \omega\}$.

Finally, to prove that $X_\kappa \times Y$ is not C^* -embedded in $X_\kappa \times vY$, we consider the countable family $\{U_n : n \in \omega\}$ of open sets in Y where $U_n = \{b(n, \alpha) : \alpha < \kappa\}$. Clearly, U_n has type κ . We observe

that $\{U_n : n \in \omega\}$ is locally finite in Y but each neighborhood of p hits infinitely many U_n 's. Thus, $\{U_n : n \in \omega\}$ is not locally finite in vY . \square

We have been unable to come up with a condition, even in the case of \mathbb{Q} , which does not explicitly mention vY . We anticipated that there may be a condition relating locally finite families to disjoint zero sets arising from the fact that points of $p \in vY$ are naturally viewed as (countably complete) ultrafilters of zero sets of Y . In particular consider the condition

if a zero set Z is completely separated from each member of a locally finite family $\{U_n : n \in \omega\}$ of cozero sets, then Z is completely separated from the union.

The intuition is somewhat compelling: if a point $p \in vY$ is a potential point of accumulation of the family $\{U_n : n \in \omega\}$, but is not a member of the closure of any given U_n , then for each n , there would be a member Z_n of the ultrafilter p which is completely separated from U_n . By the above condition, the zero set $Z = \bigcap_n Z_n$ would be completely separated from the family $\{U_n : n \in \omega\}$ ensuring that p is not a point of accumulation. There are two flaws. One is that we have assumed that $p \in \overline{U_n}$ for each n but perhaps a given family $\{U_n : n \in \omega\}$ can be replaced by one satisfying this and still give a valid conclusion. The second flaw is that there is no guarantee that a failure of this condition would give rise to a suitable point in vY to contradict the local finiteness of the family $\{U_n : n \in \omega\}$ in vY .

This next pair of examples is to demonstrate that the above condition is neither necessary nor sufficient. Note that weakening the restriction that the zero set be completely separated from each of the U_n 's to simply being disjoint would be far too restrictive on Y since it would basically guarantee that the cozero sets are clopen.

Example 2. *There is an example of a space Y in which there is a locally finite pairwise disjoint family $\{U_n : n \in \omega\}$ of cozero sets which is not locally finite in vY , i.e., $v(\mathbb{Q} \times Y) \neq \mathbb{Q} \times vY$. Additionally, Y has the property that whenever a zero set Z is completely separated from each member of a countable locally finite family of cozero sets, it is completely separated from the union.*

Proof: In this example we use a version of the well-known Tychonoff spiral (see [8, 1]). A building block for our spiral is a version of the Tychonoff plank. The plank \mathbb{P} will denote the space $(\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$ considered as a subspace of the product (with the corner point removed). We view this plank as having three distinguished closed copies of ω_1 , namely the top edge $T = \omega_1 \times \{\omega_1\}$, the right edge $R = \{\omega_1\} \times \omega_1$, and the diagonal $D = \{(\alpha, \alpha) : \alpha \in \omega_1\}$. For each $n \in \omega$, let \mathbb{P}_n denote the n -th copy of \mathbb{P} , $\{n\} \times \mathbb{P}$ with edges and diagonal denoted T_n, R_n and D_n .

The plank \mathbb{P} is interesting because of the property that if E is a non-compact closed subset of \mathbb{P} which has compact intersection with each of T, R then it will intersect D in a non-compact subset.

To see this, first note that for each $\alpha \in \omega_1$, $[0, \alpha] \times \omega_1 + 1$ and $\omega_1 \times [0, \alpha]$ are compact subsets of \mathbb{P} . Now assume that E meets each of T, R and D in a compact set. We may then fix an α_0 so that E is disjoint from

$$(T \cup D \cup R) \cap ([\alpha_0, \omega_1] \times [\alpha_0, \omega_1]).$$

Further, it follows that for each $\alpha > \alpha_0$, $E \cap ((\{\alpha\} \times \omega_1) \cup (\{\alpha\} \times \omega_1))$ is countable. This can be restated as that for each $\alpha < \omega_1$, there is a pair $\beta, \gamma \in (\alpha, \omega_1)$ such that $(\beta, \gamma) \in E$. We may then recursively construct a sequence of pairs $(\beta_n, \gamma_n) \in E$ so that $\alpha_0 < \max\{\beta_n, \gamma_n\} < \min\{\beta_{n+1}, \gamma_{n+1}\}$. It follows that $\{\beta_n : n \in \omega\}$ and $\{\gamma_n : n \in \omega\}$ have a common supremum, β ; hence, (β, β) is a member of E , contradicting the assumption defining α_0 .

Therefore, if W is any open set containing co-countably many points of D , then the closure of W contains co-countably many points of each of R and T . Note that it then follows that no uncountable subset of T can be completely separated from any uncountable subset of R .

We construct the spiral \mathbb{S} by forming a quotient space of the free union $\bigcup_n \mathbb{P}_n$. For each n , identify the points of T_n with the points of R_{n+1} (and imagine this space “spiralling” upwards in a manner like the covering space for the log function).

We will need to compute $v\mathbb{S}$. We attach a point p with a neighborhood basis given by putting one “corner point” simultaneously into all the component planks:

$\{\{p\} \cup W : W \text{ an open subset of } \mathbb{S},$
and $\mathbb{P}_n \setminus W \text{ is compact for each } n\}$.

We show below that \mathbb{S} is C -embedded in $\mathbb{S} \cup \{p\}$. Since the complement of each clopen neighborhood of p can be expressed as a countable union of compact open sets, it follows from lemmas 4.1 and 4.2 that this space is realcompact; hence, there are no other points of $\beta\mathbb{S}$ in $v\mathbb{S}$. Now to see that \mathbb{S} is C -embedded in $\mathbb{S} \cup \{p\}$, assume that $f \in C(\mathbb{S})$ and for each n , let r_n denote the real number witnessing that f is eventually constant on D_n (from Proposition 2.3). Since D_n cannot be completely separated from any uncountable subset of T_n and D_{n+1} cannot be completely separated from any uncountable subset of $R_{n+1} = T_n$, it follows that $r_n = r_{n+1}$ for all n . Now we set $f(p) = r = r_1$ and show that f is continuous at p . Fix any $\epsilon > 0$ and let $W = f^{-1}(r - \epsilon, r + \epsilon)$. We need to show that $T_n \setminus W$ is compact for each n . Since W contains the closure of $f^{-1}(r - \epsilon/2, r + \epsilon/2)$, it follows that W is an open set which contains a co-countable subset of T_n , R_n , and D_n for each n . From the remarks above it follows that the closed sets $\mathbb{P}_n \setminus W$ are compact for each n .

Finally, we construct our space Y by attaching countably many more copies of the Tychonoff plank. For each n , let $\tilde{U}_n = \{n\} \times (\omega_1 + 1) \times (\omega + 1) \setminus \{(n, \omega_1, \omega)\}$ be a copy of the standard Tychonoff plank. For each n , identify the top edge of \tilde{U}_n , i.e., $\{n\} \times \omega_1 \times \{\omega\}$, with the diagonal D_n . The space vY is completely analogous to $v\mathbb{S}$ in that a single point p is added such that for each neighborhood W of p and each plank P , whether some \mathbb{P}_n or some \tilde{U}_n , $P \setminus W$ is compact.

For each n , let U_n be the subset $\{n\} \times (\omega_1 + 1) \times \omega$. It is easily seen that U_n is a cozero set in Y since, for each $m \in \omega$, $\{n\} \times (\omega_1 + 1) \times \{m\}$ is a clopen subset.

We leave the reader to check that $\{U_n : n \in \omega\}$ is locally finite in Y , and that every neighborhood of p in vY meets each U_n .

To finish our discussion of this example, we must consider a zero set Z together with a locally finite collection $\{C_n : n \in \omega\}$ of cozero sets such that Z is completely separated from each C_n . Note that for each of our planks, because the family $\{C_n : n \in \omega\}$ is locally finite, there will be only finitely many that meet the plank. Also, since Z is completely separated from each C_n , it follows that either

$Z \cap (U_n \cup \mathbb{P}_n)$ is compact for each n , or for each k and n , $C_k \cap (U_n \cup \mathbb{P}_n)$ has compact closure.

Set K to be the union of the closures of the C_n 's (which, because the family is locally finite, is closed).

We wish to show that Z and K are completely separated. Again, we have that either Z or K meets each $U_n \cup \mathbb{P}_n$ in a compact set (recall that K intersected with each plank will be a finite union of closures of some of the C_n). The situation is quite symmetric, so assume for definiteness that Z meets each plank in a compact set. Each point of Z has a clopen compact neighborhood that misses K ; hence, for each plank, Z intersect that plank is contained in a compact open set which is disjoint from K and which meets, at most, four planks (of course we leave this to the reader). Finally, since the collection of planks is locally finite in Y , it follows that this family of clopen sets is locally finite, and hence has clopen union. This union avoids K and contains Z , thus showing that they are completely separated. \square

Example 3. *There is an example of a realcompact space Y in which there is a locally finite family $\{U_n : n \in \omega\}$ of cozero sets together with a zero set Z which is completely separated from each of the U_n 's but which is not completely separated from the union.*

Proof: An interesting compact first countable space is the so-called Alexandroff duplicate of the unit interval. This space, call it \mathbb{D} , is obtained by taking $[0, 1] \times \{0, 1\}$ with the topology as follows: for each $x \in [0, 1]$ and $\epsilon > 0$, each of the following sets is basic open,

$$\{(x, 1)\} \quad \text{and} \quad (x - \epsilon, x + \epsilon) \times \{0, 1\} \setminus \{(x, 1)\}.$$

Note that $[0, 1] \times \{1\}$ is open dense and discrete, while $[0, 1] \times \{0\}$ is homeomorphic to $[0, 1]$. The space is compact because every open cover by basic open sets of $[0, 1] \times \{0\}$ will have a finite subcover, thus omitting only finitely many points of $[0, 1] \times \{1\}$.

The space $X = (\omega + 1) \times \mathbb{D}$ is compact first countable and we let Y be the subspace $X \setminus (\{\omega\} \times ([0, 1] \times \{0\}))$. That is we remove the non-isolated points from the top copy of \mathbb{D} . By Proposition 4.3, Y is realcompact. It is routine to see that $\{\omega\} \times \mathbb{D}$ is a zero-set of X ; hence, $Z = \{\omega\} \times \mathbb{D} \cap Y$ is a zero-set of Y . We now identify a locally finite family $\{U_n : n \in \omega\}$ of cozero sets of Y such that Z is completely separated from each, but not from the union.

For each n , let A_n be a countable dense subset of $[0, 1]$ such that $A_n \cap A_m$ is empty for $n \neq m$. Let $U_n = \{n\} \times (A_n \times \{1\})$. Each U_n consists of countably many isolated points of Y , therefore is easily seen to be a cozero set. Since each basic open set containing a non-isolated point of \mathbb{D} will contain points of $A_n \times \{1\}$ (because A_n is dense in $[0, 1]$), it follows that $\{n\} \times ([0, 1] \times \{0\})$ is contained in the closure of U_n . Since $\{n\} \times \mathbb{D}$ is clopen in Y , it follows that Z is completely separated from U_n . Finally, we check that Z is not completely separated from $\bigcup\{U_n : n \in \omega\}$. Indeed, we show that if W is any open set containing $\overline{U_n}$ for each n , then \overline{W} will meet Z . Fix such a W and recall that for each n , $\{n\} \times ([0, 1] \times \{0\}) \subset W$; hence, $\{n\} \times ([0, 1] \times \{1\}) \setminus W$ is finite. Therefore, there is a countable set $A \subset [0, 1]$ such that for all n and all $a \in A$, $(n, (a, 1)) \in W$. It follows then, that $(\omega, (a, 1)) \in Z \cap \overline{W}$. \square

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