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SEIFERT-TAKENS-SINGHOF FILLINGS OF 3-MANIFOLDS

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Abstract. We give a short description of some results obtained recently about Seifert-Takens-Singhof fillings of 3-manifolds and show how these decompositions are related to other classical concepts used in the study of 3-manifold theory.

1. Seifert-Takens-Singhof Fillings

The motivation for defining Seifert-Takens-Singhof fillings (*STS-fillings*) comes from Seifert fiber spaces, which can be decomposed into solid tori T_i such that $T_i \cap T_j = \partial T_i \cap \partial T_j$ is empty or consists of essential annuli (for $i \neq j$). However, the pieces for an STS-filling need not be solid tori and the manifolds admitting STS-fillings need not be Seifert fiber spaces. The concept of a filling was introduced by Takens [14] in his investigation of the minimal number of critical points of a function on a compact manifold and was later used by Singhof [13] as a central tool in his study of minimal coverings of manifolds with balls. Following these ideas of Seifert, Takens and Singhof we make the following more general definition.

Let $\mathcal{V}$ be a family of compact, connected 3-manifolds with non-empty boundary. We say that a compact, connected 3-manifold M

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has a $(\mathcal{V}, \text{N})$-*STS-filling* if M admits a decomposition into $V_1, \ldots, V_{\text{N}}$ where each V_i is homeomorphic to some manifold of $\mathcal{V}$ and such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ for $i \neq j$. Furthermore, if a point p in *int* (M) meets k different $V_{i_1}, \ldots, V_{i_k}$ for some $2 \leq k$, then we require that a neighborhood of p is as in Fig 1 (here $k = 4$). In this case we say that p *has order* k.

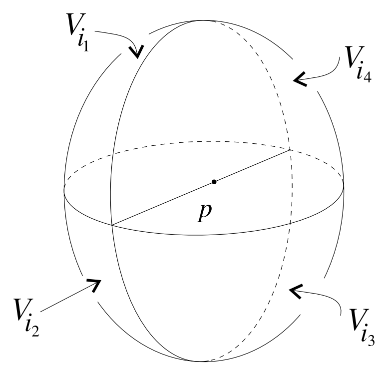

FIGURE 1.

Note that the set of points of order 2 constitutes the interior of compact surfaces whose boundary components are empty or consist of simple closed curves such that all points in a boundary curve C are of the same order $k \geq 3$. Such a curve C is called an *intersection curve of order* k.

Furthermore, if $\partial M \neq \emptyset$ and $p \in \partial M$ meets k different $V_{i_1}, \ldots, V_{i_k}$ for some $k \geq 2$, then a neigborhood of p is required to be as in Fig 2 (here $k = 3$). Again in this case we say that p has order k and note that the set of points on ∂M of order $k \geq 2$ constitutes simple closed curves. All points in one of these simple closed curves C are of the same order $k \geq 2$ and we call such a curve an intersection curve of order k. Observe that each of these $C \subset \partial M$ is also a boundary curve of (at least) one surface formed by interior points of M of order 2.

We note that if M admits a $(\mathcal{V}, \text{N})$-STS-filling $M = \bigcup_{j=1}^{k} V_{i_j}$ where $V_{i_j} \in \mathcal{V}$ and V_{i_j} is *orientable* $(j = 1, \ldots, k)$ then *we can always assume that the order of an intersection curve C is three (resp. two) if $C \subset M - \partial M$ (resp. $C \subset \partial M$).* This can be obtained in

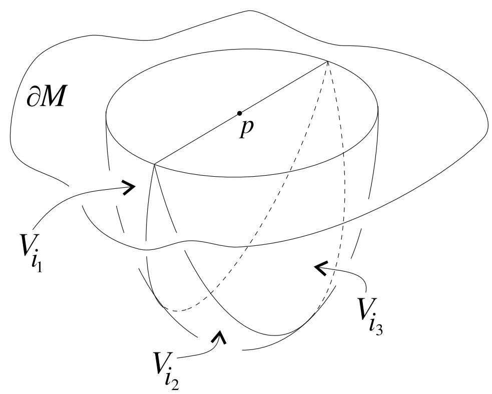

FIGURE 2.

the following way: first, it is easy to see (e.g., [8]) that a regular neighborhood of C, $N(C)$, is a solid torus. Now, perform an isotopy in $N(C)$ as Figure 3(a) (resp. Figure 3(b)) shows when $C \subset M - \partial M$ (resp. $C \subset \partial M$). Note that we may increase the number of

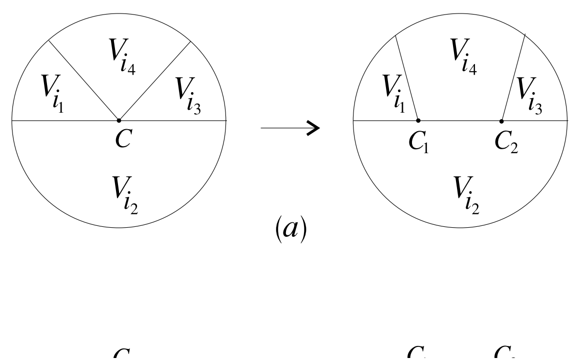

FIGURE 3.

intersection curves after the isotopy but the order of the new curves is as required. Also, note that the family $\{V_{i_j}\}_{j=1}^{k}$ is not changed.

As typical examples (cf. Corollary 1 and Corollary 2 below) consider the case when $\mathcal{V}$ consists of one ball or one solid torus. Note that if $\mathcal{V}$ is the family of (orientable or non-orientable) handlebodies then a $(\mathcal{V}, 2)$-STS- filling of a closed, connected 3-manifold M is precisely a Heegaard splitting of M. It was pointed out in [5] that if $\mathcal{V}$ is the family of *orientable* handlebodies, then every closed, connected (non-orientable) 3-manifold M has a $(\mathcal{V}, 3)$-STS-filling. A classification of all closed, connected 3-manifolds admitting a $(\mathcal{V}, 3)$-STS-filling where $\mathcal{V}$ consists of a 3-ball, a solid torus, a solid Klein bottle, and $P^2 \times I$, was obtained in [6].

On the other hand, note that the existence of a Heegaard splitting shows that every closed, connected 3-manifold admits a decomposition into four 3-balls meeting only along their boundaries. However, the local intersection pattern is different from that of an STS-filling since one is forced to accept points p in M with a local intersection pattern as in Fig 4, where the B_i's are balls.

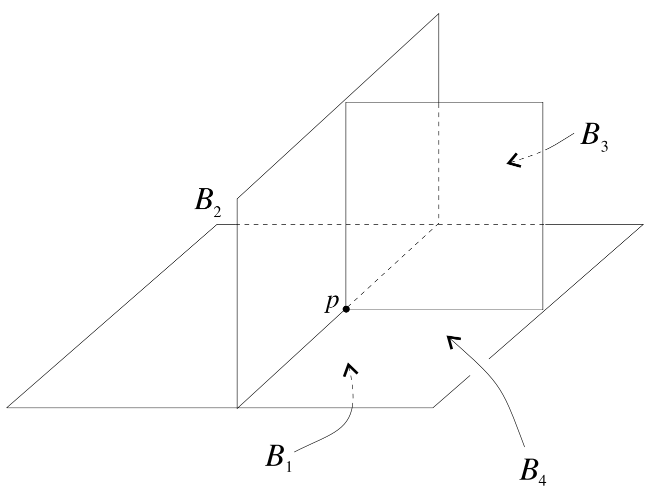

FIGURE 4.

2. SOME RESULTS ABOUT STS-FILLINGS

Now let $\mathcal{T}$ be the family of punctured balls and punctured solid tori (any number of punctures) and let M be a closed, connected 3-manifold. We show in [8] that if M has a $(\mathcal{T}, \text{N})$-STS-filling, then

M is a connected sum of Seifert fiber spaces (for any $N \geq 2$). More precisely, let $\hat{S}(m)$ denote any closed Seifert fiber space (orientable or non-orientable) with arbitrary orbit surface and at most m exceptional fibers, and different from a lens space. Let L or L_i denote a lens space, possibly S^3, but different from $S^2 \times S^1$. Let $\mathcal{B}$ denote the family of connected sums of S^3 and S^2-bundles over S^1 (with arbitrarily many factors). For a 3-manifold M and collections $\mathcal{A}_1$ and $\mathcal{A}_2$ of 3-manifolds, we denote by $M\#\mathcal{A}_1$, resp. $\mathcal{A}_1\#\mathcal{A}_2$, the collection of manifolds that are connected sums of M with a manifold from $\mathcal{A}_1$, resp. manifolds that are a connected sum $A_1\#A_2$ for $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$.

Then in [8] we prove:

Theorem 1. *If the closed 3-manifold M admits a $(\mathcal{T}, N{+}M)$-STS-filling with N punctured solid tori and M punctured balls then M is of the form*

$$M = \hat{S}(m_1)\#\cdots\#\hat{S}(m_k)\#L_1\#\cdots\#L_r\#\mathcal{B}.$$

Furthermore, each $m_i \geq 3$ and $\sum_{i=1}^{k} m_i + r \leq N$.

As a consequence we have the following results:

Corollary 1. *Let $\mathcal{V} = \left\{B^3\right\}$, B^3 a 3-ball. If M has a $(\mathcal{V}, N)$-STS-filling, then $M \in \mathcal{B}$.*

Corollary 2. *Let $\mathcal{V} = \{T\}$, T a solid torus. If M has a $(\mathcal{V}, 3)$-STS-filling, then M is of the form $L\#\mathcal{B}$, $L_1\#L_2\#\mathcal{B}$, $L_1\#L_2\#L_3\#\mathcal{B}$, or $\hat{S}(3)\#\mathcal{B}$.*

To obtain a result similar to Theorem 1 for manifolds with non-empty boundary we let $S(m)$ denote a Seifert fiber space (orientable or non-orientable) with arbitrary orbit surface and at most m exceptional fibers, and with *non empty boundary.*

A *Seifert-web* $W(m_1, \ldots, m_k)$ is a connected manifold obtained from the disjoint union $S(m_1) \cup \cdots \cup S(m_k)$ by attaching 1-handles (either to the same or distinct $S(m_i)$'s). The $S(m_i)$'s are called the *nodes* of the Seifert-web.

We also use the following notations:

- $\hat{\mathcal{S}}$ is the collection of all connected sums of closed Seifert fiber spaces;

$\mathcal{W}$ is the collection of all connected sums of Seifert-webs;
$\mathcal{L}$ is the collection of 3-manifolds that are connected sums of lens spaces L_i;
$\mathcal{F}$ is the collection of all 3-manifolds of the form $\mathcal{B}\#H_1\#\cdots\#H_k\#H$, where $H_1,\ldots,H_k$ are handlebodies (orientable or non-orientable, *not a ball*), H is a punctured ball (with arbitrary many punctures) and k is any non-negative integer.

As before let $\mathcal{T}$ be the family of punctured solid tori and punctured balls (with any number of punctures). A manifold M is a *Seifert union of* $\mathcal{T}$ if M admits a $(\mathcal{T},\textsc{n})$-STS-filling with no points p of order $k\geq 3$. In [7, Theorem 8] we proved

Theorem 2. *A Seifert union of punctured solid tori and punctured balls is in* $\mathcal{F}\#\mathcal{L}\#\mathcal{W}$*. Conversely, every manifold* M *of* $\mathcal{F}\#\mathcal{L}\#\mathcal{W}$ *with* $\partial M\neq\emptyset$ *can be realized as a Seifert union of tori and balls (without punctures).*

Note in Theorem 2 that if $M\in\mathcal{F}\#\mathcal{L}\#\mathcal{W}$, and $\partial M=\emptyset$, then M is a connected sum of S^2-bundles over S^1 and lens spaces, and is therefore clearly a Seifert union of punctured solid tori and punctured balls.

We now obtain the following generalization of Theorem 1.

Theorem 3. *Let* M *be a compact, connected* 3*-manifold (orientable or non-orientable, closed or with non-empty boundary). If* M *has a* $(\mathcal{T},\textsc{n})$*-STS-filling, then* M *is in* $\mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$*.*

Conversely, every manifold M *of* $\mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$ *has a* $(\mathcal{T},\textsc{n})$*-STS-filling.*

Proof: Suppose M has a $(\mathcal{T},\textsc{n})$-STS-filling. If M is closed then Theorem 1 shows that $M\in\mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$. So assume that $\partial M\neq\emptyset$.

As we pointed out before, we may assume that the order of an intersection curve $C\subset M-\partial M$ is three and is two if $C\subset\partial M$. Also, a regular neighborhood of an intersection curve $C\subset M-\partial M$ of order $=3$ is a solid torus. Let R be the manifold obtained from M by removing the interiors of regular neighborhoods of all intersection curves of order $=3$. Then R is a Seifert union of manifolds from $\mathcal{T}$. By Theorem 2, R is of the form

$$R=\mathcal{B}\#L_1\#\cdots\#L_r\#H_1\#\cdots\#H_s\#H\#W_1\#\cdots\#W_t$$

where the H_i's are handlebodies and the W_j's are Seifert-webs. Now M is obtained from R by attaching solid tori to R along tori. If a solid torus is attached to an H_i then this factor of R is changed to a lens space factor. In [10] it was shown that attaching a solid torus to a Seifert fiber space results in a Seifert fiber space, or a connected sum of lens spaces, S^2-bundles over S^1 and handlebodies of genus 1. (The proof given in [10] for orientable Seifert fiber spaces with orientable orbit space works also for non-orientable Seifert fiber spaces with orientable or non-orientable orbit space.) From this it is easy to see that attaching solid tori to a W_k results in a manifold in $\mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$. Hence, $M \in \mathcal{F}\#\hat{\mathcal{S}}\#\mathcal{W}$.

The converse is proved similarly to the proof of Theorem 8 (b) in [7]. □

We now consider the case that a compact, connected 3-manifold M which admits an STS-filling as in Theorem 3 embeds in S^3. If M is irreducible then the only possibilities are that M is an orientable handlebody, B^3, S^3, or a Seifert-web. The Seifert-webs that embed in S^3 are classified in [9] and can be described as follows.

A *tree* of links $l_1, \ldots, l_m$ is a graph Γ in S^3 obtained from a finite tree T that is embedded in S^3 with vertices $v_1, \ldots, v_m$ lying in disjoint balls $B_1, \ldots, B_m$ by replacing each vertex v_i in B_i with the link l_i in B_i and replacing each edge e_k of T from v_i to v_j by an arc α_k in S^3 that joins a component of l_i to a component of l_j. A *Burde-Murasugi link* l is a link in S^3, not the unlink of more than one component, whose components are components of a torus link on an unknotted torus T^2, possibly together with curves α and β as in the Fig. 5. As we know from [2] and [1], these Burde-

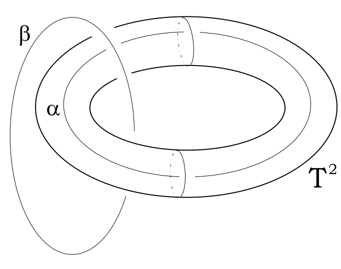

FIGURE 5.

Murasugi links are precisely those for which the space of the link is a Seifert fiber space. Here, as usual, the space of a graph Γ in S^3 is $\overline{S^3 - N(\Gamma)}$ for a regular neighborhood of Γ.

In [9] we show that an irreducible 3-manifold M with $\partial M \neq \emptyset$ that admits an STS-filling and embeds in S^3 is either a ball or homeomorphic to the space of a tree of Burde-Murasugi links. From this we have

Theorem 4. *If a compact, connected 3-manifold M has a $(\mathcal{T}, \textsc{n})$-STS-filling and embeds in S^3 then $M = M_1 \# \cdots \# M_k$, where each M_i is either S^3 or a ball or homeomorphic to the space of a tree of Burde-Murasugi links.*

We now would like to compile a table that gives an overview of the STS-fillings of closed, connected 3-manifolds with orientable handlebodies (see Table 1). For this table, let H_n denote the orientable handlebody of genus n, $n \geq 1$, and H_0 denote a ball.

N \ n	H_0	H_1	H_2	$\cdots$	H_n	$\cdots$	Σ
2	S^3	L	$\mathcal{H}_2$	$\cdots$	$\mathcal{H}_n$	$\cdots$	$\mathcal{M}_0$
3	$\mathcal{B}$ Corol. 1	Corol. 2					$\mathcal{M}$
$\vdots$	$\vdots$	$\vdots$					$\vdots$
N	$\mathcal{B}$ Corol. 1	Thm. 1					$\mathcal{M}$
$\vdots$	$\vdots$	$\vdots$					$\vdots$
Σ	$\mathcal{B}$	$\hat{\mathcal{S}}$					$\mathcal{M}$

TABLE 1. $(\{H_n, \textsc{n}\})$-STS-fillings

In Table 1, $\mathcal{M}$ (resp. $\mathcal{M}_0$) denotes the family of all closed, connected (resp. closed, connected orientable) 3-manifolds. Also $\mathcal{H}_n$ denotes the family of all closed, connected orientable 3-manifolds of Heegaard genus less than or equal to n. We note that all manifolds in the i^{th} row and j^{th} column are contained in the $(i+1)$st row and j^{th} column and also in the i^{th} row and $(j+1)$st column, so that each row is contained in its succeeding row and each column in its succeeding column. Observe that the union of all manifolds in the 1^{st} row (for N$=2$) is $\mathcal{M}_0$. Using the observation that every non-orientable 3-manifold is a union of three orientable handlebodies [5], it is not hard to see that the union of all manifolds in row 2 consists of $\mathcal{M}$.

Observe that the union of all manifolds in the 1^{st} column (for $n=0$) are all the manifolds in $\mathcal{B}$, and (since S^2-bundles over S^1 are Seifert fiber spaces) the union of all manifolds in the 2^{nd} column is $\hat{\mathcal{S}}$. It is an open question what the union of the manifolds in the m^{th} column is, when $m \geq 3$ (i.e., $n \geq 2$).

3. STS-fillings and Some Classical Invariants

Let w_n be a core of H_n, i.e., w_n is a wedge of n circles for $n \geq 1$, and w_0 is a point. Following Clapp and Puppe [3], we define the w_n-category of M as follows. A subset $A \subset M$ is *w_n-categorical* if the inclusion map $i : A \longrightarrow M$ factors homotopically through w_n. The *w_n-category of M*, denoted by $cat_n(M)$, is the smallest number of sets, open and w_n-categorical, needed to cover M. (This is denoted by $\{w_n\}$-cat(M) in [3].)

Clearly $\text{cat}_0(M) = \text{cat}(M)$, where $\text{cat}(M)$ denotes the Lusternik-Schnirelmann category of M, i.e., the smallest number of sets, open and contractible in M needed to cover M. Sometimes $\text{cat}_1(M)$ is called the *round category of M* (cf. [12]).

By $C_n(M)$ we denote the minimal number of sets homeomorphic to $\overset{\circ}{H}_n$, needed to cover M, e.g., $C_0(M)$ is the minimal number of open balls needed to cover M (cf. [11]).

We now consider invariants associated to smooth functions $f : M \longrightarrow \mathbb{R}^1$. The number of critical points of f is denoted by $\mu_M(f)$ and $F_0(M) = \min_f \{\mu_M(f)\}$. Following Khimshiashvilli and Siersma [12], we say that f is a *round function* if the critical set $C(f)$ is a smooth link in M; the components of $C(f)$ are called the *critical loops* of f. In [12], it is shown that round function exists on closed, connected orientable 3-manifolds. The *round complexity*, $F_1(M)$, is the minimal number of critical loops, taken over all round functions on M.

In Table 2, we show the results that are known about the invari-

N \ n	w_0	w_1
	$F_0(M) \leq$ N; [14]	$F_1(M) \leq$ N, M orientable; [12]
2	S^3	L
3	$\mathcal{B}$	M is orientable and admits an STS-filling with three solid tori
4	$\mathcal{M}$	$\mathcal{M}_0$
	$C_0(M) \leq$ N; [11]	$C_1(M) \leq$ N
2	S^3	
3	$\mathcal{B}$	
4	$\mathcal{M}$	$\mathcal{M}$
	$\mathrm{cat}_0(M) \leq$ N; [4]	$\mathrm{cat}_1(M) \leq$ N
2	$\pi_1(M) = 0 \Leftrightarrow M \simeq S^3$	
3	$\pi_1(M)$ is a free group $\Leftrightarrow M \simeq M' \in \mathcal{B}$	
4	$\mathcal{M}$	$\mathcal{M}$

Table 2

ants F_0, F_1, C_0 and cat_0. It is clear that from Table 1 that several interesting questions can be proposed about C_1 and cat_1. In upcoming joint work with F. González-Acuña, we will explore some of these questions.

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