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Σ-PRODUCTS OF PARACOMPACT $\mathcal{DC}$-LIKE SPACES

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Abstract. In this paper, we shall discuss Σ-products of paracompact $\mathcal{DC}$-like spaces and show the following: (1) Let Σ be a Σ-product of paracompact $\mathcal{DC}$-like spaces. If Σ has countable tightness, then it is collectionwise normal; (2) If Σ is a Σ-product of first countable, paracompact (subparacompact) $\mathcal{DC}$-like spaces, then it is shrinking (subshrinking).

1. Introduction

Since the concept of Σ-products was introduced by Corson [1], the normality of Σ-products has been studied by several authors. In particular, since the normality of Σ-products of metric spaces was proved by Gul'ko [4] and Rudin [9], the result has been extended for generalized metric spaces (cf. Kombarov [6] and Yajima [18, 19]). Furthermore, Rudin [10] proved the shrinking property of Σ-products of metric spaces. So, the shrinking property of Σ-products of generalized metic spaces has been another interesting subject (Yajima [20]).

On the other hand, Telgársky [14] defined C-scattered spaces, which is a generalization of scattered spaces and locally compact spaces, for paracompactness of products. Furthermore, using topological games, Telgársky [15] defined $\mathcal{DC}$-like spaces for it. The

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class of $\mathcal{DC}$-like spaces includes all spaces which admit a σ-closure-preserving closed cover by compact sets, and all subparacompact, σ-C-scattered spaces. Telgársky [15] and Yajima [17] proved that if X is a paracompact (subparacompact) $\mathcal{DC}$-like space, then $X \times Y$ is paracompact (subparacompact) for every paracompact (subparacompact) space Y, and the second author [11, 12] proved that if $\{X_n : n \in \omega\}$ is a countable collection of paracompact (subparacompact) $\mathcal{DC}$-like spaces, then the product $\prod_{n\in\omega} X_n$ is paracompact (subparacompact).

C-scattered spaces are not generalized metric spaces. However, as the spaces consisting of ordinals (with the usual order topology) are scattered, many important examples using ordinals are scattered. So, the study of Σ-products of paracompact C-scattered spaces is meaningful. The second author and Yajima [13] showed the following: (1) Let Σ be a Σ-product of paracompact C-scattered spaces. If Σ has countable tightness, then it is collectionwise normal; (2) If Σ is a Σ-product of first countable, paracompact (subparacompact) C-scattered spaces, then it is shrinking (subshrinking).

It seems to be natural to consider Σ-products of paracompact $\mathcal{DC}$-like spaces. So, we shall discuss normality and shrinking property of Σ-products of paracompact $\mathcal{DC}$-like spaces and prove analogues of the above (1) and (2).

All spaces are assumed to be regular T_1 spaces. Let ω denote the set of natural numbers and $|A|$ denote the cardinality of a set A. Undefined terminology can be found in Engelking [2].

2. Topological games

A space X is said to be *scattered* if every nonempty (closed) subset A of X has an isolated point in A. A space X is said to be *C-scattered* if for every nonempty closed subset A of X, there is an $x \in A$ such that x has a compact neighborhood in A. Then scattered spaces and locally compact spaces are C-scattered. A space X is said to be *σ-scattered* (*σ-C-scattered*) if X is the union of countably many scattered (C-scattered) closed subspaces. A space X has a *σ-closure-preserving closed cover by compact sets* if $X = \cup_{n\in\omega}(\cup\mathcal{F}_n)$, where every $\mathcal{F}_n$ is a closure-preserving closed collection, consisting of compact subsets of X.

Let $\mathcal{DC}$ be the class of all spaces which have a discrete cover by compact subsets. For a space X, topological game $\mathrm{G}(\mathcal{DC}, X)$ is defined as follows. There are two players, I and II. They alternatively choose closed subsets of II's previous play. Player I's choice must be in the class $\mathcal{DC}$ and II's choice must be disjoint from I's. We say that Player I *wins* if the intersection of II's choices is empty. A space X is said to be a *$\mathcal{DC}$-like* space if Player I has a winning strategy in $\mathrm{G}(\mathcal{DC}, X)$. Recall in Galvin and Telgársky [3] that a space X is a $\mathcal{DC}$-like space if and only if Player I has a stationary winning strategy in $\mathrm{G}(\mathcal{DC}, X)$. That is,

Lemma 2.1 [3]. *A space X is a $\mathcal{DC}$-like space if and only if there is a function s from 2^X into $2^X \cap \mathcal{DC}$, where 2^X denotes the set of all closed subsets of X, satisfying*
(i) *$s(F) \subset F$ for each $F \in 2^X$,*
(ii) *if $\{F_n : n \in \omega\}$ is a decreasing sequence of closed subsets of X such that $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\cap_{n\in\omega} F_n = \emptyset$.*

A space X is said to be *totally paracompact* if every base of X has a locally finite subcover. A space X is said to be *subparacompact* if every open cover of X has a σ-discrete closed refinement. Clearly every paracompact space is subparacompact. Let $X_1, X_2, \ldots, X_n$ be spaces. A subset $A = \prod_{i=1}^n A_i$ is said to be *open (closed) rectangle* in $\prod_{i=1}^n X_i$ if A_i is an open (closed) subset in X_i for each $i = 1, 2, \ldots, n$.

Lemma 2.2 [8], [15]. (i) *If a space X has a σ-closure-preserving closed cover by compact sets, then X is a $\mathcal{DC}$-like space.*
(ii) *If X is a subparacompact, σ-C-scattered space, then X is a $\mathcal{DC}$-like space.*

Lemma 2.3 [15], [16], [17]. (i) *If X and Y are paracompact $\mathcal{DC}$-like spaces, then $X \times Y$ is a paracompact $\mathcal{DC}$-like space.*
(ii) *Every paracompact $\mathcal{DC}$-like space is totally paracompact.*
(iii) *If X is a subparacompact $\mathcal{DC}$-like space and Y is a subparacompact space, then every open cover of $X \times Y$ has a σ-discrete refinement, consisiting of closed rectangles and hence, $X \times Y$ is subparacompact.*

Remark 2.4. It is well known that the space ω^ω of irrationals is not totally paracompact. Hence, the countable products of paracompact $\mathcal{DC}$-like spaces need not be $\mathcal{DC}$-like.

3. Normality of Σ-products

Let $\{X_\lambda\colon \lambda \in \Lambda\}$ be a collection of spaces. We may assume that the index set Λ is uncountable and every X_λ contains at least two points. Let $X = \prod_{\lambda\in\Lambda} X_\lambda$ and take a point $x^* = (x^*_\lambda) \in X$. The subspace

$$\Sigma = \{x = (x_\lambda) \in X : |\mathrm{Supp}(x)| \leq \omega\}$$

of X is called a Σ-*product* of spaces $X_\lambda, \lambda \in \Lambda$, where $\mathrm{Supp}(x) = \{\lambda \in \Lambda : x_\lambda \neq x^*_\lambda\}$. The $x^* \in \Sigma$ is called a *base point* of Σ. The mention of base point x^* is often omitted.

For a set Λ, we denote $[\Lambda]^{<\omega}$ the set of all finite subsets of Λ. For each $R \in [\Lambda]^{<\omega}$, we denote X_R the finite subproduct $\prod_{\lambda\in R} X_\lambda$ of Σ, and denote by p_R the projection of Σ onto X_R. In particular, $p_{\{\lambda\}}$ is denoted by p_λ for each $\lambda \in \Lambda$. Furthermore, we denote by $p^{R'}_R$ the projection of $X_{R'}$ onto X_R for $R, R' \in [\Lambda]^{<\omega}$ with $R \subset R'$. For each $R \in [\Lambda]^{<\omega}$, let $\Sigma_{\Lambda-R}$ be the Σ-product of spaces $X_\lambda, \lambda \in \Lambda - R$, with the base point $x^*|(\Lambda - R) = (x^*_\lambda)_{\lambda\in\Lambda-R}$.

Let Θ be an index set such that $\theta, \delta \in \Theta$ assign $R_\theta, R_\delta \in [\Lambda]^{<\omega}$. Then $X_{R_\delta}, X_{R_\theta}, X_{R_\theta - R_\delta}, p_{R_\delta}, p_{R_\theta}$ and $p^{R_\theta}_{R_\delta}$ are abbreviated by $X_\delta, X_\theta, X_{\theta-\delta}, p_\delta, p_\theta$ and p^θ_δ, respectively.

Let $R \in [\Lambda]^{<\omega}$. A subset A is said to be R-*cylindrically open (closed)* in Σ if $A = \prod_{\lambda\in R} A_\lambda \times \Sigma_{\Lambda-R}$, where $\prod_{\lambda\in R} A_\lambda$ is an open (closed) rectangle in X_R and A is said to be *cylindrically open (closed)* in Σ if A is R-cylindrically open (closed) in Σ for some $R \in [\Lambda]^{<\omega}$. The set of all cylindrically open subsets in Σ is a base in Σ. The space $\Sigma_{\Lambda-R_\theta}$ is abbreviated by $\Sigma_{\Lambda-\theta}$. Notice that for every R-cylindrically open (closed) set A in Σ, A is homeomorphic to $p_R^{-1} p_R(A)$.

A space X *has countable tightness* if for each $A \subset X$ and $x \in \overline{A}$, there is a countable subset $B \subset A$ such that $x \in \overline{B}$. Every first countable space has countable tightness. Kombarov and Malykhin [7] proved that a Σ-product Σ has countable tightness if and only if every finite subproduct of Σ has countable tightness.

A collection $\mathcal{D}$ of subsets of a space X is *discrete* at $x \in X$ (*discrete in* $A \subset X$) if (for each $x \in A$), there is an open neighborhood U of x in X such that $|\{D \in \mathcal{D}\colon D \cap U \neq \emptyset\}| \leq 1$. A space X is *collectionwise normal* if every discrete collection $\mathcal{D}$ of closed subsets in X can be separated by disjoint open subsets.

Let $\mathcal{U}$ be a collection of subsets of a space X and $A \subset X$. Denote by $\mathcal{U}|A$ the collection $\{U \cap A : U \in \mathcal{U}\}$ of subsets of A.

Now, we are ready to prove our main result.

Theorem 3.1. *Let Σ be a Σ-product of paracompact $\mathcal{DC}$-like spaces. If Σ has countable tightness, then it is collectionwise normal.*

Proof: Let Σ be a Σ-product of paracompact $\mathcal{DC}$-like spaces $X_\lambda, \lambda \in \Lambda$, with a base point $x^* = (x^*_\lambda) \in \Sigma$. For each $x \in \Sigma$, we denote $\mathrm{Supp}(x) = \{\lambda_{x,i}\colon\ i \in \omega\}$ and for each $n \in \omega$, let $\langle \mathrm{Supp}(x)\rangle_n = \{\lambda_{x,0}, \lambda_{x,1}, \lambda_{x,2}, \dots, \lambda_{x,n}\}$. Let $\mathcal{D}$ be a discrete collection of closed subsets in Σ. For each $\lambda \in \Lambda$, by Lemma 2.1, let s_λ be a stationary winning strategy for Player I in $\mathrm{G}(\mathcal{DC}, X_\lambda)$.

Define a collection $\mathcal{B}$ as follows: $(H, \{C_{H,\lambda} : \lambda \in R_H\}) \in \mathcal{B}$ if and only if H is an R_H-cylindrically open subset in Σ, that is $H = \prod_{\lambda \in R_H} H_\lambda \times \Sigma_{\Lambda - R_H}$, and for each $\lambda \in R_H, C_{H,\lambda}$ is a compact subset in $\overline{H_\lambda}$ ($C_{H,\lambda} = \emptyset$ may be occur).

Let $(H, \{C_{H,\lambda}\colon\ \lambda \in R_H\}) \in \mathcal{B}$, where $H = \prod_{\lambda \in R_H} H_\lambda \times \Sigma_{\Lambda - R_H}$. If $C_{H,\lambda} \neq \emptyset$, let $C_{\gamma(H,\lambda)} = C_{H,\lambda}, \Gamma(H,\lambda) = \{\gamma(H,\lambda)\}$ and $\mathcal{C}(H,\lambda) = \{C_\gamma : \gamma \in \Gamma(H,\lambda)\} = \{C_{H,\lambda}\}$. Put $W_{\gamma(H,\lambda)} = H_\lambda$ and $\mathcal{W}(H,\lambda) = \{W_\gamma\colon\ \gamma \in \Gamma(H,\lambda)\} = \{W_{\gamma(H,\lambda)}\}$. If $C_{H,\lambda} = \emptyset$, then there is a discrete collection $\mathcal{C}(H,\lambda) = \{C_\gamma : \gamma \in \Gamma(H,\lambda)\}$ of compact subsets of $\overline{H_\lambda}$ such that $s_\lambda(\overline{H_\lambda}) = \cup\mathcal{C}(H,\lambda)$. Since $\overline{H_\lambda}$ is paracompact, there is a locally finite collection $\mathcal{W}'(H,\lambda) = \{W'_\gamma : \gamma \in \Gamma(H,\lambda)\}$ of open subsets of $\overline{H_\lambda}$ such that $\overline{H_\lambda} = \cup\mathcal{W}'(H,\lambda)$, $C_\gamma \subset W'_\gamma$ for each $\gamma \in \Gamma(H,\lambda)$ and $\overline{W'_\gamma} \cap C_{\gamma'} = \emptyset$ for $\gamma, \gamma' \in \Gamma(H,\lambda)$ and $\gamma \neq \gamma'$. For each $\gamma \in \Gamma(H,\lambda)$, let $W_\gamma = W'_\gamma \cap H_\lambda$ and $\mathcal{W}(H,\lambda) = \{W_\gamma : \gamma \in \Gamma(H,\lambda)\}$. Let $\Delta_H = \prod_{\lambda \in R_H} \Gamma(H,\lambda)$. Fix a $\delta = (\gamma(\delta,\lambda)) \in \Delta_H$. Let $C_\delta = \prod_{\lambda \in R_H} C_{\gamma(\delta,\lambda)} \times \prod_{\lambda \in \Lambda - R_H} \{x^*_\lambda\}$. For each $x \in C_\delta$, there is an $R(x)$-cylindrically open neighborhood $B(x)$ of x in Σ such that $B(x)$ meets at most one member of $\mathcal{D}$. Since C_δ is compact, there is a finite subset $\{x_0, x_1, \dots, x_n\}$ of C_δ such that $C_\delta \subset \cup_{i=0}^n B(x_i)$. Let $R_\delta = R_H \cup (\cup_{i=0}^n R(x_i))$, $W_\delta = \prod_{\lambda \in R_H} W_{\gamma(\delta,\lambda)} \times X_{\delta - R_H}$ and

$$\Phi = \{x \in \overline{W_\delta} : p_\delta(\mathcal{D}) \text{ is not discrete at } x \text{ in } X_\delta\}.$$

Then $p_\delta(C_\delta) \subset \cup_{i=0}^n p_\delta(B(x_i))$ and hence, $p_\delta(C_\delta) \cap \Phi = \emptyset$. For each $x = (x_\lambda) \in \Phi$, take an open rectangle $U(x) = \prod_{\lambda \in R_\delta} U(x_\lambda)$, which is a neighborhood of x in X_δ, such that if $x_\lambda \notin p_\lambda(C_\delta), \lambda \in R_\delta$, then $\overline{U(x_\lambda)} \cap p_\lambda(C_\delta) = \emptyset$. By Lemma 2.3, every finite product of paracompact $\mathcal{DC}$-like spaces is totally paracompact. There is a locally finite (in X_δ) collection $\mathcal{V}_\delta = \{V_\xi : \xi \in \Xi_\delta\}, V_\xi = \prod_{\lambda \in R_\delta} V_{\xi,\lambda}$ for each $\xi \in \Xi_\delta$, of open rectangles in $\overline{W_\delta}$ satisfying that $\cup\mathcal{V}_\delta = \cup\{V_\xi : \xi \in \Xi_\delta\} = \overline{W_\delta}$, and $\{\overline{V_\xi} : \xi \in \Xi_\delta\}$ refines $\{U(x) : x \in \Phi\} \cup \{p_\delta(B(x_0)), \cdots, p_\delta(B(x_n))\}$. Put

$$\Xi_\delta^+ = \{\xi \in \Xi_\delta : \overline{V_\xi} \text{ meets at most one member of } p_\delta(\mathcal{D})\}$$

and $\Xi_\delta^- = \Xi_\delta - \Xi_\delta^+$. Let

$$\Theta_H^+ = \{(\delta, \xi) : \xi \in \Xi_\delta^+ \text{ and } \delta \in \Delta_H\} \text{ and}$$
$$\Theta_H^- = \{(\delta, \xi) : \xi \in \Xi_\delta^- \text{ and } \delta \in \Delta_H\}.$$

For each $\theta = (\delta, \xi) \in \Theta_H^+$, $\delta \in \Delta_H$ and $\xi \in \Xi_\delta^+$, let $G_\theta = p_\delta^{-1}(V_\xi \cap W_\delta)$ and for each $\theta = (\delta, \xi) \in \Theta_H^-$, $\delta \in \Delta_H$ and $\xi \in \Xi_\delta -$, let $H_\theta = p_\delta^{-1}(V_\xi \cap W_\delta)$. Put $H_\theta = \prod_{\lambda \in R_\delta} H_{\theta,\lambda} \times \Sigma_{\Lambda - \delta}$. Let

$$\mathcal{G}_H = \{G_\theta : \theta \in \Theta_H^+\}.$$

Then $\mathcal{G}_H$ is a locally finite collection of cylindrically open subsets in Σ such that $\overline{G}$ meets at most one member of $\mathcal{D}$ for each $G \in \mathcal{G}_H$. For each $\theta \in \Theta_H^-$, there is an $x_\theta = (x_{\theta,\lambda}) \in \Phi$ such that $V_\xi \subset U(x_\theta)$. Since X_δ has countable tightness, it follows from [18, Lemma 2] that there is a countable subset $Y_\theta = \{y_{\theta,i} : i \in \omega\}$ of $\cup\mathcal{D}$ such that $p_\delta(\mathcal{D}|Y_\theta)$ is not discrete at x_θ. Let

$$A_\theta = \{\lambda \in R_\delta : \overline{p_\lambda(H_\theta)} \cap p_\lambda(C_\delta) = \emptyset\}.$$

Then $H - \cup\mathcal{G}_H \subset \cup\{H_\theta : \theta \in \Theta_H^-\}$. If $\theta = (\delta, \xi) \in \Theta_H^-$, $\delta = (\gamma(\delta, \lambda)) \in \Delta_H, \xi \in \Xi_\delta^-, \lambda \in R_\delta$ and $x_{\theta,\lambda} \notin p_\lambda(C_\delta)$, then $\overline{U(x_{\theta,\lambda})} \cap p_\lambda(C_\delta) = \emptyset$. So, $\overline{p_\lambda(H_\theta)} \cap p_\lambda(C_\delta) = \emptyset$. Thus $\lambda \in A_\theta$.

Claim 1. *Let $\theta \in \Theta_H^-$ and $\lambda \in R_H$ with $C_{H,\lambda} = \emptyset$. If $\lambda \in A_\theta$, then $s_\lambda(\overline{H_\lambda}) \cap \overline{H_{\theta,\lambda}} = \emptyset$.*

Proof: Let $\theta = (\delta, \xi) \in \Theta_H^-$, $\delta = (\gamma(\delta, \lambda)) \in \Delta_H$ and $\xi \in \Xi_\delta^-$. Since $\lambda \in R_H$ with $C_{H,\lambda} = \emptyset$, we take a discrete collection $\mathcal{C}(H, \lambda)$ of compact subsets of $\overline{H_\lambda}$ such that $s_\lambda(\overline{H_\lambda}) = \cup\mathcal{C}(H, \lambda)$. Then we have $s_\lambda(\overline{H_\lambda}) \cap \overline{W_{\gamma(\delta,\lambda)}} = C_{\gamma(\delta,\lambda)}$. By $\lambda \in A_\theta, \overline{p_\lambda(H_\theta)} \cap p_\lambda(C_\delta) = \overline{H_{\theta,\lambda}} \cap C_{\gamma(\delta,\lambda)} = \emptyset$. So it follows that $s_\lambda(\overline{H_\lambda}) \cap \overline{H_{\theta,\lambda}} = \emptyset$. □

Let $\mathcal{G}_0 = \{\emptyset\}, \Theta_0 = \{\emptyset\}, H_\emptyset = \Sigma$. Take an arbitrary $\lambda_0 \in \Lambda$. Let $R_\emptyset = \{\lambda_0\}$ and $C_{\emptyset,\lambda_0} = \emptyset$. Then $(H_\emptyset, \{C_{\emptyset,\lambda_0}\}) \in \mathcal{B}$.

By the above construction, for each $n \in \omega$, we obtain a collection $\mathcal{G}_n$ of cylindrically open subsets of Σ and an index set Θ_n of n-th level of a tree with the height ω such that each $\theta \in \Theta_n$ assigns an index set Δ_θ, finite subsets $R_\delta, R_\theta, A_\theta \in [\Lambda]^{<\omega}$, an index set Ξ_δ^-, a compact set C_δ, a point $x_\theta \in X_\delta$, a countable subset $Y_\theta \subset \Sigma$, and for each $\lambda \in R_\theta$, a compact set $C_{\theta,\lambda}$, satisfying the following conditions (1) - (3): for each $n \in \omega$,

(1) $\mathcal{G}_n$ is locally finite in Σ such that $\overline{G}$ meets at most one member of $\mathcal{D}$ for each $G \in \mathcal{G}_n$,
(2) $\{H_\theta : \theta \in \Theta_n\}$ is locally finite in Σ,
(3) for each $\theta \in \Theta_n$, $n \geq 1$,
  (a) $\theta = (\delta, \xi)$ for $\delta = (\delta(\lambda))_{\lambda \in R_{\theta_-}} \in \Delta_{\theta_-}, \xi \in \Xi_\delta^-$ and $\theta_- \in \Theta_{n-1}$. Define the order $<$ as follows: for $\theta \in \Theta_n$ and $\mu \in \Theta_{n-1}$, $\mu < \theta$ if and only if $\mu = \theta_-$ and $\theta = (\delta, \xi)$ for some $\delta \in \Delta_\mu$ and $\xi \in \Xi_\delta^-$,
  (b) $R_{\theta_-} \subset R_\delta$,
  (c) H_{θ_-} is an R_{δ_-}-cylindrically open subset in Σ such that $(H_{\theta_-}, \{C_{\theta_-,\lambda} : \lambda \in R_{\theta_-}\}) \in \mathcal{B}$, where if $\theta \in \Theta_1$ and hence $\theta_- = \emptyset$, let $\delta_- = \emptyset$, $R_{\delta_-} = \{\lambda_0\}$ and $H_{\theta_-} = \prod_{\lambda \in R_{\delta_-}} H_{\theta_-,\lambda} \times \Sigma_{\Lambda - \delta_-}$,
  (d) $H_{\theta_-} - \cup\mathcal{G}_n \subset \cup\{H_\mu : \mu \in \Theta_n \text{ with } \theta_- < \mu\} \subset H_{\theta_-}$,
  (e) $x_\theta = (x_{\theta,\lambda}) \in \overline{p_\delta(H_{\theta_-})}$
  (f) $Y_\theta = \{y_{\theta,i} : i \in \omega\}$ is a countable subset of $\cup\mathcal{D}$ such that $p_\delta(\mathcal{D}|Y_\theta)$ is not discrete at x_θ,
  (g) a set A_θ and for a $\delta = (\delta(\lambda)) \in \Delta_{\theta_-}$, a nonempty compact subset $C_\delta = \prod_{\lambda \in R_{\theta_-}} C_{\delta(\lambda)} \times \prod_{\lambda \in \Lambda - R_{\theta_-}} \{x^*_\lambda\} \subset \overline{H_{\theta_-}}$ satisfy the following:
    (g-1) if $\lambda \in R_{\theta_-}$ and $C_{\theta_-,\lambda} \neq \emptyset$, let $C_{\delta(\lambda)} = C_{\theta_-,\lambda}$,
    (g-2) if $\lambda \in R_{\theta_-}$ and $C_{\theta_-,\lambda} = \emptyset$, let $C_{\delta(\lambda)} \in \mathcal{C}(\overline{H_{\theta_-}, \lambda})$ and $s_\lambda(\overline{H_{\theta_-,\lambda}}) \cap \overline{H_{\theta,\lambda}} \subset C_{\delta(\lambda)}$,
    (g-3) for $x \in p_\delta(C_\delta), \overline{p_\delta(\mathcal{D})}$ is discrete at x in X_δ,
    (g-4) $A_\theta = \{\lambda \in R_\delta : \overline{p_\lambda(H_\theta)} \cap p_\lambda(C_\delta) = \emptyset\}$ and hence, if $\lambda \in R_{\theta_-}$ with $C_{\theta_-,\lambda} = \emptyset$ and $\lambda \in A_\theta$, then $s_\lambda(\overline{H_{\theta_-,\lambda}}) \cap \overline{H_{\theta,\lambda}} = \emptyset$,
    (g-5) if $\lambda \in R_\delta$ and $x_{\theta,\lambda} \notin p_\lambda(C_\delta)$, then $\lambda \in A_\theta$,
  (h) $R_\theta = \cup\{\langle \mathrm{Supp}(y_{\mu,j}) \rangle_k : \mu \leq \theta \text{ and } j, k \leq n\} \cup R_\delta$,

(i) for each $\lambda \in R_\theta$,
let $C_{\theta,\lambda} = \overline{p_\lambda(H_\theta)} \cap p_\lambda(C_\delta) = \overline{H_{\theta,\lambda}} \cap C_{\delta(\lambda)}$
if $\lambda \in R_{\theta\_} - A_\theta$;
let $C_{\theta,\lambda} = \{x^*_\lambda\}$ if $\lambda \in R_\delta - (R_{\theta\_} \cup A_\theta)$; and
let $C_{\theta,\lambda} = \emptyset$, otherwise.

(j) $(H_\theta, \{C_{\theta,\lambda} : \lambda \in R_\theta\}) \in \mathcal{B}$.

Let $\mathcal{G} = \cup_{n\in\omega}\mathcal{G}_n$. By (1), it suffices to prove that $\mathcal{G}$ covers Σ. Assume that $\mathcal{G}$ does not cover Σ. Take an $x = (x_\lambda) \in \Sigma - \cup\mathcal{G}$. Then, by (3) (d), we can inductively choose a sequence $\{\theta_n \colon n \in \omega\}$ such that $\theta_n = (\delta_n, \xi_n) \in \Theta_n, \delta_n \in \Delta_{\theta_{n-1}}, \xi_n \in \Xi^-_{\delta_n}, \theta_{n-1} < \theta_n, n \geq 1$ and $x \in H_{\theta_n}$ for each $n \in \omega$. Let $Q = \cup_{n\in\omega} R_{\theta_n}$. Then the following is obvious.

Claim 2. (i) $R_{\theta_n} \subset R_{\delta_{n+1}} \subset R_{\theta_{n+1}}$ *for each* $n \in \omega$ *and hence, for each finite subset* $F \subset Q$*, there is an* $n \geq 1$ *such that* $F \subset R_{\delta_n}$*,*
(ii) $\cup\{\mathrm{Supp}(y) \colon y \in Y_{\theta_n}, n \geq 1\} \subset Q$.

After this in proof, we omit the index letter θ and δ for simplicity. That is, $x_{\theta_n}, H_{\theta_n}, R_{\delta_n}, p^{\delta_n}_{\delta_k}, p_{\delta_k}, A_{\theta_n}$ and C_{δ_n} are abbreviated by $x_n, H_n, R_n, p^n_k, p_k, A_n$ and C_n, respectively.

Claim 3. *For each* $\lambda \in Q$*,* $|\{n \geq 1 \colon \lambda \in A_n\}| < \omega$.

Proof: Assume that there is a $\lambda \in Q$ such that $|\{n \geq 1 \colon \lambda \in A_n\}| = \omega$. Let $\{n \geq 1 \colon \lambda \in A_n\} = \{n_t \colon t \geq 1\}$. We shall show that for each $t \geq 1$, $s_\lambda(\overline{H_{n_t,\lambda}}) \cap \overline{H_{n_{t+1},\lambda}} = \emptyset$. Since $\lambda \in A_{n_t}$, note by (i) that for each $t \geq 1, C_{n_t,\lambda} = \emptyset$. Let $t \geq 1$. If $n_{t+1} = n_t + 1$, then it follows from (3) (g-4) that $s_\lambda(\overline{H_{n_t,\lambda}}) \cap \overline{H_{n_{t+1},\lambda}} = \emptyset$. Assume that $n_{t+1} > n_t + 1$. By $\lambda \in A_{n_{t+1}}$ and (3) (g-4), $\overline{H_{n_{t+1},\lambda}} \cap C_{n_{t+1},\lambda} = \emptyset$. Then, by (3) (d) and (g-2), $\overline{H_{n_t,\lambda}} \supset \overline{H_{n_t+1,\lambda}} \supset \cdots \supset \overline{H_{n_{t+1}-1,\lambda}} \supset \overline{H_{n_{t+1},\lambda}}$ and $s_\lambda(\overline{H_{n_t,\lambda}}) \cap \overline{H_{n_t+1,\lambda}} \subset C_{n_t+1,\lambda}$. For each i with $n_t < i < n_{t+1}$, $\lambda \notin A_i$. By (3) (g-1) and (i), for each i with $n_t < i < n_{t+1}, \overline{H_{i+1,\lambda}} \cap C_{i,\lambda} = C_{i+1,\lambda}$. So it follows that $s_\lambda(\overline{H_{n_t,\lambda}}) \cap \overline{H_{n_{t+1},\lambda}} = \emptyset$.

Since s_λ is a stationary winning strategy in $\mathrm{G}(\mathcal{DC}, X_\lambda)$, $\cap_{t\geq 1}\overline{H_{n_t,\lambda}} = \emptyset$. But $x_\lambda \in \cap_{t\geq 1} H_{n_t,\lambda}$, which is a contradiction. Therefore, for each $\lambda \in Q$, $|\{n \geq 1 \colon \lambda \in A_n\}| < \omega$. □

Claim 4. *For each* $\lambda \in Q$*, there is an* $m_\lambda \geq 1$ *such that if* $n \geq m_\lambda$*, then* $x_{n,\lambda} \in p_\lambda(C_{m_\lambda})$.

Proof: Let $\lambda \in Q$. By Claim 3, there is an $m_\lambda \geq 1$ such that $\lambda \in R_{m_\lambda - 1}$, and if $n \geq m_\lambda$, then $\lambda \notin A_n$. If there is an $n \geq m_\lambda$ such that $x_{n,\lambda} \notin C_{n,\lambda}$, then, by (3) (g-5), $\lambda \in A_n$, which is a contradiction. So, by (3) (g-1), (g-4) and (i), we have that $x_{n,\lambda} \in C_{n,\lambda} \subset C_{m_\lambda,\lambda} = p_\lambda(C_{m_\lambda})$ for each $n \geq m_\lambda$. □

For $m, k \geq 1$ with $m > k$, let $C_k^m = \{p_k^n(x_n) : n \geq m\}$. Since $x_n \in X_{\delta_n}$, notice that $p_k^n : X_{\delta_n} \to X_{\delta_k}$. Choose $m_k > k$ with $m_k > \max\{m_\lambda : \lambda \in R_k\}$. It follows from Claim 4 that $p_k^n(x_n)_\lambda = x_{n,\lambda} \in p_\lambda(C_{m_\lambda})$ for each $\lambda \in R_k$ and for $n \geq m_k$. Hence $\overline{C_k^m}$ is compact for each $m \geq m_k$. Let $K_k = \cap_{m \geq k} \overline{C_k^m} (= \cap_{m \geq m_k} \overline{C_k^m})$ for each $k \geq 1$. Then every K_k is compact such that $p_k^{k+1}(K_{k+1}) \subset K_k$ for each $k \geq 1$. Thus, $\{K_k, p_k^{k+1}\}$ is an inverse sequence of nonempty compact spaces. Hence, there is a point $(z_k) \in \varprojlim\{K_k, p_k^{k+1}\}$. Define a point $z = (z_\lambda) \in \Sigma$ such that $p_k(z) = z_k$ for each $k \geq 1$ and $z_\lambda = x_\lambda^*$, otherwise.

We can show that $\mathcal{D}$ is not discrete at z. However, by Claim 4 and (3) (f), this is verified in the same manner as the proof of [5, Theorem 1]. □

Corollary 3.2. (i) *Let Σ be a Σ-product of paracompact spaces which admit a σ-closure-preserving closed cover by compact sets. If Σ has countable tightness, then it is collectionwise normal.*
(ii) *Let Σ be a Σ-product of paracompact σ-C-scattered spaces. If Σ has countable tightness, then it is collectionwise normal.*

Proof: This immediately follows from Theorem 3.1 and Lemma 2.2. □

4. Shrinking property of Σ-products

A space X is said to be *shrinking* if for every open cover $\{U_\lambda\colon \lambda \in \Lambda\}$ of X, there is a closed cover $\{F_\lambda\colon \lambda \in \Lambda\}$ of X such that $F_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$. A space X is said to be *subshrinking* if for every open cover $\{U_\lambda\colon \lambda \in \Lambda\}$ of X, there is a closed cover $\{F_{\lambda,n}\colon \lambda \in \Lambda \text{ and } n \in \omega\}$ of X such that $F_{\lambda,n} \subset U_\lambda$ for each $\lambda \in \Lambda$ and $n \in \omega$. It is well known that a space X is shrinking if and only if it is normal and subshrinking. So, it often happens to discuss the subshrinking property instead of the shrinking one.

Let $\{X_\lambda\colon \lambda \in \Lambda\}$ be a collection of spaces and Σ be a Σ-product of them. We may take a point $y_\lambda \in X_\lambda$ different from x_λ^* for each $\lambda \in$

Λ. For each $r \in [\Lambda - R]^{<\omega}$, an open neighborhood W_r of $x^*|(\Lambda - R)$ in $\Sigma_{\Lambda - R}$ is said to be *r-basic* if $W_r = \prod_{\lambda \in r} W_\lambda \times \Sigma_{\Lambda-(R \cup r)}$, where W_λ is an open neighborhood of x^*_λ in X_λ with $y_\lambda \notin W_\lambda$ for each $\lambda \in r$.

Let $\mathcal{G} = \{G_\upsilon : \upsilon \in \Upsilon\}$ be an open cover of Σ. For $R \in [\Lambda]^{<\omega}$ and a subset F in X_R, let

$$M(F) = \{r \in [\Lambda - R]^{<\omega} : \text{there is an r-basic open neighborhood}$$
$$W_r \text{ of } x^*|(\Lambda - R) \text{ such that } \overline{F} \times \overline{W_r} \subset G_\upsilon \text{ for some } \upsilon \in \Upsilon\}.$$

Lemma 4.1 [13]. *Let Σ be a Σ-product of spaces $X_\lambda, \lambda \in \Lambda$, and let $\mathcal{G} = \{G_\upsilon : \upsilon \in \Upsilon\}$ be an open cover of Σ. If there is a σ-discrete closed cover $\{E_\theta : \theta \in \Theta^+\}$ of Σ such that each E_θ, $\theta = (\delta, \xi) \in \Theta^+$, $\delta \in \Delta_{\theta_-}$, $\xi \in \Xi^-_\delta$, $\theta_- \in \Theta$, is an R_δ-cylindrically closed set in Σ, satisfying*

$$E_\theta \subset \cup\{p^{-1}_{\Lambda-\delta}(W_r) : r \in M(p_\delta(E_\theta))\},$$

where $R_\delta \in [\Lambda]^{<\omega}$ and $p_{\Lambda-\delta}$ denotes the projection of Σ onto $\Sigma_{\Lambda-\delta}$, then there is a closed cover $\{F_{\upsilon,n} : \upsilon \in \Upsilon$ and $n \in \omega\}$ of Σ such that $F_{\upsilon,n} \subset G_\upsilon$ for each $\upsilon \in \Upsilon$ and $n \in \omega$.

Theorem 4.2. *If Σ is a Σ-product of first countable, subparacompact $\mathcal{DC}$-like spaces, then it is subshrinking.*

Proof: Let Σ be a Σ-product of first countable, subparacompact $\mathcal{DC}$-like spaces $X_\lambda, \lambda \in \Lambda$, with a base point $x^* = (x^*_\lambda) \in \Sigma$. Let $\mathcal{G} = \{G_\upsilon : \upsilon \in \Upsilon\}$ be an open cover of Σ. For each $\lambda \in \Lambda$, let s_λ be a stationary winning strategy for Player I in $\mathrm{G}(\mathcal{DC}, X_\lambda)$.

Define a collection $\mathcal{B}$ similarly: $(E, \{C_{E,\lambda} : \lambda \in R_E\}) \in \mathcal{B}$ if and only if E is an R_E-cylindrically closed subset in Σ, that is $E = \prod_{\lambda \in R_E} E_\lambda \times \Sigma_{\Lambda - R_E}$ and for each $\lambda \in R_E, C_{E,\lambda}$ is a compact subset in E_λ ($C_{E,\lambda} = \emptyset$ may be occur).

Let $\Theta_0 = \Theta^-_0 = \{\emptyset\}, E_\emptyset = \Sigma$. Take an arbitrary $\lambda_0 \in \Lambda$. Let $R_\emptyset = \{\lambda_0\}$ and $C_{\emptyset,\lambda_0} = \emptyset$. Then $(E_\emptyset, \{C_{\emptyset,\lambda_0}\}) \in \mathcal{B}$.

For each $n \geq 1$, we will construct an index set $\Theta_n = \Theta^+_n \oplus \Theta^-_n$ of n-th level of a tree with height ω such that each $\theta \in \Theta_n$ assigns a cylindrically closed set $E_\theta \subset \Sigma$, an index set Δ_θ, finite subsets $R_\delta, R_\theta, A_\theta, \in [\Lambda]^{<\omega}$, an index set $\Xi_\delta = \Xi^+_\delta \oplus \Xi^-_\delta$, a compact subset C_δ, points $x_\theta \in X_\delta, y_{\delta,k} \in \Sigma$, basic open sets $U_\delta(\cdot), U_\delta(\cdot, k) \subset X_\delta$, satisfying the following conditions (1) - (4): for $n \in \omega$,

(1) $\{E_\theta : \theta \in \Theta_n\}$ is σ-discrete in Σ,
and $n \geq 1$,

(2) $\Theta_n^+ = \{\theta = (\delta, \xi) : \delta = (\delta(\lambda))_{\lambda \in R_{\theta_-}} \in \Delta_{\theta_-}, \xi \in \Xi_\delta^+, \theta_- \in \Theta_{n-1}^-\}$, $\Theta_n^- = \{\theta = (\delta, \xi) : \delta = (\delta(\lambda))_{\lambda \in R_{\theta_-}} \in \Delta_{\theta_-}, \xi \in \Xi_\delta^-, \theta_- \in \Theta_{n-1}^-\}, \Theta_n = \Theta_n^+ \oplus \Theta_n^-$, and for each $\theta = (\delta, \xi) \in \Theta_n$,
  (a) $\theta_- < \theta$,
  (b) $R_{\theta_-} \subset R_\delta$,
  (c) E_θ is an R_δ-cylindrically closed set in Σ, where $E_\theta = \prod_{\lambda \in R_\delta} E_{\theta,\lambda} \times \Sigma_{\Lambda - \delta}$,

(3) for each $\theta \in \Theta_n^+, E_\theta \subset \cup\{p_{\Lambda-\delta}^{-1}(W_r) : r \in M(p_\delta(E_\theta))\}$,

(4) for each $\theta = (\delta, \xi) \in \Theta_n^-, \delta \in \Delta_{\theta_-}, \xi \in \Xi_\delta^-, \theta_- \in \Theta_{n-1}^-$,
  (a) $(E_{\theta_-}, \{C_{\theta_-,\lambda} : \lambda \in R_{\theta_-}\}) \in \mathcal{B}$,
  (b) $E_{\theta_-} = \cup\{E_\mu : \mu \in \Theta_n \text{ with } \theta_- < \mu\}$,
  (c) for each $x \in X_\delta, U_\delta(x)$ is a basic open neighborhood of x in X_δ,
  (d) for each $x \in X_\delta, \{U_\delta(x,k) : k \in \omega\}$ is an open neighborhood base at x in X_δ such that $U_\delta(x) = U_\delta(x,0)$ and $U_\delta(x,k+1) \subset U_\delta(x,k)$ for each $k \in \omega$,
  (e) $x_\theta = (x_{\theta,\lambda}) \in p_\delta(E_{\theta_-})$,
  (f) $p_\delta(E_\theta) \subset U_\delta(x_\theta)$,
  (g) $p_{\delta_-}^\delta(U_\delta(x_\theta)) \subset U_{\delta_-}(x_{\theta_-})$, where $n \geq 2$ and $\theta = (\delta, \xi)$ and $\theta_- = (\delta_-, \xi')$ for $\delta_- \in \Delta_\mu, \xi \in \Xi_\delta^-, \mu \in \Theta_{n-2}$, and $\xi' \in \Xi_\mu^-$,
  (h) $y_{\theta,k} \in p_\delta^{-1}(U_\delta(x_\theta,k)) - \cup\{p_{\Lambda-\delta}^{-1}(W_r) : r \in M(U_\delta(x_\theta,k))\}$,
  (i) a set A_θ and for a $\delta = (\delta(\lambda)) \in \Delta_{\theta_-}$, a nonempty compact subset $C_\delta = \prod_{\lambda \in R_{\theta_-}} C_{\delta(\lambda)} \times \prod_{\Lambda - R_{\theta_-}} \{x_\lambda^*\} \subset E_{\theta_-}$, satisfy the following:
    (i-1) if $\lambda \in R_{\theta_-}$ and $C_{\theta_-,\lambda} \neq \emptyset$, let $C_{\delta(\lambda)} = C_{\theta_-,\lambda}$,
    (i-2) if $\lambda \in R_{\theta_-}$ and $C_{\theta_-,\lambda} = \emptyset$, let $\mathcal{C}(E_{\theta_-}, \lambda)$ be a discrete collection of compact subsets of $E_{\theta_-,\lambda}$ such that $s(E_{\theta_-,\lambda}) = \cup\mathcal{C}(E_{\theta_-}, \lambda)$. Take a compact set $C_{\delta(\lambda)}$ satisfying $s_\lambda(E_{\theta_-,\lambda}) \cap E_{\theta,\lambda} \subset C_{\delta(\lambda)}$ as follows: $C_{\delta(\lambda)} \subset C$ for some $C \in \mathcal{C}(E_{\theta_-}, \lambda)$ or $|C_{\delta(\lambda)}| = 1$,

(i-3) $A_\theta = \{\lambda \in R_\delta : p_\lambda(E_\theta) \cap p_\lambda(C_\delta) = \emptyset\}$ and hence, if $\lambda \in R_{\theta_-}$ with $C_{\theta_-,\lambda} = \emptyset$ and $\lambda \in A_\theta$, then $s_\lambda(E_{\theta_-,\lambda}) \cap E_{\theta,\lambda} = \emptyset$,

(i-4) if $\lambda \in R_\delta$ and $x_{\theta,\lambda} \notin p_\lambda(C_\delta)$, then $\lambda \in A_\theta$,

(j) $R_\theta = \cup\{\langle \mathrm{Supp}(y_{\mu,j})\rangle_k : \mu \le \theta \text{ and } j, k \le n\} \cup R_\delta$,

(k) for each $\lambda \in R_\theta$,
let $C_{\theta,\lambda} = p_\lambda(E_\theta) \cap p_\lambda(C_\delta) = E_{\theta,\lambda} \cap C_{\delta(\lambda)}$ if $\lambda \in R_{\theta_-} - A_\theta$;
let $C_{\theta,\lambda} = \{x^*_\lambda\}$ if $\lambda \in R_\delta - (R_{\theta_-} \cup A_\theta)$; and
let $C_{\theta,\lambda} = \emptyset$, otherwise.

(l) $(E_\theta, \{C_{\theta,\lambda} : \lambda \in R_\theta\}) \in \mathcal{B}$.

Assume that the construction above has been already performed for no greater than $n \ge 1$.

Take a $\theta \in \Theta^-_n$, $\theta = (\delta, \xi), \delta \in \Delta_{\theta_-}, \xi \in \Xi^-_\delta, \theta_- \in \Theta^-_{n-1}$, $n \ge 1$ and let $(E_\theta, \{C_{\theta,\lambda} : \lambda \in R_\theta\}) \in \mathcal{B}$, where $E_\theta = \prod_{\lambda \in R_\delta} E_{\theta,\lambda} \times \Sigma_{\Lambda - \delta}$. For each $\lambda \in R_\theta$, take a discrete collection $\mathcal{C}(\theta, \lambda) = \{C_\gamma : \gamma \in \Gamma(\theta, \lambda)\}$ of compact subsets of $E_{\theta,\lambda}$ as before. Then there is a σ-discrete closed cover $\mathcal{F}_{\theta,\lambda} = \cup_{n\in\omega} \mathcal{F}_{\theta,\lambda,n}, \mathcal{F}_{\theta,\lambda,n} = \{F_\psi : \psi \in \Psi_{\theta,\lambda,n}\}$, of $E_{\theta,\lambda}$ such that for each $\psi \in \Psi_{\theta,\lambda,n}, n \in \omega, F_\psi$ meets at most one member of $\mathcal{C}(\theta, \lambda)$. For each $\psi \in \Psi_{\theta,\lambda,n}, n \in \omega$, let $C_\psi = F_\psi \cap C_\gamma$ if there is a (unique) $\gamma \in \Gamma(\theta, \lambda)$ such that $F_\psi \cap C_\gamma \ne \emptyset$, or $C_\psi = \{f\}$, where $f \in F_\psi$, otherwise. Take a $\epsilon = (m_\lambda) \in \omega^{|R_\theta|}$. Define $\Delta_{\theta,\epsilon} = \prod_{\lambda \in R_\theta} \Psi_{\theta,\lambda,m_\lambda}$ and $\Delta_\theta = \cup\{\Delta_{\theta,\epsilon} : \epsilon \in \omega^{|R_\theta|}\}$. For each $\delta \in \Delta_\theta$, we denote $\delta = ((\psi(\delta, \lambda))$. Define $C_\delta = \prod_{\lambda \in R_\theta} C_{\psi(\delta,\lambda)} \times \prod_{\Lambda - R_\theta} \{x^*_\lambda\}$. Since C_δ is compact, there is a finite collection $\mathcal{B}_\delta$ of cylindrically open subsets in Σ such that for each $B \in \mathcal{B}_\delta$, there is a $G \in \mathcal{G}$ such that $\overline{B} \subset G$, and $C_\delta \subset \cup\mathcal{B}_\delta$. Define R_δ as before and $F_\delta = \prod_{\lambda \in R_\theta} F_{\psi(\delta,\lambda)} \times X_{\delta-\theta}$. Let

$$\Omega = \{P : P \text{ is an open subset in } X_\delta \text{ meeting } F_\delta \text{ such that}$$

$$p_\delta^{-1}(P) \subset \cup\{p_{\Lambda-\delta}^{-1}(W_r) : r \in M(P)\}\} \text{ and}$$

$$\Phi_\delta = F_\delta - \cup\Omega.$$

It is clear that $p_\delta(C_\delta) \cap \Phi_\delta = \emptyset$. By Lemma 2.3 (iii), every open cover of F_δ has a σ-discrete refinement, consisting of closed rectangles. The rest of the construction is similar to that in the proofs of [13, Theorem 3.1 and Theorem 4.2] and [20, Theorem 4].

Let $\Theta = \cup_{n\in\omega}\Theta_n$, $\Theta^+ = \cup_{n\geq 1}\Theta_n^+$ and $\mathcal{E} = \{E_\theta : \theta \in \Theta^+\}$. It follows from (1) and (3) that $\mathcal{E}$ is a σ-discrete collection of cylindrically closed sets in Σ and for each $\theta \in \Theta^+, E_\theta \subset \cup\{p^{-1}_{\Lambda-\delta}(W_r) : r \in M(p_\delta(E_\theta))\}$. By Lemma 4.1, it suffices to prove that $\mathcal{E}$ is a cover of Σ. Assume that there is a point $x = (x_\lambda) \in \Sigma - \cup\mathcal{E}$. By (4) (b), we can inductively choose a sequence $\{\theta_n : n \in \omega\}$ such that $\theta_n = (\delta_n, \xi_n) \in \Theta_n, \delta_n \in \Delta_{\theta_{n-1}}, \xi_n \in \Xi^-_{\delta_n}, \theta_{n-1} < \theta_n, n \geq 1$ and $x \in E_{\theta_n}$ for each $n \in \omega$. Let $Q = \cup_{n\in\omega}R_{\theta_n}$. As the same way as Claim 4 in Theorem 3.1, for each $\lambda \in Q$, there is an $m_\lambda \geq 1$ such that if $n \geq m_\lambda$, then $x_{n,\lambda} \in p_\lambda(C_{m_\lambda})$. (We similarly use C_{m_λ}.) By the proof of Theorem 3.1, we define C^m_k and K_k for each $k, m \geq 1$ with $m > k$. Then we can find a point $(z_k) \in \varprojlim\{K_k, p^{k+1}_k\}$. Define a point $z = (z_\lambda) \in \Sigma$ such that $p_k(z) = z_k$ for each $k \geq 1$ and $z_\lambda = x^*_\lambda$ otherwise. Then we have a contradiction in the same argument as [20, Lemma 7]. □

By theorems 3.1 and 4.2 and Lemma 2.2, we have

Theorem 4.3. *If Σ is a Σ-product of first countable, paracompact $\mathcal{DC}$-like spaces, then it is shrinking.*

Corollary 4.4. (i) *If Σ is a Σ-product of first countable, paracompact (subparacompact) spaces which admit a σ-closure-preserving closed cover by compact sets, then it is shrinking (subshrinking).*
(ii) *If Σ is a Σ-product of first countable, paracompact (subparacompact) σ-C-scattered spaces, then it is shrinking (subshrinking).*

A space X is said to be *subnormal* if for any disjoint closed sets E and F in X, there are disjoint G_δ-sets G and H in X such that $E \subset G$ and $F \subset H$. Clearly subparacompact spaces and subshrinking spaces are subnormal.

The proof of the following result is similar to that of Theorem 3.1. (We also use the similar method in the proof of Theorem 4.2.)

Theorem 4.5. *Let Σ be a Σ-product of subparacompact $\mathcal{DC}$-like spaces. If Σ has countable tightness, then it is subnormal.*

Corollary 4.6. (i) *Let Σ be a Σ-product of subparacompact spaces which admit a σ-closure-preserving closed cover by compact sets. If Σ has countable tightness, then it is subnormal.*
(ii) *Let Σ be a Σ-product of subparacompact σ-C-scattered spaces. If Σ has countable tightness, then it is subnormal.*

References

[1] H. H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. **81** (1959), 785–796.

[2] R. Engelking, *General Topology*, Heldermann, Berlin, 1989.

[3] F. Galvin and R. Telgársky, *Stationary strategies in topological games*, Topology and Appl. **22** (1986), 51–69.

[4] S. P. Gul'ko, *On the properties of subsets of* Σ*-products*, Soviet Math. Dokl. **18** (1977), 1438–1442.

[5] A. P. Kombarov, *On* Σ*-products of topological spaces*, Soviet Math. Dokl. **13** (1971), 1101–1104.

[6] A. P. Kombarov, On tightness and normality of Σ-products, Soviet Math. Dokl. **19** (1978), 403–407.

[7] A. P. Kombarov and V. I. Malykhin, *On* Σ*-products*, Soviet Math. Dokl. **14** (1973), 1780–1783.

[8] H. B. Potoczny, *Closure-preserving families of compact sets*, General Top. Appl. **3** (1973), 243–248.

[9] M. E. Rudin, Σ*-products of metric spaces are normal*, preprint

[10] M. E. Rudin, *The shrinking property*, Canad. Math. Bull. **28** (1983), 385–388.

[11] H. Tanaka, *A class of spaces whose countable product with a perfect paracompact space is paracompact*, Tsukuba J. Math. **16** (1992), 503–512.

[12] H. Tanaka, *Covering properties in countable products*, Tsukuba J. Math. **17** (1993), 565–587.

[13] H. Tanaka and Y. Yajima, Σ*-products of paracompact C-scattered spaces*, to appear in Topology and Appl.

[14] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math. **73** (1971), 59–74.

[15] R. Telgársky, *Spaces defined by topological games*, Fund. Math. **88** (1975), 193–223.

[16] Y. Yajima, *Topological games and products II*, Fund. Math. **117** (1983), 47–60.

[17] Y. Yajima, *Topological games and products III*, Fund. Math. **117** (1983), 223–238.

[18] Y. Yajima, *On* Σ*-products of* Σ*-spaces*, Fund. Math. **123** (1984), 29–37.

[19] Y. Yajima, *On* Σ*-products of semi-stratifiable spaces*, Topology and Appl. **25** (1987), 1–11.

[20] Y. Yajima, *The shrinking property of* Σ*-products*, Tsukuba J. Math. **13** (1989), 83–98.

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