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ON ARCWISE ACCESSIBILITY IN HYPERSPACES

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Dedicated to Alfredo Macías on the occasion of his 75th birthday

ABSTRACT. The purpose of this paper is to show that in the results of Section 4 of [3], concerning arcwise accessibility of singletons from $\mathcal{C}_2(X) \setminus \mathcal{C}(X)$ for a continuum X, we can replace $\mathcal{C}_2(X)$ by $\mathcal{C}_n(X)$ for any $n > 2$. We also present more results about arcwise accessibility in these hyperspaces.

1. Introduction

Professor Janusz J. Charatonik asked the author if in the results of [3, Section 4], concerning arcwise accessibility of singletons from $\mathcal{C}_2(X) \setminus \mathcal{C}(X)$, we can replace $\mathcal{C}_2(X)$ by $\mathcal{C}_n(X)$ for some $n > 2$. The purpose of this paper is to give a positive answer to this question (see 3.4, below), and present more results about arcwise accessibility in these hyperspaces. In section 2, we give the basic definitions for understanding the paper. In section 3, we present the main results.

2. Definitions

If (Z, d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, and the closure of A is denoted by $Cl(A)$.

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A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a space Z is a continuum contained in Z. A continuum is said to be *decomposable* provided it can be written as the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable. A continuum X is *hereditarily indecomposable* provided that each subcontinuum of X is indecomposable. A *map* means a continuous function. An *arc* in a continuum X is a one–to–one map $\alpha\colon [0,1] \to X$.

Given a continuum X and a point x in X, the *composant* of X containing x, is the union of all proper subcontinua of X containing x (see [5, p. 208]).

Given a continuum X, we define its *hyperspaces* as the following sets:

$$2^X = \{A \subset X \ \mid A \text{ is closed and nonempty}\},$$

$$\mathcal{C}(X) = \{A \in 2^X \mid A \text{ is a connected}\},$$

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},\ n \in \mathbb{N},$$

$$\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}.$$

We agree that $\mathcal{C}(X) = \mathcal{C}_1(X)$. Let us observe that for each $n \in \mathbb{N}$,

$$\mathcal{F}_n(X) \subset \mathcal{C}_n(X),$$

$$\mathcal{C}_n(X) \subset \mathcal{C}_{n+1}(X),$$

$$\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X).$$

On the other hand, it is known that 2^X is a metric space with the Hausdorff metric, $\mathcal{H}$, defined as follows:

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\},$$

(see [9, (0.1)]). It is known that 2^X and $\mathcal{C}(X)$ are arcwise connected continua (see [9, (1.13)]), and for each $n \in \mathbb{N}$, $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are continua (for $\mathcal{C}_n(X)$ see [6, 3.1] and for $\mathcal{F}_n(X)$ see [1, p. 877]). On the other hand, 2^X can be topologized with the *Vietoris Topology,* defined as follows: given a finite collection, $U_1, U_2, \ldots, U_m$, of open sets of X, we define

$$\langle U_1, \ldots, U_m\rangle =$$

$$\left\{ A \in 2^X \mid A \subset \bigcup_{k=1}^{m} U_k \text{ and } A \cap U_k \neq \emptyset \text{ for each } k \in \{1, \ldots, m\} \right\}.$$

It is known that the family of all subsets of 2^X of the form $\langle U_1, \ldots, U_m\rangle$, as defined above, forms a basis for a topology for 2^X (see [9, (0.11)]) called *Vietoris Topology,* and that the Vietoris Topology and the Topology induced by the Hausdorff metric coincide (see [9, (0.13)]).

An *order arc* in 2^X is an arc $\alpha\colon [0,1] \to 2^X$ such that if $0 \leq s < t \leq 1$ then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$.

Let X be a continuum. Let Σ_1 and Σ_2 be two arcwise connected closed subsets of 2^X, such that $\Sigma_2 \subset \Sigma_1$. A member A of Σ_2 is said to be *arcwise accessible from* $\Sigma_1 \setminus \Sigma_2$ *beginning with* K if and only if there is an arc $\alpha\colon [0,1] \to \Sigma_1$ such that $\alpha(0) = K$, $\alpha(1) = A$, and $\alpha(t) \in \Sigma_1 \setminus \Sigma_2$ for all $t < 1$.

3. Theorems

We begin observing that in the proof of Theorem (2.2) of [8], Nadler actually showed the following:

Theorem 3.1. *Let X be a continuum. If A is a nondegenerate subcontinuum of X and q is any point of A, then there exist a point $p \in A\setminus\{q\}$ and an order arc $\alpha\colon [0,1] \to \mathcal{C}(A)$ such that $\alpha(0) = \{q\}$, $\alpha(1) = A$ and $p \notin \alpha(t)$ for any $t < 1$.*

Using this result we have the following one.

Theorem 3.2. *Let n be a positive integer greater than one. Let X be a continuum and A be an element of $\mathcal{C}_n(X)$ having exactly n components and at least one of them is nondegenerate. Then A is arcwise accessible from $\mathcal{C}_{n+1}(X) \setminus \mathcal{C}_n(X)$, beginning with an element in $\mathcal{F}_{n+1}(X) \setminus \mathcal{F}_n(X)$.*

Proof: Suppose $A_1, \ldots, A_n$ are the components of A and that A_n is not degenerate. For each $j \in \{1, \ldots, n\}$, let $q_j \in A_j$. By 3.1, there exist $q_{n+1} \in A_n$ and an order arc $\alpha_n\colon [0,1] \to \mathcal{C}(A_n)$ such that $\alpha_n(0) = \{q_n\}$, $\alpha_n(1) = A_n$, and $q_{n+1} \notin \alpha_n(t)$ for any $t < 1$. For each $j \in \{1, \ldots, n-1\}$, let $\alpha_j\colon [0,1] \to \mathcal{C}(A_j)$ be an order arc such that $\alpha_j(0) = \{q_j\}$, $\alpha_j(1) = A_j$ (see [9, (1.8)]).

Let $\gamma\colon [0,1] \to \mathcal{C}_{n+1}(X)$ be given by $\gamma(t) = \alpha_1(t) \cup \ldots \cup \alpha_n(t) \cup \{q_{n+1}\}$. Since for each $j \in \{1, \ldots, n\}$, α_j is continuous and the union is also continuous (see [9, (1.48)]), then γ is continuous. Also, we have that $\gamma(0) = \{q_1, \ldots, q_{n+1}\}$, $\gamma(1) = A$, and $\gamma(t) \in \mathcal{C}_{n+1}(X)\setminus \mathcal{C}_n(X)$ for each $t < 1$. Therefore, A is arcwise accessible from

$C_{n+1}(X) \setminus \mathcal{C}_n(X)$, beginning with an element in $\mathcal{F}_{n+1}(X) \setminus \mathcal{F}_n(X)$. □

Corollary 3.3. *Let n be a positive integer greater than two. Let X be a continuum. If x is a point of X such that $\{x\}$ is arcwise accessible from $\mathcal{C}_2(X) \setminus \mathcal{C}(X)$ with an arc α such that $\alpha([0,1]) \cap (\mathcal{C}_2(X) \setminus \mathcal{F}_2(X)) \neq \emptyset$, then $\{x\}$ is arcwise accessible from $\mathcal{C}_n(X) \setminus \mathcal{C}(X)$.*

Remark 3.4. Since in the proofs of 4.1, 4.2, 4.3, 4.5, and 4.7, and in Example 4.4 of [3], the arcs used for the arcwise accessibility of singletons have, at some point, a nondegenerate component, we may replace $\mathcal{C}_2(X)$ by $\mathcal{C}_n(X)$ for each positive integer $n > 2$.

Theorem 3.5. *Let n be a positive integer greater than two. Let X be a continuum and let a be a point of X such that $\{a\}$ is arcwise accessible from $2^X \setminus \mathcal{C}(X)$. Then each element A of $\mathcal{F}_n(X) \setminus \mathcal{F}_{n-1}(X)$ containing a is arcwise accessible from $2^X \setminus \mathcal{C}_n(X)$.*

Proof: Let $\{a_1, \dots, a_n\} \in \mathcal{F}_n(X) \setminus \mathcal{F}_{n-1}(X)$ containing a. Without loss of generality we may assume that $a = a_n$. Let U be an open set of X such that $a_n \in U$, and for each $j \in \{1, \dots, n-1\}$, $a_j \notin U$. Since $\{a_n\}$ is arcwise accessible from $2^X \setminus \mathcal{C}(X)$, there exists an arc $\alpha\colon [0,1] \to 2^X$ such that $\alpha(1) = \{a_n\}$ and $\alpha(t) \in 2^X \setminus \mathcal{C}(X)$ for each $t < 1$. By continuity, there exists $t_1 \in [0,1)$ such that $\alpha(t) \in U$ for each $t \geq t_1$. Let $\beta\colon [0,1] \to 2^X$ be given by $\beta(s) = \{a_1, \dots, a_{n-1}\} \cup \alpha((1-s)t_1 + s)$. Since α is continuous and the union function is also continuous (see [9, (1.48)]), we have that β is continuous. By construction, we also have that $\beta(1) = \{a_1, \dots, a_n\}$, and $\beta(t) \in 2^X \setminus \mathcal{C}_n(X)$ for each $t < 1$. Therefore, $\{a_1, \dots, a_n\}$ is arcwise accessible from $2^X \setminus \mathcal{C}_n(X)$. □

As we can see from the proof of the previous theorem, we have the following result.

Corollary 3.6. *Let n and m be positive integers, n greater than two. Let X be a continuum and let a be a point of X such that $\{a\}$ is arcwise accessible from $\mathcal{C}_m(X) \setminus \mathcal{C}(X)$. Then each element A of $\mathcal{F}_n(X) \setminus \mathcal{F}_{n-1}(X)$ containing a is arcwise accessible from $\mathcal{C}_{m+n-1}(X) \setminus \mathcal{C}_n(X)$.*

Theorem 3.7. *Let n be a positive integer greater than one. Let X be a continuum. Suppose that for each $j \in \{1, \dots, n\}$, a_j is a* *point of X such that $\{a_j\}$ is not arcwise accessible from $2^X \setminus \mathcal{C}(X)$. Then $\{a_1, \ldots, a_n\}$ is not arcwise accessible from $2^X \setminus \mathcal{C}_n(X)$.*

Proof: Suppose that $\{a_1, \ldots, a_n\}$ is arcwise accessible from $2^X \setminus \mathcal{C}_n(X)$, so there exists an arc $\alpha\colon [0,1] \to 2^X$ such that $\alpha(1) = \{a_1, \ldots, a_n\}$ and $\alpha(t) \in 2^X \setminus \mathcal{C}_n(X)$ for each $t < 1$. Let $U_1, \ldots, U_n$ be open sets of X such that for each $j \in \{1, \ldots, n\}$, $a_j \in U_j$, and $Cl(U_j) \cap Cl(U_k) = \emptyset$ if and only if $j, k \in \{1, \ldots, n\}$ and $j \neq k$. By continuity, there exists $t_1 \in [0,1)$ such that $\alpha(t) \in \langle U_1, \ldots, U_n \rangle$ for each $t \geq t_1$.

For each $j \in \{1, \ldots, n\}$, let $\beta_j\colon [0,1] \to 2^X$ be given by $\beta_j(s) = \alpha((1-s)t_1+s) \cap U_j$. Since the family $\{U_1, \ldots, U_n\}$ of open sets has pairwise disjoint closures, β_j is well defined for each $j \in \{1, \ldots, n\}$. To see that each β_j is continuous, let $j \in \{1, \ldots, n\}$ and $\varepsilon > 0$ such that $d(Cl(U_k), Cl(U_\ell)) > \varepsilon$, for each $k, \ell \in \{1, \ldots, n\}$ and $k \neq \ell$. Let $s_0 \in [0,1]$. By continuity, there exists $\delta > 0$ such that if $|s_0 - s_1| < \delta$ then $\mathcal{H}(\alpha((1-s_0)t_1+s_0), \alpha((1-s_1)t_1+s_1)) < \varepsilon$. Let $x \in \alpha((1-s_0)t_1+s_0) \cap U_j$. Since $\mathcal{H}(\alpha((1-s_0)t_1+s_0), \alpha((1-s_1)t_1+s_1)) < \varepsilon$, there exits $y \in \alpha((1-s_1)t_1+s_1)$ such that $d(x,y) < \varepsilon$. Since $y \in \bigcup_{k=1}^{n} U_k$, there exists $\ell \in \{1, \ldots, n\}$ such that $y \in U_\ell$. Since $d(x,y) < \varepsilon$ and $d(Cl(U_k), Cl(U_\ell)) > \varepsilon$ for each $k \in \{1, \ldots, n\}$ and $k \neq \ell$, we have that $\ell = j$ and $y \in U_j$. Therefore, $y \in \alpha((1-s_1)t_1+s_1) \cap U_j$, and $\alpha((1-s_0)t_1+s_0) \cap U_j \subset \mathcal{V}^d_\varepsilon(\alpha((1-s_1)t_1+s_1) \cap U_j)$. Similarly $\alpha((1-s_1)t_1+s_1) \cap U_j \subset \mathcal{V}^d_\varepsilon(\alpha((1-s_0)t_1+s_0) \cap U_j)$. Thus, $\mathcal{H}(\alpha((1-s_0)t_1+s_0) \cap U_j, \alpha((1-s_1)t_1+s_1) \cap U_j) = \mathcal{H}(\beta(s_0), \beta(s_1)) < \varepsilon$. Hence, β_j is continuous.

On the other hand, for each $j \in \{1, \ldots, n\}$, $\beta_j(1) = \{a_j\}$. Since, for each $j \in \{1, \ldots, n\}$, $\{a_j\}$ is not arcwise accessible from $2^X \setminus \mathcal{C}(X)$, we have that, for each $j \in \{1, \ldots, n\}$, there exists $s_j \in [0,1)$ such that $\beta_j(s) \in \mathcal{C}(X)$ for every $s \geq s_j$. Let $s_* = \max\{s_1, \ldots, s_n\}$. Then, for each $j \in \{1, \ldots, n\}$, $\beta_j(s) \in \mathcal{C}(X)$ for every $s \geq s_*$.

Let $s \geq s_*$, then $\bigcup_{j=1}^{n} \beta_j(s) = \bigcup_{j=1}^{n} \alpha((1-s)t_1+s) \cap U_j = \alpha((1-s)t_1+s) \cap \bigcup_{j=1}^{n} U_j = \alpha((1-s)t_1+s)$. On the other hand, if $s \geq s_*$,

$\bigcup_{j=1}^{n} \beta_j(s) \in \mathcal{C}_n(X)$. Thus, for each $s \geq s_*$, $\alpha((1-s)t_1+s) \in \mathcal{C}_n(X)$, a contradiction. Therefore, $\{a_1, \dots, a_n\}$ is not arcwise accessible from $2^X \setminus \mathcal{C}_n(X)$. □

Corollary 3.8. *Let n be a positive integer greater than one. If X is a hereditarily indecomposable continuum, then $\{a_1, \dots, a_n\}$ is not arcwise accessible from $2^X \setminus \mathcal{C}_n(X)$ for each $\{a_1, \dots, a_n\} \in \mathcal{F}_n(X)$.*

Proof: Since X is hereditarily indecomposable, no singleton is arcwise accessible from $2^X \setminus \mathcal{C}(X)$ (see [8, (3.4)]). Thus, the result follows from Theorem 3.7. □

Remark 3.9. Nadler has shown that if X is the union of two hereditarily indecomposable Y and Z such that $Y \cap Z$ is a nondegenerate continuum, then no singleton is arcwise accessible from $2^X \setminus \mathcal{C}(X)$ (see [8, (3.5)]). By Theorem 3.5, we have that for this continuum X, and for each positive integer greater than one, no element of $\mathcal{F}_n(X)$ is arcwise accessible from $2^X \setminus \mathcal{C}_n(X)$.

The proof of the following result is similar to the one given in 3.10 of [7]; we include it for completeness.

Theorem 3.10. *Let n be a positive integer. Let X be an indecomposable continuum. Suppose $A \in \mathcal{C}_n(X)$ and let $\kappa_1, \dots, \kappa_\ell$ be the composants of X whose union contains A. Let $K \in 2^X \setminus \mathcal{C}_n(X)$ and suppose there exists an arc $\alpha\colon [0,1] \to 2^X$ such that $\alpha(0) = K$, $\alpha(1) = A$, and $\cup\alpha([0,1]) \neq X$. Then $K \subset \bigcup_{j=1}^{\ell} \kappa_j$ and $K \cap \kappa_j \neq \emptyset$ for each $j \in \{1, \dots, \ell\}$.*

Proof: Suppose first that $K \cap \kappa_j = \emptyset$, for some $j \in \{1, \dots, \ell\}$. Let $\beta\colon [0,1] \to 2^X$ be given by $\beta(t) = \cup\alpha([1-t, 1])$. Then, β is an order arc, $\beta(0) = A$, and $\beta(1) = \cup\alpha([0,1])$. Hence, for each $t \in [0,1]$, we have that $\beta(t) \in \mathcal{C}_n(X)$ (see [9, (1.8)]). On the other hand, the map $\gamma\colon [0,1] \to 2^X$ given by $\gamma(t) = \cup\alpha([0,t])$ is also an order arc, $\gamma(0) = K$, and $\gamma(1) = \beta(1) = \cup\alpha([0,1])$. Thus, there exists an order arc in 2^X from K to $\gamma(1)$, $K \cap \kappa_j = \emptyset$, and $\gamma(1) \cap \kappa_j \neq \emptyset$; this contradicts [9, (1.8)], since $\cup\alpha([0,1])$ is a proper subset of X.

A similar argument shows that if $K \cap \kappa \neq \emptyset$, then $\kappa \in \{\kappa_1, \ldots, \kappa_\ell\}$. Therefore, we obtain that $K \subset \bigcup_{j=1}^{\ell} \kappa_j$ and $K \cap \kappa_j \neq \emptyset$ for each $j \in \{1, \ldots, \ell\}$. □

The proof of the following result, which we include for the convenience of the reader, is a modification of the one given in [8, (3.1)].

Theorem 3.11. *Let n be a positive integer greater than one. Let X be an indecomposable continuum and let Λ be a locally connected subcontinuum of 2^X. If $\cup\Lambda = X$ and $\Lambda \cap \mathcal{C}_n(X) \neq \emptyset$, then $X \in \Lambda$.*

Proof: Since Λ is a locally connected continuum, there is a map $\lambda\colon [0,1] \to \Lambda$ (see [4, Theorem 3–30]). Since $\Lambda \cap \mathcal{C}_n(X) \neq \emptyset$, there exists $t_0 \in [0,1]$ such that $\lambda(t_0) \in \mathcal{C}_n(X)$. Let $a = \sup\{t \leq t_0 \mid \cup \lambda([t,1]) = X\}$, and let $b = \inf\{t_0 \leq t \mid \cup\lambda([a,t]) = X\}$. Then, by the continuity of λ and of the union function (see [9, (1.48)]), we have that $\cup\lambda([a,b]) = X$. Also observe that $[a,b]$ is minimal with respect of the property of being an interval containing t_0 and having the union of its images be all of X.

Suppose $a < t_0$. Let $s_0 \in (a, t_0]$, then $\lambda([s_0,b])$ is a subcontinuum of 2^X meeting $\mathcal{C}_n(X)$; a similar argument to the one given in 7.2 of [6] shows that $\cup\lambda([s_0,b]) \in \mathcal{C}_n(X)$. By the minimality of $[a,b]$, we have that $\cup\lambda([s_0,b]) \neq X$. Observe that, by 3.10, $\cup\lambda([s_0,b])$ is contained in the union of the composants of X containing $\lambda(t_0)$. Thus, $\cup\lambda([s_0,b])$ is a nowhere dense subset of X (see [5, Theorem 2, p. 207]). The same is true for $\cup\{\cup\lambda([s,b]) \mid a < s \leq t_0\} = \cup\lambda((a,b])$. It follows from $\lambda([a,b]) = X$ and the compactness of $\lambda(a)$ that $\lambda(a) = X$. A similar argument shows that if we assume that $t_0 < b$, then $\lambda(b) = X$. Finally, if $a = t_0 = b$, then $\lambda(t_0) = X$. □

Remark 3.12. Let m and n be positive integers such that $m > n$. Suppose an element $A \in \mathcal{F}_n(X)$ is arcwise accessible from $\mathcal{F}_m(X) \setminus \mathcal{F}_n(X)$, beginning at a point $B \in \mathcal{F}_m(X) \setminus \mathcal{F}_n(X)$. Let $\alpha\colon [0,1] \to \mathcal{F}_m(X)$ be an arc such that $\alpha(0) = B$ and $\alpha(1) = A$. Then, $\cup\alpha([0,1])$ is a locally connected closed subset of X (see [2, 2.2]). Thus, given a point $a \in A$, there exist a point $b \in B$ and an arc in X joining them and vice versa, i. e., given a point $b' \in B$, there exist a point $a' \in A$ and an arc joining them.

Question 3.13. Let n be a positive integer greater that two. Let X be a continuum and x a point of X. If $\{x\}$ is arcwise accessible from $\mathcal{C}_n(X) \setminus \mathcal{C}(X)$, then is $\{x\}$ arcwise accessible from $\mathcal{C}_2(X) \setminus \mathcal{C}(X)$?

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