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n – FOLD HYPERSPACES, CONES AND PRODUCTS

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ABSTRACT. Given a continuum X and an integer n greater than one, we consider the n–fold hyperspace, $\mathcal{C}_n(X)$, of X. We show that if X is finite dimensional, then $\mathcal{C}_n(X)$ is not homeomorphic to Cone(X). We also show that if X is hereditarily indecomposable, then $\mathcal{C}_n(X)$ is not homeomorphic to the Cone(Z) for any finite – dimensional continuum Z. On the other hand, we show that if S^1 is the unit circle, then $\mathcal{C}_2(S^1)$ is not homeomorphic to the product, $Y \times D$, for any one – dimensional continuum D.

1. Introduction

The question of when the hyperspace of subcontinua of a continuum X is homeomorphic to its cone or to the product of two continua Y and Z has been studied (see Chapters VIII and X of [11] and Sections 40, 79 and 80 of [6]). It is natural to ask if the n – fold hyperspaces of X, for an integer $n > 1$, are homeomorphic to the cone over X or to the product of two continua. We give some partial answers to these questions. In section 3, we consider

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the case of cones. In section 4, we present the case of products, mainly $\mathcal{C}_2(S^1)$, where S^1 is the unit circle.

2. Definitions

If (Y, d) is a metric space, then given $A \subset Y$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior of A is denoted by $\mathrm{Int}_Y(A)$, and its closure is denoted by $\mathrm{Cl}_Y(A)$. A closed subset A of Y is a *$Z-$set* in Y if and only if for every $\varepsilon > 0$, there exists a continuous function $f_\varepsilon \colon Y \to Y \setminus A$ such that $d(y, f_\varepsilon(y)) < \varepsilon$ for any $y \in Y$. The symbol $\mathbb{N}$ denotes the set of positive integers.

A *continuum* is a nonempty, compact, connected, metric space. For the definitions of special types of continua, see [12]. We note that a finite−dimensional continuum X is a *Cantor manifold* if for any subset B of X, such that $\dim(B) \leq \dim(X) - 2$, $X \setminus B$ is connected.

Given a continuum X, a subcontinuum B of X is said to be *terminal in* X provided that if Y is a subcontinuum of X such that $Y \cap B \neq \emptyset$ then either $Y \subset B$ or $B \subset Y$.

We denote the cone over a space Z by $\mathrm{Cone}(Z)$ and the vertex of $\mathrm{Cone}(Z)$ by $v(Z)$. The symbol π denotes the projection $\pi \colon \mathrm{Cone}(Z) \setminus \{v(Z)\} \to Z$ given by $\pi((z, t)) = z$.

We use Z^n to denote the cartesian product of n copies of Z with itself. An *$n-$cell* is any space homeomorphic to $[0, 1]^n$.

A *map* is a continuous function.

Given a continuum X, we consider the following hyperspaces of X:

$$\begin{aligned}
2^X &= \{A \subset X \mid A \text{ is closed and nonempty}\} \\
\mathcal{C}(X) &= \{A \in 2^X \mid A \text{ is a connected}\} \\
\mathcal{C}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},\ n \in \mathbb{N} \\
\mathcal{F}_n(X) &= \{A \in 2^X \mid A \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}
\end{aligned}$$

each with the Hausdorff metric (see [11] and [6] for general information about hyperspaces).

The space $\mathcal{F}_n(X)$ is called the *$n-$fold symmetric product* of X; $\mathcal{C}_n(X)$ is called the *$n-$fold hyperspace* of X.

3. Cones

We prove that for any continuum X and any integer n greater than one, $\mathcal{C}_n(X)$ is not homeomorphic to Cone(X). Then we assume that X is a continuum whose $n-$fold hyperspace is homeomorphic to the cone over a finite$-$dimensional continuum Z; we present some properties of X and Z under these assumptions.

Lemma 3.1. *If X is a continuum such that each of its proper subcontinuum is indecomposable, then X is indecomposable. Hence, X is hereditarily indecomposable.*

Proof: Suppose X is decomposable, then there exist two proper subcontinua A and B of X such that $X = A \cup B$. Note that $X \setminus B$ is an open subset of X contained in A. On the other hand, there exists a proper subcontinuum, H, of X containing A [12, 5.5]. Hence, H is an indecomposable continuum containing a proper subcontinuum with nonempty interior, which is impossible [7, Theorem 2, p. 207]. Therefore, X is indecomposable. □

Theorem 3.2. *Let X be a finite$-$dimensional continuum. Then for each integer $n > 1$, $\mathcal{C}_n(X)$ is not homeomorphic to Cone(X).*

Proof: Let $n > 1$ be an integer and suppose $\mathcal{C}_n(X)$ is homeomorphic to Cone(X). Since X is of finite dimension, Cone(X) is of finite dimension, too. In fact, $\dim(\text{Cone}(X)) = \dim(X)+1$ (see [3, p. 34] or [11, 8.0]). Since $\mathcal{C}(X) \subset \mathcal{C}_n(X)$ and $\mathcal{C}_n(X)$ is homeomorphic to Cone(X), we have that $\dim(\mathcal{C}(X)) < \infty$. Hence, by the dimension theorem $\dim(X) = 1$ [8, Theorem 2.1]. Thus, $\dim(\text{Cone}(X)) = 2$, and $\dim(\mathcal{C}_n(X)) = 2$. Therefore, since $\mathcal{C}_n(X)$ contains an $n-$cell [9, 3.4], $n = 2$. We consider two cases.

Case 1: *X contains a proper decomposable subcontinuum.* Then, with an argument similar to the one given in the proof of 3.5 in [9], it is easy to show that $\mathcal{C}_2(X)$ contains a $3-$cell, a contradiction to the fact that $\dim(\mathcal{C}_2(X)) = 2$.

Case 2: *All proper subcontinua of X are indecomposable.* Then, by 3.1, X is hereditarily indecomposable. Hence, Cone(X) is uniquely arcwise connected. On the other hand, since $\mathcal{C}_2(X)$ contains 2−cells [9, 3.4], $\mathcal{C}_2(X)$ is not uniquely arcwise connected.

Therefore, $\mathcal{C}_n(X)$ is not homeomorphic to Cone(X). □

Theorem 3.3. *Let X be a continuum and let $n \geq 2$ be an integer. If Z is a finite−dimensional continuum such that Cone(Z) is homeomorphic to $\mathcal{C}_n(X)$, then dim$(X) = 1$ and X contains at most one nondegenerate indecomposable continuum. Hence, X is not hereditarily indecomposable.*

Proof: Let $n > 1$ be an integer. Let $h\colon \mathcal{C}_n(X) \to$ Cone(Z) be a homeomorphism. The proof of the fact that dim$(X) = 1$ is similar to the one given in 3.2.

Suppose X contains two nondegenerate indecomposable continua, A and B. Then $\mathcal{C}(X)\setminus\{A\}$ and $\mathcal{C}(X)\setminus\{B\}$ have infinitely many arc components [11, (v), p. 312]. Thus, $\mathcal{C}_n(X) \setminus \{A\}$ and $\mathcal{C}_n(X) \setminus \{B\}$ have infinitely many arc components ([3, Theorem 3−46], [11, (v), p. 312], [9, 6.6] and [10, 3.9]). Hence, Cone$(Z) \setminus \{h(A)\}$ and Cone$(Z) \setminus \{h(B)\}$ both have infinitely many arc components. On the other hand, for each $p \in$ Cone$(Z) \setminus \{v(Z)\}$, Cone$(Z) \setminus \{p\}$ has at most two arc components. Therefore, X contains at most one nondegenerate indecomposable subcontinuum. □

Lemma 3.4. *Let X be a continuum such that $\mathcal{C}(X)$ is finite dimensional. If A is a nondegenerate indecomposable proper subcontinuum of X, then at most a finite number of composants of A have the property that some subcontinuum of X contains a point of $X \setminus A$ and a point of the composant but does not contain A; also, $\mathcal{C}_n(X) \setminus \{A\}$ has uncountably many arc components.*

Proof: The first part follows by the proof of (*) of [11, p. 312]. By the proof of (v) of [11, p. 312], $\mathcal{C}(X) \setminus \{A\}$ has uncountably many arc components. Hence, $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected [9, 6.6] and has uncountably many arc components by [12, 11.15] and [10, 3.9]. □

Theorem 3.5. *Let X be a continuum and let $n \geq 2$. Then every 2−cell in $\mathcal{C}_n(X)$ is nowhere dense.*

Proof: First, suppose $n \geq 3$. Let $\mathcal{U}$ be any nonempty open set in $\mathcal{C}_n(X)$. Then there exists $A \in \mathcal{U}$ such that A has exactly n components, $A_1, \ldots, A_n$ [9, 3.3]. By [12, 5.5], for each $j \in \{1, \ldots, n\}$, there is a subcontinuum B_j of X such that B_j contains A_j properly, $B_j \cap B_\ell = \emptyset$ if $j \neq \ell$ and $B = \bigcup_{j=1}^{n} B_j \in \mathcal{U}$.

For each $j \in \{1, \ldots, n\}$, let $\alpha_j \colon [0,1] \to \mathcal{C}(X)$ be an order arc such that $\alpha_j(0) = A_j$ and $\alpha_j(1) = B_j$ [11, (1.8)]. Define $\gamma((t_1, \ldots, t_n)) = \bigcup_{j=1}^{n} \alpha_j(t_j)$. Then $\gamma([0,1]^n)$ is an $n-$cell contained in $\mathcal{U}$. Since $n \geq 3$, $\mathcal{U}$ cannot be a $2-$cell.

Next, suppose $n = 2$. Suppose that $\mathcal{D}$ is a $2-$cell in $\mathcal{C}_2(X)$ with nonempty interior in $\mathcal{C}_2(X)$. Then there exists $D \in \mathrm{Int}_{\mathcal{C}_2(X)}(\mathcal{D})$ such that D has two nondegenerate components, D_1 and D_2 ([9, 3.3] and [12, 5.5]). Let $\varepsilon > 0$ be such that if $E \in \mathcal{C}_2(X)$ and $\mathcal{H}(E, D) < \varepsilon$, then $E \in \mathrm{Int}_{\mathcal{C}_2(X)}(\mathcal{D})$.

Let $\mu \colon \mathcal{C}(D_2) \to [0,1]$ be a Whitney map. Let $\varphi \colon \mathcal{C}(D_2) \to \mathcal{C}_2(X)$ be given by $\varphi(B) = D_1 \cup B$. Then, φ is an embedding of $\mathcal{C}(D_2)$ into $\{G \in \mathcal{C}_2(X) \mid G \subset D\}$. Let $t_0 \in (0,1)$ be such that if $\mu(B) \geq t_0$, then $\mathcal{H}(\varphi(B), D) < \varepsilon$. Let $\mathcal{B} = \{\varphi(B) \mid t_0 < \mu(B) < 1\}$. Then $\mathcal{B}$ is a $2-$dimensional subset of $\mathrm{Int}_{\mathcal{C}_2(X)}(\mathcal{D})$, $\dim(\mathcal{B}) = 2$ is seen using the fact that no zero$-$dimensional set separates $\mathcal{C}(D_2)$ by [11, (2.15)]. Hence, $\mathrm{Int}_{\mathcal{C}_2(X)}(\mathcal{B}) \neq \emptyset$ [3, IV 3, p. 44]. Thus, letting $B_0 \in \mathcal{C}(D_2)$ such that $\varphi(B_0) \in \mathrm{Int}_{\mathcal{C}_2(X)}(\mathcal{B})$, we have that $\varphi(B_0)$ is not arcwise accessible from $\mathcal{C}_2(X) \setminus \mathcal{B}$. However, let $\beta \colon [0,1] \to \mathcal{C}(D_1)$ be an order arc such that $\beta(0) \in \mathcal{F}_1(D_1)$ and $\beta(1) = D_1$ [11, (1.8)]; then $\beta(s) \cup B_0 \notin \mathcal{B}$ for any $s \in [0,1)$ and $\beta(1) \cup B_0 = D_1 \cup B_0 = \varphi(B_0) \in \mathcal{B}$, a contradiction.

Therefore, every $2-$cell in $\mathcal{C}_n(X)$ is nowhere dense. □

Theorem 3.6. *Let X be a continuum containing a nondegenerate indecomposable subcontinuum A. Let $n \geq 2$ be an integer, and let Z be a finite$-$dimensional continuum such that Cone(Z) is homeomorphic to $\mathcal{C}_n(X)$. If $h \colon \mathcal{C}_n(X) \to$ Cone(Z) is a (surjective) homeomorphism, then*

(1) $h(A) = v(Z)$;

(2) *Z has uncountably many arc components. In particular, Z is not locally connected*;
(3) *$dim(\mathcal{C}_n(X)) \geq 2n$ and $dim(Z) \geq 2n-1$*;
(4) *Each point z of Z is contained in an arc in Z and some points of Z belong to locally connected subcontinua of Z whose dimension is at least $2n-1$*;
(5) *No point of Cone$(Z)\setminus\{v(Z)\}$ arcwise disconnects Cone(Z)*;
(6) *If $A = X$, then X does not contain a nondegenerate proper terminal subcontinuum*;
(7) *Z is not irreducible. In particular, Z is decomposable.*

Proof: (1) The proof is similar to the proof of 3.3.

(2) As can be seen in the proof of 3.3, $\mathcal{C}_n(X) \setminus \{A\}$ has uncountably many arc components. Since $h(A) = v(Z)$ (by (1)), Cone$(Z) \setminus \{v(Z)\}$ has uncountably many arc components. Thus, since Cone$(Z)\setminus\{v(Z)\}$ is homeomorphic to $Z \times [0,1)$, we conclude that Z has uncountably many arc components.

(3) By 3.3, each subcontinnum of X, distinct from A, is decomposable; thus, $\mathcal{C}_n(X)$ contains a $2n-$cell [9, 3.5]. Hence, dim$(\mathcal{C}_n(X))$ $\geq 2n$. Since Cone(Z) is homeomorphic to $\mathcal{C}_n(X)$ and dim(Cone(Z)) $=$ dim$(Z)+1$ (see [3, p. 34] or [11, (8.0)]), we have that dim$(Z) \geq$ $2n-1$.

(4) Let z be any point of Z. We consider two cases (π is as in section 2).

First, suppose there exists $t_0 \in [0,1)$ such that $h^{-1}((z,t_0)) \in \mathcal{C}_n(X)\setminus\mathcal{C}(X)$. Let $B = h^{-1}((z,t_0))$ and let $B_1,\ldots,B_k$ be the components of B, where $2 \leq k \leq n$. By [12, 5.5], for each $j \in \{1,\ldots,k\}$, there exists a subcontinuum C_j of X containing B_j properly. We assume without loss of generality that $C_j \cap C_\ell = \emptyset$ if $j \neq \ell$.

For each $j \in \{1,\ldots,k\}$, let $\alpha_j\colon [0,1] \to \mathcal{C}(X)$ be an order arc [11, (1.8)] such that $\alpha_j(0) = B_j$ and $\alpha_j(1) = C_j$. Let $\alpha\colon [0,1]^k \to \mathcal{C}_n(X)$ be given by $\alpha((t_1,\ldots,t_k)) = \displaystyle\bigcup_{j=1}^{k} \alpha_j(t_j)$. Let $\mathcal{D} = \alpha\left([0,1]^k\right)$. Then $\mathcal{D}$ is a $k-$cell such that $B \in \mathcal{D}$ and $A \notin \mathcal{D}$. Thus, $h(\mathcal{D})$ is a $k-$cell containing the point (z,t_0) and not containing $v(Z)$. Hence, $\pi(h(\mathcal{D}))$ is a locally connected subcontinuum of Z containing z; since $k \geq 2$, $\pi(h(\mathcal{D}))$ is nondegenerate. Thus, z is contained in an arc by [12, 8.23].

Next, suppose that $h^{-1}((z,t)) \in \mathcal{C}(X)$ for each $t \in [0,1)$. Since $h(A) = v(Z)$, there exists $t' \in [0,1)$ be such that $h^{-1}((z,t')) \neq A$ and $h^{-1}((z,t')) \notin \mathcal{F}_1(X)$. Let $E = h^{-1}((z,t'))$. Since $E \neq A$ and E is nondegenerate, E is a decomposable continuum (by 3.3). Hence, there are two proper subcontinua K and H of E such that $E = H \cup K$.

Suppose, first, that A is not contained in E. Let $x_1 \in H \setminus K$ and $x_2 \in K \setminus H$. Let $\beta_j\colon [0,1] \to \mathcal{C}(X)$ be an order arc [11, (1.8)] such that $\beta_j(0) = \{x_j\}$, $j \in \{1,2\}$, $\beta_1(1) = H$ and $\beta_2(1) = K$. Let $\beta\colon [0,1]^2 \to \mathcal{C}_n(X)$ be given by $\beta((t_1,t_2)) = \beta_1(t_1) \cup \beta_2(t_2)$. Let $\mathcal{G} = \beta\left([0,1]^2\right)$. Then $\mathcal{G}$ is a locally connected subcontinuum of $\mathcal{C}_n(X)$ such that $\mathcal{G}$ contains a $2-$cell and such that $E \in \mathcal{G}$ and $A \notin \mathcal{G}$. Thus, $h(\mathcal{G})$ is a locally connected subcontinuum of Cone(Z) containing a $2-$cell, such that $(z,t') \in h(\mathcal{G})$ and $v(Z) \notin h(\mathcal{G})$. Hence, $\pi(h(\mathcal{G}))$ is a nondegenerate locally connected subcontinuum of Z containing z. Thus, z is in an arc by [12, 8.23].

Suppose next that A is contained in E. Since A is indecomposable, $E \neq A$; hence, there exists a point $x_1 \in E \setminus A$. Suppose that $x_1 \in H$. Choose a point $x_2 \in K \setminus \{x_1\}$. Then, we just repeat the argument in the preceding paragraph to construct a nondegenerate locally connected subcontinuum of Z containing z.

This completes the proof of the first part of (4). We prove the second part of (4) as follows:

By [9, 3.5], there exists a $2n-$cell $\mathcal{E}$ in $\mathcal{C}_n(X)$. We may choose $\mathcal{E}$ such that $A \notin \mathcal{E}$. Let $B \in \mathcal{E}$. Then, $h(\mathcal{E})$ is a $2n-$cell such that $h(B) \in \mathcal{E}$. Hence, $\pi(h(\mathcal{E}))$ is a locally connected subcontinuum of Z containing $\pi(h(B))$, and $\dim(\pi(h(\mathcal{E}))) \geq 2n-1$ (by Remark in [3, p. 34] since $\pi(h(\mathcal{E})) \times [0,1)$ contains $h(\mathcal{E})$ and, thus, has dimension at least $2n$).

(5) By (4), each point of Cone(Z) lies in the cone over an arc. Hence, (5) follows easily.

(6) This is a consequence of 3.3, (5) and [9, 6.4].

(7) Suppose there exist two points z_1 and z_2 of Z such that Z is irreducible between them.

First, we prove that both z_1 and z_2 belong to $\pi(h(\mathcal{F}_n(X)))$. Suppose this is not true. Note that $\mathcal{F}_n(X)$ intersects all the arc components of $\mathcal{C}_n(X) \setminus \{A\}$. Hence, there are arcs α_1 and α_2 in Z such that one end point of α_j is z_j and the other point of α_j is in $\pi(h(\mathcal{F}_n(X)))$, for $j \in \{1,2\}$. Hence, by irreducibility, $Z = \alpha_1 \cup \alpha_2 \cup \pi(h(\mathcal{F}_n(X)))$.

On the other hand, Z cannot contain free arcs (otherwise, $\mathcal{C}_n(X)$ contains $2-$cells with nonempty interior, which contradicts 3.5). Thus, we have proved that z_1 and z_2 belong to $\pi(h(\mathcal{F}_n(X)))$.

Let t_1 and t_2 be points of $[0,1)$ such that (z_1, t_1) and (z_2, t_2) belong to $h(\mathcal{F}_n(X))$. Let B_1 and B_2 be the elements of $\mathcal{F}_n(X)$ such that $h(B_1) = (z_1, t_1)$ and $h(B_2) = (z_2, t_2)$. Let $x_1 \in B_1$ and $x_2 \in B_2$. Let

$$\mathcal{B}_1 = \{\{x_1\} \cup B \mid B \in \mathcal{F}_{n-1}(X)\}$$

and

$$\mathcal{B}_2 = \{\{x_2\} \cup B \mid B \in \mathcal{F}_{n-1}(X)\}.$$

Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are subcontinua of $\mathcal{C}_n(X)$ containing B_1 and B_2, respectively; also, $\mathcal{B}_1 \cap \mathcal{F}_{n-1}(X) \neq \emptyset$ and $\mathcal{B}_2 \cap \mathcal{F}_{n-1}(X) \neq \emptyset$. Hence, $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}_{n-1}(X)$ is a subcontinuum of $\mathcal{C}_n(X)$ which does not intersect all the arc components of $\mathcal{C}_n(X) \setminus \{A\}$, which we prove as follows: By 3.4, $\mathcal{C}_n(X) \setminus \{A\}$ has uncountably many arc components. Let $a_1, \ldots, a_n$ be n points of A in n distinct composants, $\kappa_1, \ldots, \kappa_n$, of A such that $\{x_1, x_2\} \cap \bigcup_{j=1}^{n} \kappa_j = \emptyset$; if $A \neq X$, by 3.4, we may take n composants that are not accessible from $X \setminus A$. Let $\mathcal{G}$ be the arc component of $\mathcal{C}_n(X) \setminus \{A\}$ containing $\{a_1, \ldots, a_n\}$. Then, $\mathcal{G} \cap (\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}_{n-1}(X)) = \emptyset$.

Since $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}_{n-1}(X)$ does not intersect all the arc components of $\mathcal{C}_n(X) \setminus \{A\}$, we have that $h(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}_{n-1}(X))$ is a subcontinuum of $\mathrm{Cone}(Z)$ containing both (z_1, t_1) and (z_2, t_2), which does not intersect all the arc components of $\mathrm{Cone}(Z) \setminus \{v(Z)\}$. Then, $\pi(h(\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}_{n-1}(X)))$ is a proper subcontinuum of Z containing z_1 and z_2, a contradiction. Therefore, Z is not irreducible. □

Question 3.7. Does there exist an indecomposable continuum X such that $\mathcal{C}_n(X)$ is homeomorphic to the cone over a finite$-$dimensional continuum for some integer $n > 1$?

Question 3.8. Does there exist a hereditarily decomposable continuum X that is not an arc such that $\mathcal{C}_n(X)$ is homeomorphic to the cone over a finite$-$dimensional continuum for some integer $n > 1$?

4. Products

Recently, R. Schori proved that $\mathcal{C}_2([0,1])$ is homeomorphic to $[0,1]^4$ (see [5, Lemma 1] for a proof). It is natural to ask if there are other continua X for which their n–fold hyperspaces are homeomorphic to a product of nondegenerate continua. The question seems very difficult. We obtain partial answers.

We show that if the n–fold hyperspace of a continuum X is homeomorphic to a product of two continua, then X must be hereditarily decomposable without nondegenerate proper terminal subcontinua. We also show that if S^1 is the unit circle, then $\mathcal{C}_2(S^1)$ is not homeomorphic to a product when one of the factors is one–dimensional (4.9). Along the way we obtain results of independent interest (e.g., 4.6).

Theorem 4.1. *Let X be a finite–dimensional continuum and let n be an integer greater than one. If $\mathcal{C}_n(X)$ is homeomorphic to the product of two nondegenerate continua, then X is hereditarily decomposable and X has no nondegenerate proper terminal continua.*

Proof: Suppose that $\mathcal{C}_n(X)$ is homeomorphic to $Y \times Z$, where Y and Z are continua. Since $\mathcal{C}_n(X)$ is arcwise connected [9, 3.1], we have that both Y and Z are arcwise connected continua.

Suppose X contains an indecomposable subcontinuum A. Note that $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected [9, 6.3, 6.4] and [11, (v), p. 312]. On the other hand, it is easy to show that no point arcwise disconnects the product of two arcwise connected continua, a contradiction.

The fact that X does not contain terminal subcontinua follows from [9, 6.4] and the fact that no point arcwise disconnects the products of two arcwise connected continua. □

Our next results are concerned with the case when $X = S^1$.

For a continuum X and an integer $n \geq 2$, we let

$$ND_n(X) = \{A \in \mathcal{C}_n(X) \mid A \text{ has } n \text{ nondegenerate components}\}.$$

For a subset Y of X, $ND_n(Y)$ is the set of elements of $ND_n(X)$ contained in Y.

Lemma 4.2. *If $A \in ND_n([0,1])$ and $\{0,1\} \cap A = \emptyset$, then A has an open $2n-$cell neighborhood in $\mathcal{C}_n([0,1])$. Hence, $ND_n((0,1))$ is a $2n-$dimensional manifold.*

Proof: We only prove the case when $n = 2$; the general case is similar.

Let $A = [a,b] \cup [c,d]$ be an element of $ND_2([0,1])$ such that $\{0,1\} \cap A = \emptyset$. Thus, $0 < a < b < c < d < 1$. Let $\alpha\colon [0,1]^4 \to \mathcal{C}_2([0,1])$ be given by

$$\begin{aligned}\alpha\left((t_1,t_2,t_3,t_4)\right) = \quad & \left[(1-t_1)\tfrac{2a+b}{3} + t_1\tfrac{2a}{3}, (1-t_2)\tfrac{a+2b}{3} + t_2\tfrac{2b+c}{3}\right] \cup \\ & \left[(1-t_3)\tfrac{b+2c}{3} + t_3\tfrac{2c+d}{3}, (1-t_4)\tfrac{c+2d}{3} + t_4\tfrac{2d+1}{3}\right].\end{aligned}$$

Then, α is an embedding of $[0,1]^4$ into $\mathcal{C}_2([0,1])$ and

$$\alpha\left(\left(\frac{b-a}{b}, \frac{b-a}{c-a}, \frac{c-b}{d-b}, \frac{d-c}{1-c}\right)\right) = A$$

Let us observe that if $\varepsilon < \frac{1}{6}\min\{a, b-a, c-b, d-c, 1-d\}$, then $\mathcal{V}^{\mathcal{H}}_{\varepsilon}(A) \subset \alpha([0,1]^4)$. Therefore, $\alpha([0,1]^4)$ is a neighborhood of A in $\mathcal{C}_2([0,1])$. □

Corollary 4.3. *If $A \in ND_n(S^1)$, then A has an open $2n-$cell neighborhood in $\mathcal{C}_n(S^1)$. Hence, $ND_n((S^1)$ is a $2n-$dimensional manifold.*

Proof: This follows easily from 4.2, since every element A of $ND_n(S^1)$ lies in the interior of an arc in S^1. □

We show in 4.6 that $\mathcal{C}_n(S^1)$ is a $2n-$dimensional Cantor manifold.

Lemma 4.4. *$ND_n(S^1)$ is arcwise connected and each element of $\mathcal{C}_2(S^1) \setminus ND_n(S^1)$ is arcwise accessible from $ND_n(S^1)$.*

Proof: We only prove the case $n = 2$; the general case is similar.

For a real number $\ell \in (0,\pi)$, let $\mathcal{A}_\ell = \{A \in ND_2(S^1) \mid$ both components of A have length ℓ and the midpoints of the components of A are antipodal points of $S^1\}$. Hence, each element A of $\mathcal{A}_\ell$ is completely determined by the midpoints of its components. Since such points are antipodal, we have that $\mathcal{A}_\ell$ is a simple closed curve.

Now, let A be any element of $ND_2(S^1)$. If both components of A have the same length, ℓ, then by rotating one of the components, there is an arc in $ND_2(S^1)$ joining A with an element of $\mathcal{A}_\ell$. Suppose, then, that the components of A have different lengths.

Rotate the component A_1, of A with smallest length, say ℓ, until its midpoint is antipodal to the midpoint of the other component A_2 of A. Then, using the map K_ρ in [11, (0.65.1)], we can find an arc $\alpha\colon [0,1] \to \mathcal{C}(S^1)$ such that $\alpha(0) = A_2$; $\alpha(1)$ is a subcontinuum C of S^1, contained in A_2, having the same midpoint as A_2 and having the same length as A_1. Hence, with the rotation and the arc α, we see that there is an arc from A to an element of $\mathcal{A}_\ell$. Therefore, $ND_2(S^1)$ is arcwise connected.

The fact that each point of $\mathcal{C}_2(S^1)\setminus ND_2(S^1)$ is arcwise accessible from $ND_2(S^1)$ follows from the fact that each subcontinuum of S^1 can be approximate from "both sides" by shrinking or stretching and then rotating disjoint arcs. □

Corollary 4.5. *No subset of* $\mathcal{C}_n(S^1) \setminus ND_n(S^1)$ *arcwise separates* $\mathcal{C}_n(S^1)$.

Proof: Let $\mathcal{K}$ be a subset of $\mathcal{C}_n(S^1) \setminus ND_n(S^1)$, and let A and B be two elements of $\mathcal{C}_n(S^1) \setminus \mathcal{K}$. By 4.4, there exist two arcs, β_A and β_B, in $\mathcal{C}_n(S^1)$ having A and B as one end point, respectively, and such that $\beta_A \setminus \{A\} \subset ND_n(S^1)$ and $\beta_B \setminus \{B\} \subset ND_n(S^1)$. Since $ND_n(S^1)$ is arcwise connected (by 4.4), there exists an arc γ in $ND_n(S^1)$ intersecting both β_A and β_B. Then $\beta_A \cup \beta_B \cup \gamma$ is a locally connected subcontinuum of $\mathcal{C}_n(S^1) \setminus \mathcal{K}$ containing A and B. Therefore, $\mathcal{C}_n(S^1) \setminus \mathcal{K}$ is arcwise connected. □

Theorem 4.6. $\mathcal{C}_n(S^1)$ *and* $\mathcal{C}_n([0,1])$ *are* $2n$ – *dimensional Cantor manifolds.*

Proof: First observe that $ND_n(S^1)$ is a connected $2n$ – dimensional manifold that is dense in $\mathcal{C}_n(S^1)$.

Let $\mathcal{A}$ be a subset of $\mathcal{C}_n(S^1)$ such that $\mathcal{A}$ separates $\mathcal{C}_n(S^1)$. We assume without loss of generality that $\mathcal{A}$ is closed in $\mathcal{C}_n(S^1)$ [14, (1.4), p. 43]. Let $\mathcal{B} = ND_n(S^1) \cap \mathcal{A}$. Then, since $ND_n(S^1)$ is clearly a dense subset of $ND_n(S^1)$ which is connected by 4.4, $\mathcal{B}$ separates $ND_n(S^1)$, say

$$ND_n(S^1) \setminus \mathcal{B} = \mathcal{E}|\mathcal{F}.$$

Now, suppose that $\dim(\mathcal{A}) \leq 2n-2$. Then, since $ND_n(S^1)$ is a $2n-$manifold, by 4.3, and $\mathcal{B}$ is closed in $ND_n(S^1)$, $\mathcal{B}$ is nowhere dense in $ND_n(S^1)$ by [3, Corollary 1, p. 46]. Hence,

$$\mathrm{Cl}_{ND_n(S^1)}(\mathcal{E}) \cup \mathrm{Cl}_{ND_n(S^1)}(\mathcal{F}) = ND_n(S^1).$$

Therefore, since $ND_n(S^1)$ is connected by 4.4, there is a point $B \in \mathrm{Cl}_{ND_n(S^1)}(\mathcal{E}) \cap \mathrm{Cl}_{ND_n(S^1)}(\mathcal{F})$. Now, let $\mathcal{Q}$ be a neighborhood of B in $ND_n(S^1)$ such that $\mathcal{Q}$ is homeomorphic to $[0,1]^{2n}$. Then $\mathcal{Q} \cap \mathcal{E} \neq \emptyset$ and $\mathcal{Q} \cap \mathcal{F} \neq \emptyset$ since $B \in \mathcal{Q} \cap \mathrm{Cl}_{ND_n(S^1)}(\mathcal{E})$ and $B \in \mathcal{Q} \cap \mathrm{Cl}_{ND_n(S^1)}(\mathcal{F})$. Thus, $\mathcal{Q} \setminus \mathcal{B}$ is not connected (since $\mathcal{B}$ separates $ND_n(S^1)$); however, this contradicts the fact that $\mathcal{Q}$ is a $2n-$dimensional Cantor manifold [3, VI 11, p. 93]. This proves the theorem for $\mathcal{C}_n(S^1)$.

The proof of $\mathcal{C}_n([0,1])$ is similar using $ND_n((0,1))$ in place of $ND_n(S^1)$. □

In connection with 4.6, we note that the $n-$fold hyperspaces of simple continua may not be Cantor manifolds for any n. For example, let X be a simple triod. Then $\dim(\mathcal{C}_n(X)) = 2n+1$ but $\dim_A(\mathcal{C}_n(X)) = 2n$, where A is singleton other than the ramification point. Thus, $\mathcal{C}_n(X)$ is not a Cantor manifold since the dimension of a Cantor manifold at each of its points is the same [3, A), pp. 93 – 94].

Now, we focus specifically on products.

Lemma 4.7. *Assume that $\mathcal{C}_2(S^1)$ is homeomorphic to $Y \times Z$, where $\dim(Z) \leq 2$. Then Y has no separating point.*

Proof: Let $h : \mathcal{C}_2(S^1) \to Y \times Z$ be a homeomorphism.

Suppose by way of contradiction that Y has a separating point p, say

$$Y \setminus \{p\} = E|F.$$

Then $\{p\} \times Z$ is a compact separator of $Y \times Z$. Hence, by 4.5,

$$(*) \quad h^{-1}(\{p\} \times Z) \cap ND_2(S^1) \neq \emptyset.$$

Next, we prove that no point of $\{p\} \times Z$ has a $4-$cell neighborhood in $Y \times Z$: Suppose, to the contrary, C is a $4-$cell neighborhood in $Y \times Z$ of a point (p,z); then C must intersect both $E \times Z$ and $F \times Z$; hence, $C \setminus (\{p\} \times Z)$ is not connected, a contradiction to the fact that C is a $4-$dimensional Cantor manifold [3, VI 11, p. 93].

Thus, since no point of $\{p\} \times Z$ has a $4-$cell neighborhood, we have by 4.3,

$$h^{-1}(\{p\} \times Z) \cap ND_2(S^1) = \emptyset.$$

Therefore, we have a contradiction to (*). □

Lemma 4.8. *Let Y be a contractible continuum, D a dendrite, and p_0 be an end point of D. If y_0 is any point of Y, then $(Y \times D) \setminus \{(y_0, p_0)\}$ is contractible.*

Proof: Let e be an end point of D distinct from p_0. As can be seen from the proof of [2, (2.16)], there exists a map $\psi \colon D \times I \to D$ such that for every point p of D, $\psi((p, 0)) = e$ and $\psi((p, 1)) = p$. Note also that, if $p \neq e$ and $t \in (0, 1)$, then $\psi((p, t)) \neq p$.

Since Y is contractible, there exists a map $\xi \colon Y \times I \to Y$ such that for each point y of Y, $\xi((y, 0)) = y$ and $\xi((y, 1)) = y_0$.

Let $\chi \colon (Y \times D) \times I \to (Y \times D)$ be given by

$$\chi((y,p),t) = \begin{cases} (y, \psi((p, 1-2t))) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (\xi((y, 2t-1)), e) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $(y, \psi((p, 1-2\frac{1}{2}))) = (y, e)$ and $(\xi((y, 2\frac{1}{2}-1)), e) = (y, e)$, we have that χ is well defined and continuous, $\chi((y,p),0) = (y,p)$ and $\chi((y,p),1) = (y_0, e)$, for each $(y, p) \in Y \times D$.

Note that, since p_0 is an end point of D, we have that if $t > 0$, then $\chi((Y \times \{p_0\} \times \{t\})) \cap (Y \times \{p_0\} \times \{0\}) = \emptyset$. Hence, if $\zeta = \chi|_{[(Y \times D)\setminus\{(y_0,p_0)\}]}$, then $\zeta \colon [(Y \times D) \setminus \{(y_0, p_0)\}] \times I \to [(Y \times D) \setminus \{(y_0, p_0)\}]$ is a map such that $\zeta((y,p),0) = (y,p)$ and $\zeta((y,p),1) = (y_0, e)$. Therefore, $[(Y \times D) \setminus \{(y_0, p_0)\}]$ is contractible. □

The last part of the proof of the following result is the same as the proof of Lemma 3.1 of [4]; we include it for completeness.

Theorem 4.9. *$\mathcal{C}_2(S^1)$ is not homeomorphic to $Y \times D$ for any one$-$dimensional continuum D and any continuum Y.*

Proof: Suppose first that $\mathcal{C}_2(S^1)$ is homeomorphic to $Y \times D$ for some one$-$dimensional continuum D and some continuum Y. Since $\mathcal{C}_2(S^1)$ is an absolute retract [15, Théorème II_m], we have that $Y \times D$ is an absolute retract. Since Y and D are $r-$images of $Y \times D$, Y and D are absolute retracts [1, (2.1), p. 101]. Hence, D is a dendrite [1, (13.5), p. 138].

Let $h\colon \mathcal{C}_2(S^1) \to Y \times D$ be a homeomorphism and assume that $h(S^1) = (y_0, p_0)$.

If p_0 is an end point of D, then $(Y \times D) \setminus \{(y_0, p_0)\}$ is an absolute neighborhood retract [1, (2.10), p. 102]. Hence, by 4.8, $(Y \times D) \setminus \{(y_0, p_0)\}$ is an absolute retract [1, (9.1), p. 96]. Thus, it is unicoherent [1, (2.4), p. 86].

Assume that p_0 is a point of order 2 of D. Since p_0 is a point of order 2, $D \setminus \{p_0\}$ has exactly two components A and B [12, 10.13], $\mathrm{Cl}_D(A) = A \cup \{p_0\}$ and $\mathrm{Cl}_D(B) = B \cup \{p_0\}$.

Let $\Gamma_A = (Y \times \mathrm{Cl}_D(A)) \setminus \{(y_0, p_0)\}$ and $\Gamma_B = (Y \times \mathrm{Cl}_D(B)) \setminus \{(y_0, p_0)\}$.

Note that Γ_A and Γ_B are contractible (by 4.8); also, they are absolute neighborhood retracts, hence absolute retracts [1, (9.1), p. 96]. Furthermore, $\Gamma_A \cap \Gamma_B = (Y \times \{p_0\}) \setminus \{(y_0, p_0)\}$. Since $Y \times \{p_0\}$ does not contain separating points (by 4.7), $\Gamma_A \cap \Gamma_B$ is connected. Therefore, by [14, (5.2), p. 221], we have that $\Gamma_A \cup \Gamma_B = (Y \times D) \setminus \{(y_0, p_0)\}$ is unicoherent.

If p_0 is a point of finite order, then using induction and [14, (5.2), p. 221], it can be shown that $(Y \times D) \setminus \{(y_0, p_0)\}$ is unicoherent.

Suppose p_0 is a point of infinite order, then $D \setminus \{p_0\}$ has countably many components [12, 10.20.1]. Let $\{K_m\}_{m=1}^{\infty}$ be the components of $D \setminus \{p_0\}$. Note that for each $m \in \mathbb{N}$, $\mathrm{Cl}_D(K_m) = K_m \cup \{p_0\}$.

For each $m \in \mathbb{N}$, let $\Gamma_m = (Y \times \mathrm{Cl}_D(K_m) \setminus \{(y_0, p_0)\}$. Then, as before, Γ_m is an absolute retract. Note that if $m \neq \ell$ then $\Gamma_m \cap \Gamma_\ell = (Y \times \{p_0\}) \setminus \{(y_0, p_0)\}$, which is connected.

For each $m \in \mathbb{N}$, let $X_m = \cup_{j=1}^{m} \Gamma_j$. Thus, by induction and [14, (5.2), p. 221], we have that each X_m is a unicoherent locally connected space. Note that for each $m \in \mathbb{N}$, $X_m \subset X_{m+1}$ and $(Y \times D) \setminus \{(y_0, p_0)\} = \cup_{m=1}^{\infty} X_m$.

Let $e\colon \mathbb{R} \to S^1$ be the exponential map. To show that $(Y \times D) \setminus \{(y_0, p_0)\}$ is unicoherent, it is enough to show that if $f\colon (Y \times D) \setminus \{(y_0, p_0)\} \to S^1$ is any map, then there exists a map $k\colon (Y \times D) \setminus \{(y_0, p_0)\} \to \mathbb{R}$ such that $e \circ k = f$ [14, (7.3), p. 227].

Since for each $m \in \mathbb{N}$, X_m is a locally connected unicoherent space, by [14, (7.3), p. 227] and the Unique Lifting Theorem [13, Theorem 2, p. 67], we can choose maps $k_1, k_2, \dots$ such that $k_m\colon X_m \to \mathbb{R}$ $k_{m+1}|_{X_m} = k_m$ and $e \circ k_m = f|_{X_m}$ for each $m \in \mathbb{N}$. Define $k\colon (Y \times D) \setminus \{(y_0, p_0)\} \to \mathbb{R}$ by $k((y,p)) = k_m((y,p))$ if $(y,p) \in X_m$. Then $e \circ k = f$.

Now, we prove that k is continuous at a point (y, p) of $(Y \times D) \setminus \{(y_0, p_0)\}$. Observe that if $p \neq p_0$, then there exists $m \in \mathbb{N}$ such that $(y, p) \in X_m$ and the continuity of k at that point follows from the continuity of k_m at that point. So, let us assume that $p = p_0$.

To prove that k is continuous at (y, p_0), let W be an open interval in $\mathbb{R}$ such that $k((y, p_0)) \in W$ and $\mathrm{diam}(W) < 2\pi$. Let $V = e(W)$. Then $f((y, p_0)) \in V$. Let $g \colon V \to W$ be the inverse map of $e|_W$. Since Y and D are locally connected, there exist connected open sets U_1 of Y and U_2 of D such that $U_1 \times U_2 \subset [(Y \times D) \setminus \{(y_0, p_0)\}]$ and $(y, p_0) \in U_1 \times U_2 \subset f^{-1}(V)$. It can be seen that $(U_1 \times U_2) \cap X_m$ is an open connected subset of X_m and $(y, p_0) \in (U_1 \times U_2) \cap X_m$ for each $m \in \mathbb{N}$. Note that $f(U_1 \times U_2) \subset V$. Then the Unique Lifting Theorem [13, Theorem 2, p. 67] implies that $k_m|_{[(U_1 \times U_2) \cap X_m]} = g \circ f|_{[(U_1 \times U_2) \cap X_m]}$ for each $m \in \mathbb{N}$. Hence, $k(U_1 \times U_2) \subset W$. Thus, k is continuous at (y, p_0). Therefore, $(Y \times D) \setminus \{(y_0, p_0)\}$ is unicoherent [14, (7.3), p. 227].

On the other hand, as can be seen from the proof of [5, Lemma 2], $\mathcal{C}_2(S^1) \setminus \{S^1\}$ is not unicoherent. Therefore, $\mathcal{C}_2(S^1)$ cannot be homeomorphic to $Y \times D$. This proves our theorem. □

Corollary 4.10. *$\mathcal{C}_2(S^1)$ is not homeomorphic to $Z \times [0, 1]^k$ for any continuum Z and any positive integer k.*

Proof: Apply 4.9 with $Y = Z \times [0, 1]^{k-1}$. □

Question 4.11. Is $\mathcal{C}_2(S^1)$ homeomorphic to the product of two continua having dimension of at least two?

An affirmative answer to 4.11 would show that $\mathcal{C}_2(S^1)$ is not a cartesian product of any two nondegenerate continua (by 4.9).

Question 4.12. Does there exist a continuum X, that is not an arc, for which there is an integer $n > 1$ such that $\mathcal{C}_n(X)$ is homeomorphic to the product of two finite-dimensional continua?

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