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WHEN ARE COMPACT, COUNTABLY COMPACT AND LINDELÖF CONVERGENCES TOPOLOGICAL?

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Abstract. Conditions on compact convergences to be topological are (and have been) investigated. New variants are obtained for both filter and cover compact-like properties such as countable compactness and Lindelöfness. The recently developed coreflectively modified continuous duality plays a crucial role in this quest. In particular, the results on compactness evoked above can be obtained either as byproducts of theorems on commutation of reflectors with product or as corollaries of internal characterizations of bireflective subcategories of convergences determined by coreflectively modified biduals. Both approaches lead to results of interest for their own.

1. Introduction

It is well known that compact topologies are minimal among Hausdorff topologies. This fact extends to pseudotopologies. Consequently, every compact topologically Hausdorff pseudotopology is a topology. This paper studies properties which imply the topologicity of compact, countably compact and Lindelöf convergences. In contrast with the well studied case of compact convergences, the results concerning countably compact and Lindelöf convergences seem to be entirely new. Moreover, the techniques used allow a unified treatment of these three cases.

One approach consists in applying the coreflectively modified duality recently developed by the author [17], [18], thanks to a new

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result on the behaviour of the reflector on regular topologies under product (obtained in section 3). Section 2 discusses existing results for compact convergences and shows how to derive new statements on compact, countably compact and Lindelöf convergences from the new result of section 3. It is interesting to note that these new statements involve bidual conditions whose internal characterizations are not needed. However, section 4 is devoted to such internal characterizations. In particular, the internal description of c-embedded spaces (in the sense of Binz [3]) by Bourdaud [5] is generalized. This leads to alternative proofs of the results of section 2.

2. Compact, countably compact and Lindelöf convergences

A *convergence* ξ is a relation

$$x \in \lim_\xi \mathcal{F},$$

between points and filters in which $\lim_\xi \mathcal{F} \supset \lim_\xi \mathcal{G}$ whenever $\mathcal{F} \geq \mathcal{G}$, every fixed ultrafilter converges to its defining point and $\lim_\xi \mathcal{F} \cap \lim_\xi \mathcal{G} = \lim_\xi(\mathcal{F} \wedge \mathcal{G})$ for every $\mathcal{F}$ and $\mathcal{G}$. The underlying set of a convergence ξ is denoted by $|\xi|$. If ξ and θ have the same underlying set, $\xi \geq \theta$ means that $\theta \supset \xi$. From now on, I will use ξ to denote the set $|\xi|$ endowed with the convergence structure ξ. In other words, ξ denotes an object of the category **Conv** of convergence spaces(1). Every topology can be considered as a convergence and there exist important naturally defined convergences that are non-topological. On the other hand, each convergence determines a topology. Indeed, a subset A of $|\xi|$ is ξ-*closed* provided that $\lim_\xi \mathcal{F} \subset A$ for every $\mathcal{F}$ such that $A \in \mathcal{F}$. The family of ξ-closed sets gives rise to a topology $T\xi$ called *topological modification of* ξ (2). The *adherence of a filter* is the union of limits of finer filters. As for a topology, a convergence is *compact* if and only if each filter has non-empty adherence. A convergence is *Hausdorff* if a filter converges to at

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[1] A map $f : \xi \to \tau$ is *continuous* provided that $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$ for every filter $\mathcal{F}$. The category with convergence spaces as objects and continuous maps as morphisms is denoted by **Conv**. This is a topological category [1], i.e., initial and final structures always exist. Hence, subspace, product, sum are defined in the usual categorical way.

[2] This is the finest topology among those which are coarser than ξ.

most one point and *T-Hausdorff* if its topological modification is Hausdorff.

A convergence is a *pseudotopology* if $x \in \lim_\xi \mathcal{F}$ provided that each ultrafilter of $\mathcal{F}$ converges to x in ξ. Pseudotopologies form a bireflective subcategory of **Conv**. I refer to [1] for undefined categorical notions and to [12], [18] for undefined notions and further details of convergence theory. It is well known that a Hausdorff compact topology is minimal among the Hausdorff topologies on the underlying set. This fact extends to pseudotopologies, e.g., [2, Theorem 1.4.10].

Theorem 2.1. *If ξ and θ are pseudotopologies such that ξ is compact, θ is Hausdorff and $\xi \geq \theta$, then $\xi = \theta$.*

Proof: Consider an ultrafilter $\mathcal{U}$ on a compact convergence ξ. By compactness $\lim_\xi \mathcal{U} \neq \varnothing$. But its superset $\lim_\theta \mathcal{U}$ is a singleton, because θ is Hausdorff. Thus, $\lim_\xi \mathcal{U} = \lim_\theta \mathcal{U}$. Hence, the pseudotopologies θ and ξ coincide on ultrafilters so that $\xi = \theta$. □

Corollary 2.2. *Each compact T-Hausdorff pseudotopology is a topology.*

Hence, the problem of minimality of compact convergences among general convergences is closely related to that of topologicity of compact convergences. The example below confirms this relation and shows that a compact convergence need not be topological in general.

Example 2.3. [2, Example 1.4.5] *Define the finest convergence on* $[0, 1]$ *that has precisely the same convergent ultrafilter as the natural topology of* $[0, 1]$ (3). *This is a compact Hausdorff convergence, but the usual topology of* $[0, 1]$ *is strictly coarser.*

Roughly speaking, I consider in this paper the problem of determining sufficient conditions for a compact convergence space to be topological. Several authors investigated this question, e.g., Cook [7], Kent and Richardson [15], Binz [4], Schroder [22], Cochran and Trail [6], Beattie and Butzmann [2]. The aim of this paper is to show how coreflectively modified continuous duality developed in [12], [18] and generalized in [20] applies to related problems,

[3] The convergent filters are the ultrafilters that converge in the usual topology of $[0, 1]$ and the finite infima of such ultrafilters converging to the same point.

namely that of topologicity of compact, countably compact and Lindelöf convergences. Recall that a topology is countably compact (respectively Lindelöf) if each countably based (respectively countably deep [4]) filter has non-empty adherence. These definitions are meaningful for arbitrary convergences. More generally, given a class $\mathfrak{J}$ of filters (e.g., of all, of countably based, of countably deep filters) a convergence is called $\mathfrak{J}$-*compact* if each filter of $\mathfrak{J}$ has non-empty adherence.

Several theorems appeared in the late sixties and in the seventies to ensure that compact pseudotopologies are topologies under the provision of certain separation axioms. An example communicated to me by Dolecki and Greco is Corollary 2.2 above. Another example is the following Theorem 2.4, proved for pretopologies by Cochran and Trail in [6], by Kent and Richardson in [15] and by Cook in [7].

Recall that a convergence ξ is a pretopology if the infimum of a family of filters converging to the same point converges to that point. Pretopologies form a reflective subcategory of **Conv** and the associated reflector$P$ is given by:

$$\lim_{P\xi}\mathcal{F} = \bigcap_{A\#\mathcal{F}} \operatorname{adh}_\xi A.$$

All (co)reflectors considered in the sequel are endo(co)reflectors of **Conv**.

A convergence is *regular* provided $\lim \mathcal{F} \subset \lim \operatorname{adh}^\natural \mathcal{F}$ for every filter $\mathcal{F}$, where $\operatorname{adh}^\natural \mathcal{F}$ is generated by $\{\operatorname{adh} F : F \in \mathcal{F}\}$. Regular convergences form a reflective subcategory of **Conv** and the associated reflector is denoted R. Regular topologies also form a reflective subcategory of **Conv** and the associated reflector is $T \wedge R$. More generally, a convergence ξ is θ-*regular* provided $\lim_\xi \mathcal{F} \subset \lim_\xi(\operatorname{adh}^\natural_\theta \mathcal{F})$. In particular, if $\Omega\xi = \bigvee_{f\in C(\xi,\mathbb{R})} f^-\mathbb{R}$ [5] is the completely regular modification of ξ (with $\mathbb{R}$ the usual topology of the real line and $C(\xi,\mathbb{R})$ the set of real valued continuous functions on ξ), a $\Omega\xi$-regular convergence ξ is called Ω-regular.

[4] A filter $\mathcal{F}$ is *countably deep* if $\bigcap \mathcal{A} \in \mathcal{F}$ for every countable family $\mathcal{A} \subset \mathcal{F}$.

[5] The inverse relation of a map f will be denoted f^- and the *initial convergence* with respect to $f : X \to \tau$ is denoted $f^-\tau$.

Theorem 2.4. *A Hausdorff regular compact pseudotopology is a topology.*

See [2, Theorem 1.4.13] for an elementary proof of Theorem 2.4.

Both Corollary 2.2 and Theorem 2.4 can be seen as corollaries of [22, Theorem 2] of Schroder which seems to be the most general known result of this kind. He calls a convergence S_0 if $y \in \lim \mathcal{F}$ and $x \in \lim(y)$ implies $x \in \lim \mathcal{F}$. Analogously, a convergence is $R_{1.2}$ if the existence of a filter $\mathcal{F}$ such that $y \in \lim \mathcal{F}$ and $x \in \lim \operatorname{adh}^{\natural} \mathcal{F}$ implies $x \in \lim(y)$.

Theorem 2.5. [22, Theorem 2] *The pseudotopological modification of a compact S_0 and $R_{1.2}$ convergence is a completely regular (not necessarily Hausdorff) topology.*

On the other hand, some of such compactness theorems involve duality conditions. One of the principal reasons for working in the context of convergences rather than that of topologies is that the category **Conv** behaves much better than the category **Top** of topological spaces (with continuous maps as morphisms) with respect to standard categorical constructions. In particular, contrary to the situation in **Top**, there always exists the coarsest convergence $[\xi, \tau]$ on the set $C(\xi, \tau)$ of continuous maps from ξ to τ that makes the evaluation map jointly continuous (e.g., [3]). In other words, **Conv** is cartesian-closed [(6)].

Binz calls a Hausdorff convergence ξ *c-embedded* if

$$\xi = i^{-}([[\xi, \mathbb{R}], \mathbb{R}]), \tag{2.1}$$

where $i : \xi \to [[\xi, \mathbb{R}], \mathbb{R}]$ is defined by $i(x)(f) = f(x)$.

Theorem 2.6. [4] *A compact c-embedded convergence is a topology.*

In view of the characterization [16, Theorem 2.4] of c-embedded convergences as Ω-Hausdorff Ω-regular pseudotopologies, this result can be seen as one more involving separation axioms. Indeed, as already observed by Schroder [22], the condition of Hausdorffness can be dropped and this refined version of Theorem 2.6 can be deduced from Theorem 2.5. However, the approach in terms of continuous duality allows one to obtain new variants that handle simultaneously compactness, countable compactness and Lindelöfness.

[6]It is moreover a quasitopoi.

Given a convergence σ, and a bicoreflector E, I defined the functor Epi_E^σ [18], [20] by

$$\mathrm{Epi}_E^\sigma \xi = i^-([E[\xi,\sigma],\sigma]). \tag{2.2}$$

Analogously, if L is a reflector,

$$\mathrm{Epi}_E^L \xi = \bigvee_{\sigma = L\sigma} \mathrm{Epi}_E^\sigma \xi.$$

As E is a bicoreflector, Epi_E^σ (resp. Epi_E^L) is a (bi)reflector [18, Proposition 4.2]. In view of (2.1), the c-modifier (reflector on (not necessarily Hausdorff) c-embedded convergences) is a particular case. More generally, I denote

$$c_{\mathfrak{J}} = \mathrm{Epi}_{\mathrm{Base}_{\mathfrak{J}}}^{\mathbb{R}}, \tag{2.3}$$

where the coreflector $\mathrm{Base}_{\mathfrak{J}}$ is defined by

$$\lim\nolimits_{\mathrm{Base}_{\mathfrak{J}}\xi} \mathcal{F} = \bigcup_{\mathcal{J} \ni \mathcal{G} \leq \mathcal{F}} \lim\nolimits_\xi \mathcal{G}.$$

A simplified version of [18, Theorem 3.1] gives [7]

Theorem 2.7. *Let L be a bireflector and let E be a bicoreflector. For convergences $\xi \geq \theta$, the following are equivalent:*

(1) *For every $\tau = E\tau$*

$$\theta \times \tau \geq L(\xi \times \tau); \tag{2.4}$$

(2) $E[\xi,\sigma] \geq [\theta,\sigma]$ *for every convergence $\sigma = L\sigma$;*

(3) $\theta \geq \mathrm{Epi}_E^L \xi$.

I say that two families $\mathcal{A}$ and $\mathcal{B}$ *mesh*, in symbol $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \varnothing$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. A family $\mathcal{A}$ of subsets of $|\xi|$ is ξ-$\mathfrak{J}$-*compactoid* (in $\mathcal{B}$) [11] if $\mathrm{adh}_\xi \mathcal{H} \neq \varnothing$ ($\mathrm{adh}_\xi \mathcal{H}\#\mathcal{B}$) for every $\mathfrak{J}$-filter $\mathcal{H}\#\mathcal{A}$. A family is ξ-$\mathfrak{J}$-*compact* if it is ξ-$\mathfrak{J}$-compactoid in itself. A convergence θ is locally ξ-$\mathfrak{J}$-compactoid if every θ-convergent filter contains a ξ-$\mathfrak{J}$-compactoid set.

A class $\mathfrak{J}$ of filters is said to be *composable* if it contains principal filters and if $\mathcal{H}\mathcal{G}$, the filter generated by $\{HG : H \in \mathcal{H}, G \in \mathcal{G}\}$ [8], is a (possibly degenerate) $\mathfrak{J}$-filter on Y whenever $\mathcal{H}$ is a $\mathfrak{J}$-filter on $X \times Y$ and $\mathcal{G}$ a $\mathfrak{J}$-filter on X. For example, the classes of principal

[7] See [18], [17] and [20] for details on Epi_E and related notions.

[8] $HG = \{y : \exists_{x \in G} (x,y) \in H\}$

filters and of countably based filters are composable, while that of filters generated by sequences is not.

Corollary 3.2 of section 3 states that, given a composable class $\mathfrak{J}$ of filters,

$$\theta \times P\tau \geq \Omega(\xi \times \tau),$$

for every $\tau = \mathrm{Base}_{\mathfrak{J}}\tau$ provided $\theta \geq \Omega\xi$ is locally ξ-$\mathfrak{J}$-compactoid. If ξ is a $\mathfrak{J}$-compact convergence, every filter on $|\xi|$ contains the ξ-$\mathfrak{J}$-compact set $|\xi|$, so that $\Omega\xi$ is locally ξ-$\mathfrak{J}$-compactoid. Consequently,

$$\Omega\xi \times \tau \geq \Omega\xi \times P\tau \geq \Omega(\xi \times \tau),$$

for every $\tau = \mathrm{Base}_{\mathfrak{J}}\tau$. In view of (2.3) and of Theorem 2.7, $\Omega\xi \geq c_{\mathfrak{J}}\xi$. Hence, we obtain the following result which generalizes and refines Theorem 2.6.

Theorem 2.8. *Let $\mathfrak{J}$ be a composable class of filters. A $\mathfrak{J}$-compact $c_{\mathfrak{J}}$-convergence is a (not necessarily Hausdorff) completely regular topology.*

An internal description of $c_{\mathfrak{J}}$-convergences is given in section 4. The interest of this bidual proof is that Theorem 3.1 applies in the same way with other reflectors $L \leq T \wedge R$ to the effect that

Theorem 2.9. *Let $\mathfrak{J}$ be a composable class of filters and let L be a bireflector coarser than $T \wedge R$. If a convergence is $\mathfrak{J}$-compact then its $\mathrm{Epi}^{L}_{\mathrm{Base}_{\mathfrak{J}}}$-reflection is a L-convergence (hence a regular topology).*

The following Figure 1 shows the relationships between reflective properties and filter-compactness properties. Let $\varphi_{\wedge\omega}$ denote the class of countably deep filters. Both Figure 1 and Figure 2 read as follows:

$L_1 \dashrightarrow L_2$ means $L_1 \geq L_2$ and $L_2 \underset{\mathrm{Hyp}}{\longrightarrow} L_1$ means that $L_1\xi = L_2\xi$ if ξ fulfills Hyp.

A family $\mathcal{S}$ is a *cover* for a convergence ξ if $\mathcal{F} \cap \mathcal{S} \neq \varnothing$ for every ξ-convergent filter $\mathcal{F}$. A convergence is *cover-compact* (resp. *cover-countably compact*) if each cover (resp. countable cover) has a finite subcover. Analogously, a convergence is *cover-Lindelöf* if each cover has a countable subcover. See [9] for general notions of cover-compactness. Each cover-(countably) compact convergence is (countably) compact, so that Theorem 2.8 applies to both cover-compactness and cover countable compactness. On the other hand, if a convergence ξ is cover-Lindelöf, its continuous dual $[\xi, \mathbb{R}]$ is first

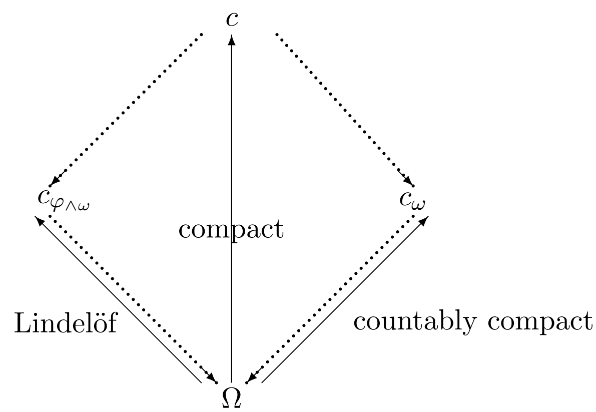

FIGURE 1. filter-compactness properties

countable by [13, Theorem 2]. Hence, $c\xi = c_\omega\xi$, by definition (2.3). Thus, we have the following variant of Theorem 2.8 for compact-like properties defined in terms of covers, which can be visualized on Figure 2, to be compared with Figure 1.

Theorem 2.10. *A cover-Lindelöf c-convergence is a c_ω-convergence.*

It is interesting to note that the general results contained in Figure 1 and Figure 2 are obtained without any information on the internal description of convergences fixed by the reflector $c_\mathfrak{J}$. Indeed, no such description was known till now apart for the c-embedded convergences. In section 4, I generalize the characterization of c-embedded convergence by Bourdaud [5] to coreflectively modified cases and I obtain explicit internal characterizations of reflectors $c_\mathfrak{J}$. This result leads to an alternative proof of Theorem 2.8. Other applications of this new characterization Theorem 4.4 and of Theorem 3.1 to preservation of topological properties and quotient properties by product (in the spirit of [12], [18]) will be investigated in a forthcoming paper [21].

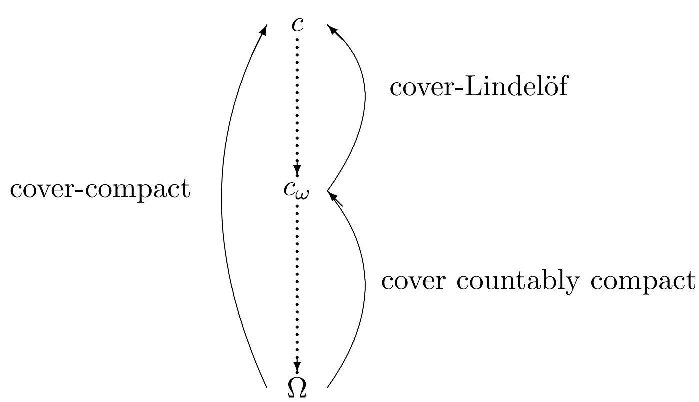

FIGURE 2. cover-compactness properties

3. A VARIANT OF THE THEOREM OF COMMUTATION OF THE TOPOLOGIZER WITH PRODUCT

The following observation holds for every reflector L.

$$x \in \lim_{L\xi} \mathcal{F} \Longrightarrow \underset{\tau}{\forall} \underset{y\in|\tau|}{\forall} (x,y) \in \lim_{L(\xi\times\tau)} \mathcal{F} \times (y). \tag{3.1}$$

The least closed set that contains A is the *closure* $\mathrm{cl}_\xi A$ of A. The topological modification can be described as follows [8]

$$\lim_{T\xi} \mathcal{F} = \bigcap_{A\#\mathcal{F}} \mathrm{cl}_\xi A. \tag{3.2}$$

Theorem 3.1. *Let $\mathfrak{J}$ be a composable class of filters and let L be a bireflector coarser than $T \wedge R$. If $\theta \geq L\xi$ is locally ξ-$\mathfrak{J}$-compactoid, then*

$$\theta \times P\tau \geq L(\xi \times \tau),$$

for every $\mathfrak{J}$-based convergence τ.

Proof: Since $L(\xi\times\tau)$ is a topology, it suffices to prove that $(x,y) \in H$, if $x \in \lim_\theta \mathcal{F}$, $y \in \lim_{P\tau} \mathcal{G}$ and if H is a $L(\xi \times \tau)$-closed set such that $H\#(\mathcal{F} \times \mathcal{G})$. Since θ is locally ξ-$\mathfrak{J}$-compactoid, there exists a ξ-$\mathfrak{J}$-compactoid set K in $\mathcal{F}$. For every element $F \subset K$ of $\mathcal{F}$, $HF\#\mathcal{G}$, so that $y \in \mathrm{adh}_\tau HF$. Thus, there exists a $\mathfrak{J}$-filter $\mathcal{L}_F$ such that $y \in \lim_\tau \mathcal{L}_F$ and $HF\#\mathcal{L}_F$. By composability, $H^-\mathcal{L}_F$ is a $\mathfrak{J}$-filter that meshes F. Thus, there exists $x_F \in \mathrm{adh}_\xi H^- \mathcal{L}_F$ and $(x_F, y) \in H$. On the other hand, $\theta \geq L\xi$ so that $x \in \lim_{L\xi} \mathcal{F}$. In view of (3.1), $(x,y) \in \lim_{L(\xi\times\tau)}(\mathcal{F} \times (y))$. Thus, there exists $F_V \in \mathcal{F}$ such that $F_V \times \{y\} \subset V$ for every $L(\xi\times\tau)$-neighborhood V of (x,y). Therefore, $(x_{F_V}, y) \in \mathrm{cl}_{\xi\times\tau}(F_V \times \{y\}) \subset V$. The filter $\mathcal{U}$ generated by $\{U_F = \{(x_{F'}, y) : F' \subset F\} : F \in \mathcal{F}\}$ is therefore $L(\xi\times\tau)$-convergent to (x,y). Since H is $L(\xi\times\tau)$-closed, $(x,y) \in H$. □

Corollary 3.2. *Let $\mathfrak{J}$ be a composable class of filters. If $\theta \geq \Omega\xi$ is locally ξ-$\mathfrak{J}$-compactoid, then*

$$\theta \times P\tau \geq \Omega(\xi \times \tau),$$

for every $\mathfrak{J}$-based convergence τ.

Other types of applications of results akin to Theorems 3.1 and Corollary 3.2 can be found in [12], [18] and [21].

4. Internal characterizations of $c_{\mathfrak{J}}$-convergences

Bourdaud characterized c-embedded spaces [5, Theorem 4.6] as pseudotopological Ω-regular convergences for which $\lim_\xi \mathcal{F}$ is $\Omega\xi$-closed for every $\mathcal{F}$ (9). The aim of this section is to generalize this result to obtain an explicit description of the $c_{\mathfrak{J}}$-reflection of a convergence for a large collection of classes $\mathfrak{J}$ of filters.

The following $\mathfrak{J}$-relativization of [5, Lemma 4.4] will be useful.

Lemma 4.1. *$x \in \lim_{c_{\mathfrak{J}}\xi} \mathcal{F}$ if and only if $x \in \lim_{\Omega\xi} \mathcal{F}$ and*

$$0 \in \lim_{\mathbb{R}} ev(\mathcal{F} \times \mathcal{G}),$$

for every $\mathfrak{J}$-filter $\mathcal{G}$ such that $0_{[\xi,\mathbb{R}]} \in \lim_{[\xi,\mathbb{R}]} \mathcal{G}$.

Proof: Assume that $x \in \lim_{\Omega\xi} \mathcal{F}$ and $0 \in \lim_{\mathbb{R}} ev(\mathcal{F} \times \mathcal{G})$ for every $\mathfrak{J}$-filter $\mathcal{G}$ such that $0_{[\xi,\mathbb{R}]} \in \lim_{[\xi,\mathbb{R}]} \mathcal{G}$. Consider a $\mathfrak{J}$-filter $\mathcal{H}$ such that $f \in \lim_{[\xi,\mathbb{R}]} \mathcal{H}$. It suffices to prove that $f(x) \in \lim_{\mathbb{R}} ev(\mathcal{F} \times \mathcal{H})$. Since $[\xi, \mathbb{R}]$ inherits the convergence group structure from $\mathbb{R}$, $0 \in \lim_{[\xi,\mathbb{R}]}(\mathcal{H} - f)$. Thus, $0 \in \lim_{\mathbb{R}} ev(\mathcal{F} \times (\mathcal{H} - f))$. Moreover, $f(x) \in \lim_\xi f(\mathcal{F})$ because $x \in \lim_{\Omega\xi} \mathcal{F}$. Notice that $ev(\mathcal{F} \times (\mathcal{H} - f)) = ev(\mathcal{F} \times \mathcal{H}) - f(\mathcal{F})$, so that $\lim_{\mathbb{R}} ev(\mathcal{F} \times \mathcal{H}) = \lim_{\mathbb{R}}(ev(\mathcal{F} \times (\mathcal{H} - f)) + f(\mathcal{F})) = 0 + f(x) = f(x)$. The converse is obvious. □

[9]Bourdaud called "Ω-closed domained" the convergences that enjoy this last property.

Recall that the *polar* F° of $F \subset |\xi|$ is the set of functions f in $C(\xi, \mathbb{R})$ such that $f(F) \subset [-1, 1]$.

Lemma 4.2. [5, Lemma 4.3] *Let $\mathcal{F}$ be a filter on $|\xi|$ and let $\mathcal{G}$ be a filter on $C(\xi, \mathbb{R})$.*

$$0 \in \lim_{\mathbb{R}} ev(\mathcal{F} \times \mathcal{G}) \iff \forall r > 0,\ \exists F \in \mathcal{F}, rF^\circ \in \mathcal{G}.$$

The following observation is essentially due to Feldman [13, Proof of Theorem 2].

Lemma 4.3. *If $\operatorname{adh}_\xi \mathcal{H} = \varnothing$ then the filter $\mathcal{G}$ generated by $\{f \in C(\xi, \mathbb{R}) : f(H^c) = 0\}_{H \in \mathcal{H}}$ converges to the zero-function in $[\xi, \mathbb{R}]$. Moreover, $ev(\mathcal{F} \times \mathcal{G}) = 0_{\mathbb{R}}$ for every ξ-convergent filter $\mathcal{F}$.*

Proof: As $\operatorname{adh}_\xi \mathcal{H} = \varnothing$, every convergent filter $\mathcal{F}$ contains H^c for some $H \in \mathcal{H}$, and the result follows. □

Given $A \in |\xi|$, $\mathcal{O}_\xi(A)$ denotes the family of ξ-open sets that contain A. Analgously, if $\mathcal{H}$ is a filter then $\mathcal{O}_\xi(\mathcal{H})$ denotes the filter generated by the families $\mathcal{O}_\xi(H)$ for $H \in \mathcal{H}$. The subclass of $\mathfrak{J}$-filters $\mathcal{H}$ for which $\mathcal{H} = \mathcal{O}_\xi(\mathcal{H})$ is denoted $\mathcal{O}_\xi(\mathfrak{J})$.

If $\mathcal{G}$ is a filter on $C(\xi, \mathbb{R})$ and if $r > 0$, then $\downarrow_r \mathcal{G}$ denotes the filter generated by the sets

$$\downarrow_r G = \{x \in |\xi| : \exists g \in G, g(x) \in [-r, r]^c\},$$

where G ranges over $\mathcal{G}$. Notice that $\downarrow_r G$ is $\Omega\xi$-open. Hence $\downarrow_r \mathcal{G} = \mathcal{O}_{\Omega\xi}(\downarrow_r \mathcal{G})$. A class $\mathfrak{J}$ of filters is said to be $\mathbb{R}$-*compatible* if it fulfills the two following properties:

(1) if $\mathcal{H}$ is a $\mathfrak{J}$-filter on $|\xi|$, then the filter generated by $\{f \in C(\xi, \mathbb{R}) : f(H^c) = 0\}_{H \in \mathcal{H}}$ is a $\mathfrak{J}$-filter on $C(\xi, \mathbb{R})$;
(2) if $\mathcal{G}$ is a $\mathfrak{J}$-filter on $C(\xi, \mathbb{R})$ and if $r > 0$ then $\downarrow_r \mathcal{G}$ is a $\mathfrak{J}$-filter on $|\xi|$.

It is easy to check that the classes of all, of countably based, of countably deep and of principal filters are $\mathbb{R}$-compatible.

Theorem 4.4. *Let $\mathfrak{J}$ be an $\mathbb{R}$-compatible class of filters. Then $x \in \lim_{c_\mathfrak{J}\xi} \mathcal{F}$ if and only if $x \in \lim_{\Omega\xi} \mathcal{F}$ and $\mathcal{F}$ is ξ-$\mathcal{O}_{\Omega\xi}(\mathfrak{J})$-compactoid.*

Proof: If $x \in \lim_{c_\mathfrak{J}\xi} \mathcal{F}$ then $x \in \lim_{\Omega\xi} \mathcal{F}$. On the other hand, consider a $\mathcal{O}_{\Omega\xi}(\mathfrak{J})$-filter $\mathcal{H}$ such that $\operatorname{adh}_\xi \mathcal{H} = \varnothing$. By $\mathbb{R}$-compatibility, the filter $\mathcal{G}$ generated by $\{f \in C(\xi, \mathbb{R}) : f(H^c) = 0\}_{H \in \mathcal{H}}$ is a $\mathfrak{J}$-filter on $C(\xi, \mathbb{R})$. Moreover, $0 \in \lim_{[\xi, \mathbb{R}]} \mathcal{G}$, by Lemma 4.3. In view of Lemma 4.1, $0 \in \lim_{\mathbb{R}} ev(\mathcal{F} \times \mathcal{G})$. By Lemma 4.2, there exists $F \in \mathcal{F}$ such that $F^{\circ} \in \mathcal{G}$. Thus, there exists $H = \operatorname{int}_{\Omega\xi} H \in \mathcal{H}$ such that $\{f : f(H^c) = 0\} \subset F^{\circ}$. Consequently, $F \subset H^c$. Otherwise, there exists $z \in F \setminus H^c$. Since $H^c = \operatorname{cl}_{\Omega\xi} H^c$, there exists $f \in C(\xi, \mathbb{R})$ such that $f(H^c) = 0$ and $f(z) = 2$, in contradiction with $\{f : f(H^c) = 0\} \subset F^{\circ}$. As $F \subset H^c$, the filters $\mathcal{F}$ and $\mathcal{H}$ do *not* mesh.

Conversely, assume that $x \in \lim_{\Omega\xi} \mathcal{F}$ and that $\mathcal{F}$ is ξ-$\mathcal{O}_{\Omega\xi}(\mathfrak{J})$-compactoid. By Lemma 4.1, it suffices to show that $0 \in \lim_{\mathbb{R}} ev(\mathcal{F} \times \mathcal{G})$ provided that $\mathcal{G}$ is a $\mathfrak{J}$-filter such that $0 \in \lim_{[\xi,\mathbb{R}]} \mathcal{G}$. Assume the contrary. There exists such a $\mathcal{G}$ and $r > 0$ such that $ev(F \times G)\#[-r,r]^c$ for every $F \in \mathcal{F}$ and every $G \in \mathcal{G}$. In other words, the filter $\downarrow_r \mathcal{G}$ meshes $\mathcal{F}$. By $\mathbb{R}$-compatibility, $\downarrow_r \mathcal{G}$ is a $\mathcal{O}_{\Omega\xi}(\mathfrak{J})$-filter. By ξ-$\mathcal{O}_{\Omega\xi}(\mathfrak{J})$-compactoidness of $\mathcal{F}$, there exists an ultrafilter $\mathcal{U}$ of $\downarrow_r \mathcal{G}$ and $z \in |\xi|$ such that $z \in \lim_{\xi} \mathcal{U}$. Consequently, $0 \in \lim_{\mathbb{R}} ev(\mathcal{U} \times \mathcal{G})$ because $0 \in \lim_{[\xi,\mathbb{R}]} \mathcal{G}$. There exist $G \in \mathcal{G}$ and $U \in \mathcal{U}$ such that $ev(U \times G) \subset [-r, r]$. Thus, $ev((U \bigcap \downarrow_r G) \times G) \subset [-r, r]$, a contradiction. □

Notice that Theorem 2.8 follows immediately from this description of $c_{\mathfrak{J}}$-convergences. Indeed, if ξ is $\mathfrak{J}$-compact, every filter is ξ-$\mathcal{O}_{\Omega\xi}(\mathfrak{J})$-compactoid so that $c_{\mathfrak{J}}\xi = \Omega\xi$.

Theorem 4.4 describes the $c_{\mathfrak{J}}$-reflection of an arbitrary convergence and not only convergences fixed by $c_{\mathfrak{J}}$. Hence, it is a stronger result than the characterization [5, Theorem 4.6] of c-embedded spaces by Bourdaud, even if $\mathfrak{J}$ is the class of all filters. Indeed, an internal characterization of convergences ξ for which $\xi = c_{\mathfrak{J}}\xi$ can easily be derived from Theorem 4.4.

Corollary 4.5. *Let $\mathfrak{J}$ be a $\mathbb{R}$-compatible class of filters. A convergence ξ is a $c_{\mathfrak{J}}$-convergence if and only if ξ is Ω-regular and $\lim_{\xi} \mathcal{F} = \lim_{\Omega\xi} \mathcal{F}$ for every $\mathfrak{J}$-compactoid filter $\mathcal{F}$.*

This is an immediate consequence of Theorem 4.4 and of the observation that $\operatorname{adh}_{\xi} \mathcal{H} = \operatorname{adh}_{\xi} \mathcal{O}_{\Omega\xi}(\mathcal{H})$ for every filter $\mathcal{H}$ provided that ξ is Ω-regular.

Notice that the condition "$\lim_{\xi} \mathcal{F} = \lim_{\Omega\xi} \mathcal{F}$ for every $\mathfrak{J}$-compactoid filter $\mathcal{F}$" implies that $\lim_{\xi} \mathcal{F}$ is $\Omega\xi$-closed for every $\mathcal{F}$ and that ξ is a pseudotopology. If $\mathfrak{J}$ is the class of all filters, the converse is easily obtained, so that Bourdaud's characterization [5, Theorem

4.6] is recovered. Moreover, the condition above is clearly sufficient for a $\mathfrak{J}$-compact convergence to be a completely regular topology.

Corollary 4.6. *A $\mathfrak{J}$-compact convergence ξ for which $\lim_\xi \mathcal{F} = \lim_{\Omega\xi} \mathcal{F}$ for every $\mathfrak{J}$-compactoid filter $\mathcal{F}$ (in particular if $\xi = c_{\mathfrak{J}}\xi$) is a completely regular topology.*

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