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THE HEREDITARILY METALINDELÖF PROPERTY OF k–SPACES WITH σ–HCP CLOSED k–NETWORKS

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ABSTRACT. In this paper, we prove that a regular k–space with a σ–HCP k–network is a hereditarily metalindelöf space. Thus, the question that appeared in [5] and [6] is answered.

1. Introduction

If X is a space, a family $\mathcal{P}$ of closed subsets of X is a *k–network* for X, if for every compact subset $K \subset X$ and an open neighborhood U of K, there is a finite $\mathcal{P}^* \subset \mathcal{P}$, such that $K \subset \cup\mathcal{P}^* \subset U$. A space X is called an *$\aleph$–space* if X has a σ–locally finite k–network (cf. [2] and [7]). A *network* for a space X is a collection $\mathcal{F}$ of subsets of X such that whenever $x \in U$ with U open, there exists $F \in \mathcal{F}$ with $x \in F \subset U$. A space X is a *σ–space* if X has a σ–discrete network (cf. [1]). A space is called a *sequential space* if a subset $F \subset X$ is closed in X if and only if F contains all limit points of sequences from F (cf. [3]). We know that k–spaces and sequential spaces are equivalent for σ–spaces (cf. [1]). A space X is a *metalindelöf space* if every open cover $\mathcal{U}$ of X has a point–countable open refinement (cf. [1]).

The properties of $\aleph$–spaces have been studied by many topologists. In [2], Foged proved that a sequential space or k–space with a σ–locally finite k–network (k–and–$\aleph$ space) is a hereditarily metalindelöf space, and a normal k–and–$\aleph$ space is a paracompact

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space. From [4], we know that a space with a σ–hereditarily closure preserving (σ–HCP) k–network need not be an $\aleph$–space. So it is necessary to study the properties of spaces with σ–HCP k–networks. By a space we mean a regular topological space.

In [5], Liu raised the following question: Are k–spaces with σ–HCP k–networks hereditarily metalindelöf? And the same question was also raised by Liu and Tanaka in [6]. Is every k–space with a σ–HCP k–network a metalindelöf space? In this paper, we prove that a k–space with a σ–HCP k–network is a hereditarily metalindelöf space. Thus, the question appearing in [5] and [6] is answered.

Lemma 1 [2]. *A k–and–$\aleph$ space is a hereditarily metalindelöf space.*

Lemma 2 [3]. *The following are equivalent for a regular space.*

(a) X is a σ–space.
(b) X has a σ–locally finite network.
(c) X has a σ–discrete network.
(d) X has a σ–closure preserving network.

By Lemma 2, we know that a space with a σ–HCP k–network is a σ–space.

Theorem 1. *A k–space with a σ–HCP k–network is a hereditarily metalindelöf space.*

Proof: A space X is hereditarily metalindelöf iff every collection $\mathcal{U}$ of open sets has a point–countable open refinement. Since X is a σ–space, it is easy to see that any $\mathcal{U}$ has a σ–closed discrete (in X) refinement. Thus, it is enough to show that every closed discrete family $\mathcal{F}$ of subsets of X may be expanded to a point–countable open family.

X is a regular space, so we may assume that X has a σ–HCP closed k–network. Let $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \omega\}$ be a σ–HCP closed k–network of X and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \omega$. Let $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ be a discrete closed family of X. Let $\mathcal{P}(\emptyset) = \mathcal{F}$. For $n \in \omega$, let $P^*_\alpha(n) = \cup\{P : P \in \mathcal{P}_n, P \cap F_\alpha \neq \emptyset, P \cap F_\beta = \emptyset$ for all $\beta \in \Lambda \setminus \{\alpha\}\}$, and let $P_\alpha(n) = P^*_\alpha(n) \setminus \cup\{P^*_\beta(n) : \beta \in \Lambda \setminus \{\alpha\}\}$. Then $\mathcal{P}(n) = \{P_\alpha(n) : \alpha \in \Lambda\}$ is a pairwise disjoint family in X, and for any $\Lambda' \subset \Lambda$, $\cup\{P^*_\beta(n) : \beta \in \Lambda'\}$ is a closed subset of X.

For any finite sequence δ of ω, suppose $\mathcal{P}(\delta)$ has been defined. That is, $\mathcal{P}(\delta) = \{P_\alpha(\delta) : \alpha \in \Lambda\}$ is a pairwise disjoint family, where $P_\alpha(\delta) = P^*_\alpha(\delta) \setminus \cup\{P^*_\beta(\delta) : \beta \in \Lambda \setminus \{\alpha\}\}$, and $\cup\{P^*_r(\delta) : r \in \Lambda'\}$ is closed for any $\Lambda' \subset \Lambda$. Now, for $n \in \omega$, we construct $\mathcal{P}(\delta n)$. Let $P^*_\alpha(\delta n) = \cup\{P : P \in \mathcal{P}_n, P \cap P_\alpha(\delta) \neq \emptyset, P \cap \cup\{P^*_\beta(\delta) : \beta \neq \alpha\} = \emptyset\}$. Then $\cup\{P^*_r(\delta n) : r \in \Lambda'\}$ is closed for any $\Lambda' \subset \Lambda$. Let $P_\alpha(\delta n) = P^*_\alpha(\delta n) \setminus \{P^*_\beta(\delta n)\colon\ \beta \neq \alpha\}$. Then $\mathcal{P}(\delta n) = \{P_\alpha(\delta n) : \alpha \in \Lambda\}$ is a pairwise disjoint family of X.

Let $U_\alpha = \cup\{P_\alpha(\delta) : \delta$ is a finite sequence in $\omega\}$. We will show that $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ is a point–countable open family of X and $F_\alpha \subset U_\alpha$ for $\alpha \in \Lambda$.

For any $x \in F_\alpha$, there is an open neighborhood V_x of X, such that $V_x \cap F_\beta = \emptyset$ for $\beta \in \Lambda \setminus \{\alpha\}$. Then there is $n \in \omega$, and $P \in \mathcal{P}_n$, such that $x \in P \subset V_x$. Since $\cup\{P_\beta(n) : \beta \in \Lambda \setminus \{\alpha\}\} \cap F_\alpha = \emptyset$, we have $x \in P_\alpha(n)$. Thus, $F_\alpha \subset U_\alpha$.

To prove U_α is open, we need only to prove that it is sequential open in X. Suppose a sequence Z converges to x, and $x \in U_\alpha$. Then there is a finite sequence δ in ω, such that $x \in P_\alpha(\delta) = P^*_\alpha(\delta) \setminus \cup\{P^*_\beta(\delta) : \beta \in \Lambda \setminus \{\alpha\}\}$. And we know that $\cup\{P^*_\beta(\delta) : \beta \in \Lambda \setminus \{\alpha\}\} = M$ is a closed subset of X. So there is an open neighborhood V_x of x, such that $V_x \cap M = \emptyset$. Then there are some $n \in \omega$ and $\mathcal{P}^* \subset \mathcal{P}_n$ $|\mathcal{P}^*| < \omega$, such that Z is eventually in $\cup\mathcal{P}^* \subset V_x$, and every $P \in \mathcal{P}^*$ contains x. So $\cup\mathcal{P}^* \subset P^*_\alpha(\delta n)$. For any $\beta \in \Lambda \setminus \{\alpha\}$, $P^*_\beta(\delta n) \cap P^*_\alpha(\delta) = \emptyset$, so $x \in X \setminus \cup\{P^*_\beta(\delta n) : \beta \in \Lambda \setminus \{\alpha\}\}$. Thus, Z is eventually in $P_\alpha(\delta n) = P^*_\alpha(\delta n) \setminus \cup\{P^*_\beta(\delta n) : \beta \in \Lambda \setminus \{\alpha\}\}$. So U_α is an open set of X.

Suppose $\{U_\alpha : \alpha \in \Lambda\}$ is not point–countable. Then there is $x \in X$ such that $|\{\alpha : x \in U_\alpha\}| > \omega$. Thus, there is a finite sequence δ in ω, satisfying $x \in P_\alpha(\delta)$, where $\alpha \in \{\beta : x \in U_\beta\}$. Thus, $\{P_\alpha(\delta) : \alpha \in \Lambda\}$ is not a pairwise disjoint family. Contradiction.

Now we have proved that $\{U_\alpha : \alpha \in \Lambda\}$ is a point–countable open family of X and $F_\alpha \subset U_\alpha$ for $\alpha \in \Lambda$. Thus, X is a hereditarily metalindelöf space. $\square$

From the proof of Theorem 1 or Theorem 6 of [8], we have the following corollary.

Corollary 1. *If X is a k–space and has a σ–HCP weak base, then X is a metalindelöf space.*

References

[1] D. K. Burke, *Covering properties*, in: Handbook of Set–Theoretic Topology, (K. Kunen and J. E. Vaughan, editors), North–Holland, Amsterdam, 1984, 349–422.

[2] L. Foged, *Normality in k– and $\aleph$ spaces*, Topology Appl., **22** (1986), 223–240.

[3] G. Gruenhage, *Generalized metric spaces*, in: Handbook of Set–Theoretic Topology, (K. Kunen and J. E. Vaughn, editors), North–Holland, Amsterdam, 1984, 425–501.

[4] S. Lin, *Generalized Metric Spaces and Mappings*, Chinese Science Publishers, Beijing, 1995.

[5] C. Liu, *Spaces with a σ–hereditarily closure–preserving k–network*, Topology Proc., **18** (1993), 179–188.

[6] C. Liu and Y. Tanaka, *Spaces having σ–compact–finite k–network and related matters*, Topology Proc., **21** (1996), 173–200.

[7] P. O' Meara, *On paracompactness in function spaces with the compact–open topology*, Proc. Amer. Math. Soc., **29** (1971), 183–189.

[8] Y. Tanaka, *σ–Hereditarily closure–preserving k–networks and g–metrizability,* Proc. Amer. Math. Soc., **112 (1)** (May 1991), 283–290.

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