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Department of Mathematics & Statistics
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COMPACT-FINITE, H-CLOSED SPACES

JACK PORTER

ABSTRACT. An infinite H–closed space is constructed in which every compact set is finite. This result answers a question by Arhangel'skii and Strecker posed to the author in 1994. This example is used to construct a Hausdorff space with the property that the remainder of each of its H–closed extensions have the same cardinality.

1. Introduction and Preliminaries

First some basic definitions (see [5]) are provided. All spaces considered in this paper are Hausdorff. A space X is **H–closed** if whenever X is a subspace of Y, X is a closed subset of Y. For a space X, this is equivalent to every open ultrafilter on X converges and to the property that for every open cover $\mathcal{C}$ of X, there is a finite subset $\mathcal{D} \subseteq \mathcal{C}$ such that $X = cl_X(\cup\mathcal{D})$.

Let X be a space and $\tau(X)(s)$ be the topology generated by the open base $\{int_X cl_X(U) : U \in \tau\}$. It is easy to check that $\tau(X)(s) \subseteq \tau(X)$ and that $(X, \tau(X)(s))$, sometimes denoted as $X(s)$, is also a Hausdorff space. A space X is **semiregular** if $\tau(X)(s) = \tau(X)$. The space $X(s)$ is semiregular. Furthermore, a space X is H–closed iff $X(s)$ is H–closed.

Construction: Let D be a dense subset of a space X and σ be the topology on X generated by $\tau(X) \cup \{D\}$, i.e., $\sigma = \{U \cup (V \cap D) : U, V \in \tau(X)\}$. Let $X(D)$ denote (X, σ) and $X(D^2) = (D \times \{0,1\}) \cup (X \backslash D)$. If $A \subseteq X$, let $A^2 = (A \backslash D) \cup ((A \cap D) \times \{0,1\})$. A subset $U \subseteq X(D^2)$ is defined to be open if $U \cap (D \times \{i\})$ is open in $D \times \{i\}$ (with the product topology) for $i = 0, 1$, and for $x \in U \cap (X \backslash D)$, there is an open set V in X such that $x \in V^2 \subseteq U$. Some of the following properties are contained in [5, 7S]; the rest are easy to verify.

Proposition 1. *Let D be a dense subset of X.*

(a) The subspace D is dense and open in $X(D)$.
(b) If $A \subseteq D$, then $cl_X A = cl_{X(D)} A$.
(c) If $A \subseteq X \backslash D$, then $cl_X A \cap (X \backslash D) = cl_{X(D)} A \cap (X \backslash D)$.
(d) For $i \in \{0, 1\}$, the function $e_i : X(D) \to X(D^2)$ defined by $e_i(x) = x$ for $x \in X \backslash D$ and $e_i(d) = (d, i)$ for $d \in D$ is a closed embedding. In particular, $X(D^2)$ is the union of two closed copies of $X(D)$.
(e) If X is compact and $X \backslash D$ is dense in X, then $X(D)$ is Urysohn, H–closed but not semiregular and $X(D^2)$ is H–closed and semiregular.

2. Modifying the space ω^*

Recall that $\beta\omega$, the Stone-Čech compactification of a countable discrete space, is extremally disconnected and every countable discrete subset of $\beta\omega$ is C*−embedded. It follows that every infinite compact subset of $\beta\omega$ contains a copy of $\beta\omega$ (in particular, has cardinality $2^{\mathfrak{c}}$) and contains an infinite number of pairwise disjoint infinite compact subsets.

Fact 2. *Let $S = \{1, 2, 3, 4\}$.*

(a) There is a partition $\{E_i : i \in S\}$ of ω^* such that for every infinite, closed subset A of ω^* and $i \in S$, $E_i \cap A$ is an infinite set.
(b) For $D = E_3 \cup E_4$, the space $\omega^*(D)$ is Urysohn, H-closed in which compact subsets are finite.
(c) For $D = E_3 \cup E_4$, the space $\omega^*(D^2)$ is H-closed and semiregular in which compact subsets are finite.

Proof: To prove (a), let $\{A_\alpha : \alpha < 2^{\mathfrak{c}}\}$ be an indexing of all the infinite closed subsets of ω^*. By induction, we select points $x^i_\alpha \in A_\alpha, i \in S$, such that $\{x^i_\alpha : \alpha < 2^{\mathfrak{c}}\} \cap \{x^j_\alpha : \alpha < 2^{\mathfrak{c}}\} = \varnothing$ for $i \neq j$. The sets $E_i = \{x^i_\alpha : \alpha < 2^{\mathfrak{c}}\}$ for $i \in S$ have the desired properties. To prove (b) and (c), note that by Proposition 1(e), as ω^* is compact and both D and $\omega^* \backslash D$ are dense in ω^*, it follows that $\omega^*(D)$ is Urysohn and H–closed and $\omega^*(D^2)$ is H–closed and semiregular. Since $\omega^*(D^2)$ is the union of two closed copies of $\omega^*(D)$, it suffices to show that compact subsets of $\omega^*(D)$ are finite. If A is an infinite, compact subset of $\omega^*(D)$, then A is an infinite, compact subset of ω^* as the topology of ω^* is contained in the topology of $\omega^*(D)$. Thus, $A \cap (\omega^* \backslash D)$ is infinite and $A \cap (\omega^* \backslash D)$ is compact in $\omega^*(D)$. But $A \cap (\omega^* \backslash D)$ is compact in ω^*. By (a), $A \cap (\omega^* \backslash D) \cap D$ is an infinite set, a contradiction. □

Comment: Since an infinite, Urysohn, H–closed and semiregular space is compact, it is not possible to have an infinite space satisfying both conclusions (b, c) of Fact 2.

3. Remainders of H–closed Extensions

Let X and Y be two spaces. A function $f : X \to Y$ is $\theta-$**continuous** if for each $p \in X$ and open set $U \in \tau(Y)$ such that $f(p) \in U$, there is an open set $V \in \tau(X)$ such that $p \in V$ and $f[cl_X V] \subseteq cl_Y U$. The function f is called **irreducible** if for each nonempty open set $U \in \tau(X)$, there is some $y \in Y$ such that $f^{\leftarrow}(y) \subseteq U$.

Let X be a space and let $\Theta X = \{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on $X\}$. For $U \in \tau(X)$, let $O(U) = \{\mathcal{U} : U \in \mathcal{U}\}$. For $U, V \in \tau(X)$, it is easy to verify (see [5]) that $O(\varnothing) = \varnothing, O(X) = \Theta X, O(U \cap V) = O(U) \cap O(V), O(U \cup V) = O(U) \cup O(V), \Theta X \backslash O(U) = O(X \backslash cl_X U)$, and $O(U) = O(int_X cl_X U)$. ΘX with the topology generated by $\{O(U) : U \in \tau(X)\}$ is an extremally disconnected compact space. The subspace $EX = \{\mathcal{U} \in \Theta X : \mathcal{U}$ is fixed$\}$ is called the **absolute** of X. The function $k : EX \to X$ defined by $k(\mathcal{U})$ is the unique convergent point of $\mathcal{U}$ is called a covering function. The subspace EX is dense in ΘX (in particular, EX is an extremally disconnected Tychonoff space and $\Theta X = \beta EX$), and the covering function $k : EX \to X$ is irreducible, $\theta-$continuous, perfect and onto. The space X is H–closed iff $EX = \beta EX$ is compact [5, 6.6(e)(1), 6.9(b)(1)], and $EX = X$ iff X is extremally disconnected and Tychonoff [5, 6.6(e)(1), 6.7(a)].

Let Y be an H-closed extension of X. Since an open ultrafilter on X is the trace of a unique open ultrafilter on Y and the trace of an open ultrafilter on Y to X is an open ultrafilter on X, there is a dense embedding of EX in EY. Since EY is an extremally disconnected compact space, it follows that EY is homeomorphic to βEX where the covering function $k_Y : EY \to Y$ corresponds to $k_Y : \beta EX \to Y : \mathcal{U} \to k_Y(\mathcal{U})$ defined by $k_Y(\mathcal{U})$ is the unique adherent point of $\mathcal{U}$ in Y. Using this identification, $k_Y^{\leftarrow}(p)$ is a compact subset of $\beta EX \backslash EX$. Thus, $P(Y) = \{k_Y^{\leftarrow}(p) : p \in Y\backslash X\}$ is a partition of compact subsets of $\beta EX\backslash EX$. Surprisingly the converse of this is true.

Fact 3. *Let X be a space, P be a partition of $\beta EX\backslash EX$ into compact sets, $Y(P) = X \cup P$, and σ denote the topology on $Y(P)$ generated by $\tau(X) \cup \{U \cup \{A\} : A \in P \text{ and } U \in \cap A\}$. Then $(Y(P), \sigma)$ is an H-closed extension of X such that $P(Y(P)) = P$.*

Proof: This fact is a variation of 7.4(a) in [5] and the proof is similar. Here is a rough sketch of the proof. First show that $\cap A$ is a free open filter on X for each $A \in P$, and use this result to show that $Y(P)$ is an H–closed extension of X. Second, show that the function $k_Y : \beta EX \to Y(P)$ defined by $k_Y|_{EX} = k$ and for $y \in A \in P$, $k_Y(y) = A$ is perfect, irreducible, $\theta-$continuous, and onto. Finally, note that since $k_Y|_{EX} = k$ and for $A \in Y(P)\backslash X$, $k_Y^{\leftarrow}(A) = A$, it follows that $P = P(Y(P))$. □

Comments. (a) An immediate consequence of Fact 3 is the existence of many H–closed extensions of a space X. For example, if P is any partition of $\beta EX\backslash EX$ into finite sets, there is an H–closed extension hX of X such that $P(hX) = P$.

(b) By 7.7(d) in [5], if Y and Z are H–closed extensions of X, then $P(Y) = P(Z)$ iff there is a $\theta-$homeomorphism $f : Y \to Z$ such that $f(x) = x$ for $x \in X$.

(c) Tikoo [6] has shown that if X is locally H–closed (every point has an H–closed neighborhood) and has an H–closed extension with an infinite remainder, then X has H–closed extensions $h_\mu(X)$ such that $|h_\mu(X)\backslash X| = \mu$ where $\mu = \omega, \mathfrak{c}$, and $2^{\mathfrak{c}}$.

(d) If $B \subset \beta\omega\backslash\omega$ is a discrete subspace and $X = \beta\omega\backslash B$, then $EX = X$, $\beta EX = \beta\omega = \beta X$, and $\beta EX\backslash EX = \beta\omega\backslash X = B$, a discrete subspace. Each H–closed extension Y of X has the property that $P(Y)$ is a partition of finite sets.

An interesting folklore question has been the existence of a space X with the properties that $\beta EX\backslash EX$ is not a discrete space and for each H–closed extension Y of X, $P(Y)$ is a partition of finite sets. The space in section 2 provides a way to construct such a space.

Construction. Let $X = \beta\omega\backslash D$, where D is the space defined in Fact 2. Note that $\omega \subset X \subset \beta\omega$, $EX = X$, $\beta X = \beta\omega$, $\beta EX\backslash EX = \beta\omega\backslash X = D$, and compact subsets of $\beta EX\backslash EX$ are finite. So, for each H–closed extension Y of X, $P(Y)$ is a partition of finite subsets and $|Y\backslash X| = 2^{\mathfrak{c}}$. That is, there are no H–closed extensions of X with small remainders.

4. Relatively Compact Subspaces

In [1] Arhangel'skii has posed some problems about relative topological properties. Gartside and Glyn [4] have answered a few of these questions. A modification of the space constructed in section 2 provides different solutions to two of the problems answered in [4].

For a subspace Y of a space X, Y is said to be **compact in** X **from the inside** if for each subspace $Z \subseteq Y$, if Z is closed in X, then Z is compact. Arhangel'skii [1, Problems 1, 5] asked if Y is compact in space X from the inside and X is Urysohn, is Y regular or Tychonoff?

Theorem 4. *The subspace* $Y = E_1 \cup E_3$ *is compact in* $X = \omega^*(E_3 \cup E_4)$ *from the inside.*

Proof: Let A be a noncompact closed subset of Y, $A_1 = A \cap E_1$ and $A_3 = A \cap E_3$. Then A_1 or A_3 is infinite. For $i \in \{1, 3\}$, if A_i is infinite, $cl_{\omega^*} A_i = C_\alpha$ for some $\alpha < 2^{\mathfrak{c}}$. If $C_\alpha = cl_{\omega^*} A_3$, then by Proposition 1, $C_\alpha = cl_{\omega^*} A_3 = cl_{\omega^*(D)} A_3$. As $C_\alpha \cap E_4 \neq \varnothing$, A is not closed in $\omega^*(D)$. Suppose $C_\alpha = cl_{\omega^*} A_1$. By Proposition 1, $\varnothing \neq C_\alpha \cap E_2 \subseteq C_\alpha \cap X\backslash D = cl_{\omega^*} A_1 \cap X\backslash D = cl_{\omega^*(D)} A_1 \cap X\backslash D$. Thus, A is not closed in $\omega^*(D)$. □

Comment. The space Y defined in Theorem 4 is not semiregular and hence not regular; yet, the H–closed space X is more than Urysohn (in particular, X is completely Hausdorff and its semiregularization is compact). The doubling process used in Fact 2(c) gives rise to a minimal Hausdorff space Z such that $Y \subseteq X \subseteq Z$. It follows that Y is also compact in Z from the inside.

References

[1] A. V. Arhangel'skii, *Some Questions and References on Relative Topological Properties, Part 1*, Topology Atlas Document **iaai-04**, (2000).

[2] A. V. Arhangel'skii, *Relative topological properties and relative topological spaces*, Top. Appl., **70** (1996), 1-13.

[3] A. V. Arhangel'skii and J. Tartir, *A characterization of compactness by a relative separation property*, Questions and Answers in General Topology, **14** (1966), 49-52.

[4] P. M. Gartside and A. Glyn, *Relative separation properties*, Topology Atlas Preprint # 438, August 17, 2000.

[5] J. R. Porter and R. G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer–Verlag, Berlin, 1988.

[6] M. L. Tikoo, *Remainders of H–closed extensions*, Top. Appl., **23** (1986), 117-128.

University of Kansas, Lawrence, Kansas 66045 USA
E-mail address: porter@math.ukans.edu