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CONTROLLING EXTENSIONS OF FUNCTIONS AND C-EMBEDDING

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Abstract. We prove that a subspace A of a space X is C-embedded in X if and only if for every continuous function $f : A \to [0,1]$ and disjoint zero-sets Z_0, Z_1 of X with $Z_i \cap A = f^{-1}(\{i\})$ $(i = 0,1)$, there exists a continuous extension $g : X \to [0,1]$ of f such that $Z_i = g^{-1}(\{i\})$ $(i = 0,1)$. This extends a result of Frantz [5] where X is normal and A is closed in X. Applying this result, we show that some results on controlling extensions of special functions, which Frantz [5] established on a closed subspace of a normal space, also hold on a C-embedded subspace of a space. Moreover, we apply the above result to give new characterization of P^γ-embedding by extending suitable collections of functions, and answer a question of Frantz [5].

1. Introduction

Throughout this paper, a space means a topological space. In [5], Frantz proved a theorem as follows:

Theorem 1.1. (Frantz [5]). *Let X be a normal space and A a closed subspace of X. Let $f : A \to [0,1]$ be a continuous function with $f^{-1}(\{i\}) \neq \emptyset$, $i = 0,1$, and suppose Z_0 and Z_1 are disjoint zero-sets of X satisfying $Z_i \cap A = f^{-1}(\{i\})$, $i = 0,1$. Then, f has a continuous extension $g : X \to [0,1]$ such that $Z_i = g^{-1}(\{i\})$ $(i = 0,1)$.*

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According to [5], this result shows that the well-known Tietze-Urysohn extension theorem admits controlling the extended function so as to take certain specified values. As for extension properties of functions on a subspace over a whole space, C- or C^*-embedding is well-known; a subspace A of a space X is said to be C (resp. C^*)-*embedded* in X if every real-valued (resp. bounded real-valued) continuous function on A can be extended to a continuous one over X. The Tietze-Urysohn extension theorem reads that every closed subspace of a normal space is C (or C^*)-embedded. Hence, these arguments suggest we consider whether the C (or C^*)-embedding of A allows us to control extensions of functions as in Theorem 1.1 without assuming the normality of X and the closedness of A. In this paper, this will be positively answered, that is, in section 2 we establish the following theorem, which improves a previous result in [15].

Theorem 1.2. *Let X be a space and A a subspace of X. Then, A is C-embedded in X if and only if for every continuous function $f : A \to [0,1]$ and disjoint zero-sets Z_0, Z_1 of X with $Z_i \cap A = f^{-1}(\{i\})$ $(i = 0,1)$, there exists a continuous extension $g : X \to [0,1]$ of f such that $Z_i = g^{-1}(\{i\})$ $(i = 0,1)$.*

In section 2, as corollaries to Theorem 1.2, we show C-embedding admits several descriptions in terms of controlling extensions of functions. One of these will be applied to characterize extending a suitable collection of functions by extending suitable cozero-set covers (Lemma 3.1). By this result and another one, we describe P^γ-embedding by introducing a new notion of sum-complete collections of functions; a collection $\{f_\alpha : \alpha \in \Omega\}$ of continuous non-negative real-valued functions on a space X is said to be *sum-complete* if $\sum_{\alpha\in\Omega} f_\alpha$ can be defined as a continuous function from X into $[0,\infty)$. Here, for an infinite cardinal γ, a subspace A of a space X is said to be P^γ-*embedded* in X if for every normal open cover $\mathcal{U}$ of A with $|\mathcal{U}| \leq \gamma$, there exists a normal open cover $\mathcal{V}$ of X such that $\{V \cap A : V \in \mathcal{V}\}$ refines $\mathcal{U}$ [1]. It is known that P^ω-embedding is equivalent to C-embedding (cf. [1]), where ω denotes the first infinite cardinal.

Theorem 1.3. (Theorems 3.4 and 3.5)**.** *For a space X and a subspace A of X, the following statements are equivalent*:

(1) *A is P^γ-embedded in X*;

(2) *for every uniformly locally finite cozero-set cover* $\{U_\alpha : \alpha \in \Omega\}$ *of* A *with* $|\Omega| \leq \gamma$, *there exists a uniformly locally finite cozero-set cover* $\{V_\alpha : \alpha \in \Omega\}$ *of* X *such that* $V_\alpha \cap A = U_\alpha$ *for every* $\alpha \in \Omega$;

(3) *for every uniformly locally finite collection* $\{f_\alpha : \alpha \in \Omega\}$ *on* A *with* $|\Omega| \leq \gamma$, *there exists a uniformly locally finite collection* $\{g_\alpha : \alpha \in \Omega\}$ *on* X *such that* $g_\alpha|A = f_\alpha$ *for each* $\alpha \in \Omega$;

(4) *for every sum-complete collection* $\{f_\alpha : \alpha \in \Omega\}$ *on* A *with* $|\Omega| \leq \gamma$, *there exists a sum-complete collection* $\{g_\alpha : \alpha \in \Omega\}$ *on* X *such that* $g_\alpha|A = f_\alpha$ *for every* $\alpha \in \Omega$.

The equivalence (1) $\Leftrightarrow$ (2) of Theorem 1.3 was proved by Alò and Shapiro [1] assuming that X is normal and A is closed. Our characterization of C-embedding works well to remove this assumption. Comparing the conditions (3) and (4) in Theorem 1.3, notice that the local finiteness is located between the uniformly local finiteness and the sum-completeness. However, we show that if we replace all "uniformly locally finite" in (3) in the above by "locally finite", then the condition is equivalent to P^γ(locally-finite)-embedding on A, which is strictly stronger than P^γ-embedding (Theorem 3.3).

We also give an answer to a question of Frantz in [5] related to controlling extensions of partitions of unity.

2. Proof of Theorem 1.2

Let X be a space and A a subspace of X. Then, A is said to be *well-embedded* in X if A is completely separated from any zero-set of X disjoint from A. A subspace A is *z-embedded* in X if every zero-set of A is the intersection of A with some zero-set of X. A subspace A is said to be C_1-*embedded* in X if every zero-set Z of A is completely separated from any zero-set of X disjoint from Z [10]. It is known that A is C-embedded in X if and only if A is C^* (or z)- and well-embedded in X (cf. [1] or [6]). It is proved in [10] that C-embedding implies C_1-embedding, and the latter implies well-embedding.

Other terminology and basic facts are referred to [1], [4], [6] or [9].

Proof of Theorem 1.2: To prove the "if" part, assume that for every continuous function $f : A \to [0,1]$ and disjoint zero-sets Z_0, Z_1 of X with $Z_i \cap A = f^{-1}(\{i\})$ $(i = 0, 1)$, there exists a continuous extension $g : X \to [0,1]$ of f such that $Z_i = g^{-1}(\{i\})$ $(i = 0,1)$. To prove C-embedding of A in X, it suffices to show that any continuous function $f : A \to (0,1)$ can be extended to a continuous function $g : X \to (0,1)$. Regard f as $f : A \to [0,1]$ and apply the condition to $Z_0 = Z_1 = \emptyset$. Then the extension g of f satisfying the condition maps X into $(0,1)$. Hence, g is the desired extension.

To prove the "only if" part, suppose A is C-embedded in X. Let $f : A \to [0,1]$ be a continuous function and Z_0, Z_1 disjoint zero-sets of X with $Z_i \cap A = f^{-1}(\{i\})$ $(i = 0,1)$. Let $\ell : X \to [0,1]$ be a continuous function satisfying $\ell^{-1}(\{i\}) = Z_i$ $(i = 0,1)$. We first prove the following claim.

Claim. *There exists a continuous extension $h : X \to [0,1]$ of f such that $Z_i \subset h^{-1}(\{i\})$ $(i = 0,1)$.*

Proof of Claim: The proof is based on that of [15, Lemma 3.2]. By induction, we shall construct continuous functions $h_n : X \to [-1/2^{n-1}, 1/2^{n-1}]$ $(n \in \mathbb{N})$ which satisfy the following conditions:

(1) $h_1^{-1}(\{i\}) \supset Z_i$ $(i = 0,1)$ and $h_n^{-1}(\{0\}) \supset Z_0 \cup Z_1$ $(n \geq 2)$; and

(2) $\left|f - \sum_{i=1}^{n} \left(h_i|A\right)\right| < 1/2^n$ $(n \in \mathbb{N})$.

Let $k_1 = f - \ell|A$. Put $F_1 = k_1^{-1}\big([-1,-1/2] \cup [1/2,1]\big)$. Then, F_1 is a zero-set of A disjoint from $Z_0 \cup Z_1$. Since A is C_1-embedded in X, there exists a continuous function $j_1 : X \to [0,1]$ such that

$$j_1^{-1}(\{1\}) \supset F_1 \quad \text{and} \quad j_1^{-1}(\{0\}) = Z_0 \cup Z_1.$$

Since A is C^*-embedded in X, there exists a continuous function $f_1 : X \to [0,1]$ such that $f_1|A = f$. Define a continuous function $h_1 : X \to [0,1]$ by

$$h_1(x) = j_1(x) \cdot f_1(x) + (1 - j_1(x)) \cdot \ell(x)$$

for every $x \in X$. Then, h_1 trivially satisfies the conditions (1) and (2).

Next assume that the continuous functions $h_1, \dots, h_n$ are defined with the properties (1) and (2) for $i = 1, \dots, n$. Put $k_{n+1} = f - \sum_{i=1}^{n}(h_i|A)$. Then, by the assumption (2), k_{n+1} takes its value in $[-1/2^n, 1/2^n]$. Put

$$F_{n+1} = k_{n+1}^{-1}\Big(\big[-\frac{1}{2^n}, -\frac{1}{2^{n+1}}\big] \cup \big[\frac{1}{2^{n+1}}, \frac{1}{2^n}\big]\Big).$$

Then, F_{n+1} is a zero-set of A disjoint from $Z_0 \cup Z_1$. Since A is C_1-embedded in X, there exists a continuous function $j_{n+1} : X \to [0,1]$ such that

$$j_{n+1}^{-1}(\{1\}) \supset F_{n+1} \text{ and } j_{n+1}^{-1}(\{0\}) = Z_0 \cup Z_1.$$

Since A is C^*-embedded in X, there exists a continuous function $f_{n+1} : X \to [-1/2^n, 1/2^n]$ such that $f_{n+1}|A = k_{n+1}$. Define a continuous function h_{n+1} by

$$h_{n+1}(x) = f_{n+1}(x) \cdot j_{n+1}(x)$$

for every $x \in X$. Then $h_{n+1} : X \to [-1/2^n, 1/2^n]$ is a continuous function satisfying (1) and (2). This completes the induction.

Put

$$h = \Big(\big(\sum_{i \in \mathbb{N}} h_i \big) \wedge 1 \Big) \vee 0.$$

It is not hard to see that h is continuous, $h|A = f$ and $Z_i \subset h^{-1}(\{i\})$ $(i = 0, 1)$. This completes the proof of Claim. □

Now, put

$$D = h^{-1}(\{0\}) \cup h^{-1}(\{1\}) - Z_0 \cup Z_1.$$

Notice that D can be represented as $D = \bigcup_{i \in \mathbb{N}} D_i$, where each D_i is a zero-set of X. Since $A \cap h^{-1}(\{i\}) = f^{-1}(\{i\}) = A \cap Z_i$ $(i = 0, 1)$, we have $A \cap D = \emptyset$ and hence $A \cap D_i = \emptyset$ $(i \in \mathbb{N})$. Since A is well-embedded in X, there exists zero-set F_i of X such that $F_i \cap D_i = \emptyset$ and $A \subset F_i$. Since $\bigcap_{i \in \mathbb{N}} F_i$ is a zero-set of X, there exists a continuous function $\varphi : X \to [0,1]$ such that $\bigcap_{i \in \mathbb{N}} F_i = \varphi^{-1}(\{1\})$. Then it follows that

$$A \subset \varphi^{-1}(\{1\}) \text{ and } \varphi^{-1}(\{1\}) \cap \big(h^{-1}(\{0\}) \cup h^{-1}(\{1\}) - Z_0 \cup Z_1 \big) = \emptyset.$$

Define a continuous function $g : X \to [0,1]$ by

$$g(x) = \varphi(x) \cdot h(x) + (1 - \varphi(x)) \cdot \ell(x)$$

for every $x \in X$. Then, g is an extension of f. Finally we shall show that $Z_i = g^{-1}(\{i\})$ $(i = 0, 1)$.

Since $Z_i = \ell^{-1}(\{i\}) \subset h^{-1}(\{i\})$ $(i = 0, 1)$, we have $Z_i \subset g^{-1}(\{i\})$ $(i = 0, 1)$.

Suppose $x \notin Z_0 \cup Z_1$. Then $0 < \ell(x) < 1$. If $\varphi(x) = 1$, then $0 < h(x) < 1$ because of the definition of φ; it follows that $0 <$

$g(x) < 1$. If $\varphi(x) < 1$, then

$$g(x) \geq (1-\varphi(x)) \cdot \ell(x) > 0 \text{ and } g(x) < \varphi(x) \cdot 1 + (1-\varphi(x)) \cdot 1 = 1;$$

it follows that $0 < g(x) < 1$. These show that

$$X - Z_0 \cup Z_1 \subset g^{-1}\big((0,1)\big).$$

Thus we have $Z_i = g^{-1}(\{i\})$ $(i = 0, 1)$. This completes the proof. □

Next, we give some variations of Theorem 1.2. The following improves [15, Lemma 3.2].

Corollary 2.1. *Let X be a space and A a subspace of X. Then, A is C-embedded in X if and only if for every continuous function $f : A \to [0,1]$ and any zero-set Z of X with $Z \cap A = f^{-1}(\{0\})$, there exists a continuous extension $g : X \to [0,1]$ of f such that $Z = g^{-1}(\{0\})$.*

Proof: The "only if" part immediately follows from Theorem 1.2. To prove the "if" part, assume that for every continuous function $f : A \to [0,1]$ and any zero-set Z of X with $Z \cap A = f^{-1}(\{0\})$, there exists a continuous extension $g : X \to [0,1]$ of f such that $Z = g^{-1}(\{0\})$. First, to prove that A is C^*-embedded in X, it suffices to show that any continuous function $f : A \to [1/2, 1]$ can be continuously extended over X. If we put $Z = \emptyset$, then f can be extended to a continuous function $g : X \to [0,1]$ by the assumption. Then $\big(g \vee (1/2)\big)$ is the desired extension of f. Next, to prove that A is well-embedded in X, let Z be a zero-set of X disjoint from A. Let $f : A \to \{1\}$ be a constant function. Then there exists a continuous extension $g : X \to [0,1]$ of f such that $Z = g^{-1}(\{0\})$ by the assumption. Then $g^{-1}(\{1\})$ is a zero-set of X containing A and disjoint from Z. Hence, Z is completely separated from A. It follows that A is C^*- and well-embedded in X, i.e., A is C-embedded in X. This completes the proof. □

Corollary 2.2. *Let X be a space and A a subspace of X. Then, A is C-embedded in X if and only if for every continuous function $f : A \to [0,\infty)$ and any zero-set Z of X with $Z \cap A = f^{-1}(\{0\})$, there exists a continuous extension $g : X \to [0,\infty)$ of f such that $Z = g^{-1}(\{0\})$.*

Proof: The "if" part is similarly proved as in Corollary 2.1. To prove the "only if" part, assume A is C-embedded in X. It suffices to show that for any continuous function $f : A \to [0,1)$ and zero-set Z of X with $Z \cap A = f^{-1}(\{0\})$, f has a continuous extension $g : X \to [0,1)$ with $Z = g^{-1}(\{0\})$. Putting $Z_0 = Z$ and $Z_1 = \emptyset$, this follows easily from Theorem 1.2. This completes the proof. □

Next, more generally, we study extensions of (not necessarily bounded) real-valued continuous functions. For a function $f : X \to \mathbb{R}$, $Coz(f)$ means $f^{-1}\big((-\infty,0)\big) \cup f^{-1}\big((0,\infty)\big)$. Our condition in (2) of the next result is a little different from the one in [5, Theorem 2]. (See (1) of Remark 2.4 below.)

Corollary 2.3. *Let X be a space and A a subspace of X. Then, the following statements are equivalent*:

(1) *A is C-embedded in X*;

(2) *for every continuous function $f : A \to \mathbb{R}$, any real numbers $r_1 < r_2 < \cdots < r_n$ and any zero-set collection $\{Z_i, Z_i^* : i = 1,2,\ldots,n\}$ of X satisfying $Z_i \cap Z_{i+1}^* = \emptyset$ $(i = 1,\ldots,n-1)$, $Z_i \cup Z_i^* = X$, $f^{-1}\big((-\infty,r_i]\big) = Z_i \cap A$ and $f^{-1}\big([r_i,\infty)\big) = Z_i^* \cap A$ $(i = 1,2,\ldots,n)$, there exists a continuous extension $g : X \to \mathbb{R}$ of f such that $g^{-1}\big((-\infty,r_i]\big) = Z_i$ and $g^{-1}\big([r_i,\infty)\big) = Z_i^*$ for $i = 1,2,\ldots,n$*;

(3) *for every continuous function $f : A \to \mathbb{R}$ and any zero-set cover $\{Z^-, Z^+\}$ of X with $f^{-1}\big((-\infty,0]\big) = Z^- \cap A$ and $f^{-1}\big([0,\infty)\big) = Z^+ \cap A$, there exists a continuous extension $g : X \to \mathbb{R}$ of f such that $g^{-1}\big((-\infty,0]\big) = Z^-$ and $g^{-1}\big([0,\infty)\big) = Z^+$*;

(4) *for every continuous function $f : A \to \mathbb{R}$ and any cozero-set U of X with $Coz(f) = U \cap A$, there exists a continuous extension $g : X \to \mathbb{R}$ of f such that $Coz(g) \subset U$.*

Proof: (1) ⇒ (2): Assume (1). Let $f : A \to \mathbb{R}$ be a continuous function, $r_1 < r_2 < \cdots < r_n$ real numbers and $\{Z_i, Z_i^* : i = 1,2,\ldots,n\}$ a zero-set collection of X satisfying $Z_i \cap Z_{i+1}^* = \emptyset$ $(i = 1,\ldots,n-1)$, $Z_i \cup Z_i^* = X$, $f^{-1}\big((-\infty,r_i]\big) = Z_i \cap A$ and $f^{-1}\big([r_i,\infty)\big) = Z_i^* \cap A$ $(i = 1,2,\ldots,n)$. Let

$$f_i = (f \vee r_i) \wedge r_{i+1} \;\; (i = 1,\ldots,n-1), \;\; f_0 = f \wedge r_1 \;\text{ and }\; f_n = f \vee r_n.$$

From Theorem 1.2 and Corollary 2.2, there exist continuous functions $g_0 : X \to (-\infty, r_1]$, $g_i : X \to [r_i, r_{i+1}]$ $(i = 1,\cdots,n-1)$ and

$g_n : X \to [r_n, \infty)$ such that $g_i|A = f_i$ $(i = 0, \dots, n)$, $g_i^{-1}(\{r_i\}) = Z_i$ $(i = 1, \dots, n)$ and $g_i^{-1}(\{r_{i+1}\}) = Z_{i+1}^*$ $(i = 0, \dots, n-1)$. Define a function $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} g_0(x) & \text{if } x \in Z_1, \\ g_i(x) & \text{if } x \in Z_i^* \cap Z_{i+1} \ (i = 1, \dots, n-1), \\ g_n(x) & \text{if } x \in Z_n^*. \end{cases}$$

Then, by the pasting lemma, g is continuous, and this is the desired extension of f.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (4): Assume (3). Let $f : A \to \mathbb{R}$ be a continuous function and U be a cozero-set of X with $\mathrm{Coz}(f) = U \cap A$. Define a function $f^* : A \to (0, 2)$ by

$$f^*(x) = \big(f(x)/(1 + |f(x)|)\big) + 1 \quad (x \in A),$$

$Z^{*-} = \emptyset$ and $Z^{*+} = X$ and apply the condition of (3) to f^* and $\{Z^{*-}, Z^{*+}\}$. There exists a continuous extension $g^* : X \to \mathbb{R}$ of f^*. Put

$$Z^- = g^{*-1}\big((-\infty, 1]\big) \cup (X - U) \ \text{ and } \ Z^+ = g^{*-1}\big([1, \infty)\big) \cup (X - U).$$

Since $\{Z^-, Z^+\}$ is a zero-set cover satisfying the condition (3), there exists a continuous extension $g : X \to \mathbb{R}$ of f such that $g^{-1}\big((-\infty, 0]\big) = Z^-$ and $g^{-1}\big([0, \infty)\big) = Z^+$. Since $g^{-1}(\{0\}) = Z^- \cap Z^+ \supset X - U$, it follows that $\mathrm{Coz}(g) \subset U$.

(4) $\Rightarrow$ (1): Assume (4). First, to prove A is C^*-embedded in X, let $f : A \to [1, 2]$ be a continuous function. Let $U = X$. From the assumption, there exists a continuous extension $g : X \to \mathbb{R}$ of f. Then, $(g \vee 1) \wedge 2$ is an extension of f. To prove A is well-embedded in X, let Z be a zero-set of X disjoint from A. Let $f : A \to \{1\}$ be a constant function and put $U = X - Z$. Let $g : X \to \mathbb{R}$ be an extension of f with $\mathrm{Coz}(g) \subset U$. Then $g^{-1}(\{1\})$ is a zero-set of X containing A and disjoint from Z, which shows A is well-embedded in X. Hence (1) holds. This completes the proof. □

Remark 2.4. (1) In [5, Theorem 2], a result concerning controlling extensions of a continuous function so as to take finitely many specified values was given. This result is required to have more assumptions. Indeed, let $X = [0, 2]$ with a subspace topology of $\mathbb{R}$, $A = \{0, 1, 2\}$ a subspace of X. Let $f : A \to \mathbb{R}$ be a function defined by $f(0) = 1$, $f(1) = 0$ and $f(2) = 2$, and $a_i = i - 1$ ($i = 1, 2, 3$). Let $\hat{A}_1 = \{1\}$, $\hat{A}_2 = \{0\}$ and $\hat{A}_3 = \{2\}$. By the intermediate value theorem, any continuous extension g of f over X must satisfy that $g^{-1}(\{a_2\}) \supsetneq \hat{A}_2$. Hence, f has no continuous extension over X such as in the theorem.

(2) As a more general observation of Corollary 2.3, we similarly have a countable case as follows: *A subspace A of a space X is C-embedded in X if and only if for every continuous function $f : A \to \mathbb{R}$, any sequence $r_1 < r_2 < \dots$ of real numbers and any zero-set collection $\{Z_i, Z_i^* : i \in \mathbb{N}\} \cup \{\hat{Z}\}$ of X satisfying $Z_i \cap Z_{i+1}^* = \emptyset$, $Z_i \cup Z_i^* = X$, $f^{-1}\big((-\infty, r_i]\big) = Z_i \cap A$, $f^{-1}\big([r_i, \infty)\big) = Z_i^* \cap A$ and $Z_i \subset \hat{Z}$ for every $i \in \mathbb{N}$ and $f^{-1}\big((-\infty, \lim r_i]\big) = \hat{Z} \cap A$ (and in addition $\bigcap_{i\in\mathbb{N}} Z_i^* = \emptyset$ if $\lim r_i = \infty$), there exists a continuous extension $g : X \to \mathbb{R}$ of f such that $g^{-1}\big((-\infty, r_i]\big) = Z_i$ and $g^{-1}\big([r_i, \infty)\big) = Z_i^*$ for every $i \in \mathbb{N}$, and $g^{-1}\big((-\infty, \lim r_i]\big) = \hat{Z}$.* Here, if $(r_i)_{i\in\mathbb{N}}$ does not converge, $\lim r_i$ means ∞.

Next we give a characterization of C-embedding in terms of extending pairwise disjoint continuous functions along the lines of Frantz [5]. In [5], continuous real-valued functions f_α ($\alpha \in \Omega$) are said to be *pairwise disjoint* if $|f_\alpha| \wedge |f_\beta| = 0$ for every $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$. Obviously, $|f_\alpha| \wedge |f_\beta| = 0$ if and only if $\mathrm{Coz}(f_\alpha) \cap \mathrm{Coz}(f_\beta) = \emptyset$. By using Corollary 2.3, we can remove the assumption of the normality of X and the closedness of A from [5, Proposition 5] as follows:

Proposition 2.5. *Let X be a space and A a subspace of X. Then, A is C-embedded in X if and only if for every collection $\{f_i : i \in \mathbb{N}\}$ of pairwise disjoint real-valued continuous functions on A, there exists a collection $\{g_i : i \in \mathbb{N}\}$ of pairwise disjoint real-valued continuous functions on X such that $g_i|A = f_i$ for each $i \in \mathbb{N}$.*

Proof: The "if" part is obvious. To prove the "only if" part, suppose A is C-embedded in X and let $\{f_i : i \in \mathbb{N}\}$ be a collection of pairwise disjoint real-valued continuous functions on A. By [2, Theorem 1], there exists a pairwise disjoint cozero-set collection $\{U_i : i \in \mathbb{N}\}$ of X such that $U_i \cap A = \mathrm{Coz}(f_i)$ for every $i \in \mathbb{N}$. Hence, by (4) of Corollary 2.3, there exists a continuous extension $g_i : X \to \mathbb{R}$ of f_i such that $\mathrm{Coz}(g_i) \subset U_i$ for every $i \in \mathbb{N}$. Then,

$\{g_i : i \in \mathbb{N}\}$ is the desired collection of continuous functions. This completes the proof. □

A subspace A of a space X is said to be T_z-*embedded* in X if every disjoint cozero-set collection of A can be extended to a disjoint cozero-set collection of X [2]. Similarly to the proof of Proposition 2.5, we give the general cardinal case of Proposition 2.5.

Proposition 2.6. *Let X be a space and A a subspace of X. Then, A is C- and T_z-embedded in X if and only if for every collection $\{f_\alpha : \alpha \in \Omega\}$ of pairwise disjoint real-valued continuous functions on A, there exists a collection $\{g_\alpha : \alpha \in \Omega\}$ of pairwise disjoint real-valued continuous functions on X such that $g_\alpha | A = f_\alpha$ for each $\alpha \in \Omega$.*

In the proof of [2, Theorem 5], it is essentially proved that every closed subspace of a hereditarily collectionwise normal space X is T_z-embedded in X. Hence, by the above proposition, we have the following which extends [5, Proposition 6] where X is metrizable and A is closed in X.

Corollary 2.7. *If X is a hereditarily collectionwise normal space, then for every closed subset A of X and every collection $\{f_\alpha : \alpha \in \Omega\}$ of pairwise disjoint real-valued continuous functions on A, there exists a collection $\{g_\alpha : \alpha \in \Omega\}$ of pairwise disjoint real-valued continuous functions on X such that $g_\alpha | A = f_\alpha$ for each $\alpha \in \Omega$.*

Note that the converse of Corollary 2.7 is false (cf. [2, Example 3]).

3. Extending functions and cozero-set covers

Throughout this section, let γ be an infinite cardinal. First we give some definitions.

A collection $\mathcal{A}$ of subsets of a space X is said to be *uniformly locally finite* if there exists a normal open cover $\mathcal{U}$ of X such that each member of $\mathcal{U}$ intersects at most finitely many members of $\mathcal{A}$ [11], [12]. A collection $\{U_\alpha : \alpha \in \Omega\}$ of subsets of a space X is uniformly locally finite in X if and only if there exist locally finite collections $\{F_\alpha : \alpha \in \Omega\}$ and $\{G_\alpha : \alpha \in \Omega\}$ of subsets of X such that F_α ($\alpha \in \Omega$) are zero-sets of X, G_α ($\alpha \in \Omega$) are cozero-sets of X and $U_\alpha \subset F_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$.

Let $\{f_\alpha : \alpha \in \Omega\}$ be a collection of continuous $[0,1]$-valued functions on X. Then, $\{f_\alpha : \alpha \in \Omega\}$ is said to be a *partition of unity* on X if $\sum_{\alpha\in\Omega} f_\alpha(x) = 1$ for every $x \in X$. Let P be the one of the following: *point-finite, locally finite, point-countable, uniformly locally finite.* A collection $\{f_\alpha : \alpha \in \Omega\}$ of non-negative real-valued continuous functions on X is said to be P if $\{f_\alpha^{-1}((0,\infty)) : \alpha \in \Omega\}$ is a collection with the property P in X.

In the terminology of [3], a subspace A of a space X is said to be $P^\gamma(\mathsf{P})$-*embedded* in X if for every P partition of unity $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \le \gamma$, there exists a P partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$. A subspace A of a space X is said to be $P(\mathsf{P})$-*embedded* in X if A is $P^\gamma(\mathsf{P})$-embedded in X for every γ. Recall that each of P^γ(locally-finite)-embedding and P^γ(point-finite)-embedding implies P^γ-embedding, and the inverse implications need not hold [3].

Besides the above definitions, we give another one. A collection $\{f_\alpha : \alpha \in \Omega\}$ of continuous non-negative real-valued functions on X is said to be a *covering collection* on X if $\{f_\alpha^{-1}((0,\infty)) : \alpha \in \Omega\}$ is a cover of X.

It should be noted that the sum-completeness of a collection of functions, which is introduced in Section 1, is related to the study of Guthrie-Henry in [7]. Recall from [7] that $\mathcal{F} = \{f_\alpha : \alpha \in \Omega\}$ is said to be *relatively complete* if for every $A \subset \Omega$, $\inf_{\alpha\in A} f_\alpha(x)$ and $\sup_{\alpha\in A} f_\alpha(x)$, $x \in X$, can be defined as real-valued continuous functions. They showed that every point-finite partition of unity is relatively complete [7]. The property of sum-completeness is located in the middle, that is, we have (i) any partition of unity is sum-complete; and (ii) any sum-complete collection is relatively complete. Indeed, statement (i) is obvious, statement (ii) can be shown by the similar proof to [7, Theorem 2], and easy examples show that (ii) need not reverse.

First, let us show the following fundamental lemmas.

Lemma 3.1. *Let X be a space and A a subspace of X. Then, the following statements are equivalent*:

(1) *for every P cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \le \gamma$, there exists a P cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A = U_\alpha$ for each $\alpha \in \Omega$*;

(2) *for every* P *covering collection* $\{f_\alpha : \alpha \in \Omega\}$ *on* A *with* $|\Omega| \leq \gamma$*, there exists a* P *covering collection* $\{g_\alpha : \alpha \in \Omega\}$ *on* X *such that* $g_\alpha|A = f_\alpha$ *for each* $\alpha \in \Omega$.

Proof: (1) $\Rightarrow$ (2): Assume (1). Let $\{f_\alpha : \alpha \in \Omega\}$ be a P covering collection on A with $|\Omega| \leq \gamma$. By (1) there exists a P cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A = f_\alpha^{-1}((0,\infty))$ for every $\alpha \in \Omega$. Notice that (1) implies that A is C-embedded in X. Hence, by Corollary 2.2, there exists a continuous extension $g_\alpha : X \to [0,\infty)$ of f such that $V_\alpha = g_\alpha^{-1}((0,\infty))$ for every $\alpha \in \Omega$. Hence (2) holds.

(2) $\Rightarrow$ (1): Obvious. This completes the proof. $\square$

Lemma 3.2. *Let* X *be a space and* A *a subspace of* X*. Then, the following statements are equivalent*:

(1) A *is* $P^\gamma(\mathsf{P})$*-embedded in* X;

(2) *for every sum-complete* P *covering collection* $\{f_\alpha : \alpha \in \Omega\}$ *on* A *with* $|\Omega| \leq \gamma$*, there exists a sum-complete* P *covering collection* $\{g_\alpha : \alpha \in \Omega\}$ *on* X *such that* $g_\alpha|A = f_\alpha$ *for each* $\alpha \in \Omega$;

(3) *for every sum-complete* P *collection* $\{f_\alpha : \alpha \in \Omega\}$ *on* A *with* $|\Omega| \leq \gamma$*, there exists a sum-complete* P *collection* $\{g_\alpha : \alpha \in \Omega\}$ *on* X *such that* $g_\alpha|A = f_\alpha$ *for each* $\alpha \in \Omega$.

Proof: (1) $\Rightarrow$ (2): Assume (1). Let $\{f_\alpha : \alpha \in \Omega\}$ be a sum-complete P covering collection on A with $|\Omega| \leq \gamma$. Put $f'_\alpha = f_\alpha / \sum_{\beta\in\Omega} f_\beta$ for every $\alpha \in \Omega$. By (1), there exists a P partition of unity $\{g'_\alpha : \alpha \in \Omega\}$ on X such that $g'_\alpha|A = f'_\alpha$ for every $\alpha \in \Omega$. Notice (1) implies that A is C-embedded in X. Hence there exists an extension $g^* : X \to (0,\infty)$ of $\sum_{\beta\in\Omega} f_\beta$ over X. Define a function $g_\alpha : X \to \mathbb{R}$ by $g_\alpha(x) = g'_\alpha(x) \cdot g^*(x)$ for every $x \in X$ and every $\alpha \in \Omega$. Then, $\{g_\alpha : \alpha \in \Omega\}$ is the required sum-complete P covering collection.

(2) $\Rightarrow$ (3): Let $\{f_\alpha : \alpha \in \Omega\}$ be a sum-complete P collection on A with $|\Omega| \leq \gamma$. Use (2) to $\{f_\alpha : \alpha \in \Omega\} \cup \{1\}$, where 1 is the constant function $1 : A \to \{1\}$.

(3) $\Rightarrow$ (1): Assume (3). Then, A is C-embedded in X. Indeed, for a continuous function $f : A \to \mathbb{R}$, applying (3) to $\{f \vee 0, -(f \wedge 0)\}$, one can show that f can be continuously extended over X. To prove (1), let $\{f_\alpha : \alpha \in \Omega\}$ be a P partition of unity on A with $|\Omega| \leq \gamma$. By (3), there exists a sum-complete P collection

$\{g'_\alpha : \alpha \in \Omega\}$ on X such that $g'_\alpha|A = f_\alpha$ for each $\alpha \in \Omega$. Since A is C-embedded in X, there exists a continuous function $h : X \to [0,1]$ such that $h((\sum_{\alpha\in\Omega} g_\alpha)^{-1}(\{0\})) = 1$ and $h(A) = 0$. Fix an $\alpha_0 \in \Omega$. Let

$$g_{\alpha_0} = (g'_{\alpha_0} + h)/(\sum_{\beta\in\Omega} g'_\beta + h) \quad \text{and} \quad g_\alpha = g'_\alpha/(\sum_{\beta\in\Omega} g'_\beta + h)$$

for every $\alpha \in \Omega - \{\alpha_0\}$. Then, $\{g_\alpha : \alpha \in \Omega\}$ is the required P partition of unity on X extending $\{f_\alpha : \alpha \in \Omega\}$. This completes the proof. □

Next, let us proceed to describe the above lemmas individually. In the case when P is "locally finite", we have:

Theorem 3.3. ((1) ⇔ (2) is in [15]). *For a space X and a subspace A of X, the following statements are equivalent*:

(1) *for every locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \le \gamma$, there exists a locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A = U_\alpha$ for each $\alpha \in \Omega$;*

(2) *A is P^γ(locally-finite)-embedded in X;*

(3) *for every locally finite collection $\{f_\alpha : \alpha \in \Omega\}$ on A with $|\Omega| \le \gamma$, there exists a locally finite collection $\{g_\alpha : \alpha \in \Omega\}$ on X such that $g_\alpha|A = f_\alpha$ for each $\alpha \in \Omega$.*

Proof: Since every locally finite collection of functions is sum-complete, the statements (2) in Lemmas 3.1 and 3.2 are equivalent. Hence, Theorem 3.3 follows from Lemmas 3.1 and 3.2. This completes the proof. □

In the case when P is "uniformly locally finite", we have the following. The equivalence (1) ⇔ (4) in the following extends [1, Theorem 12.4] where X is normal and A is closed in X.

Theorem 3.4. *For a space X and a subspace A of X, the following statements are equivalent*:

(1) *for every uniformly locally finite cozero-set cover $\{U_\alpha : \alpha \in \Omega\}$ of A with $|\Omega| \le \gamma$, there exists a uniformly locally finite cozero-set cover $\{V_\alpha : \alpha \in \Omega\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$;*

(2) *A is P^γ(uniformly locally finite)-embedded in X;*

(3) *for every uniformly locally finite collection* $\{f_\alpha : \alpha \in \Omega\}$ *on* A *with* $|\Omega| \leq \gamma$, *there exists a uniformly locally finite collection* $\{g_\alpha : \alpha \in \Omega\}$ *on* X *such that* $g_\alpha|A = f_\alpha$ *for each* $\alpha \in \Omega$;

(4) A *is* P^γ*-embedded in* X.

Proof: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): Since any uniformly locally finite covering collection is sum-complete, the statements (2) in Lemmas 3.1 and 3.2 are equivalent.

(1) $\Rightarrow$ (4): This is not difficult to see.

(4) $\Rightarrow$ (1): Assume (4). Let $\{U_\alpha : \alpha \in \Omega\}$ be a uniformly locally finite cozero-set cover of A with $|\Omega| \leq \gamma$. Since A is P^γ-embedded in X, by [8], we can take locally finite collections $\{F_\alpha : \alpha \in \Omega\}$ and $\{G_\alpha : \alpha \in \Omega\}$ of subsets of X such that F_α $(\alpha \in \Omega)$ are zero-sets of X, G_α $(\alpha \in \Omega)$ are cozero-sets of X and $U_\alpha \subset F_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$. For every $\alpha \in \Omega$, take a cozero-set H_α and a zero-set Z_α of X such that $F_\alpha \subset H_\alpha \subset Z_\alpha \subset G_\alpha$. Since A is z-embedded in X, there exists a cozero-set U^*_α of X such that $U^*_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$. Put $H^*_\alpha = U^*_\alpha \cap H_\alpha$ for every $\alpha \in \Omega$. Since A is well-embedded in X, there exists a cozero-set H^* of X such that

$$A \cap H^* = \emptyset \quad \text{and} \quad H^* \cup \bigcup_{\alpha \in \Omega} H^*_\alpha = X.$$

Replace H^*_0, Z_0 and G_0 by

$$H^*_0 = H^*_0 \cup H^*, \quad Z_0 = X \quad \text{and} \quad G_0 = X,$$

respectively. Since $H^*_\alpha \subset Z_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$, it follows that $\{H^*_\alpha : \alpha \in \Omega\}$ is uniformly locally finite in X. So, (1) holds. This completes the proof. □

Let P be "point-finite". In the case $\gamma > \omega$, the conditions in Lemma 3.1 do not imply (and are not implied by) the ones in Lemma 3.2 ([16]). In the case $\gamma = \omega$, the conditions in Lemmas 3.1 and 3.2 are equivalent ([16]).

In the case when P is "point-countable", we have:

Theorem 3.5. *For a space* X *and a subspace* A *of* X, *the following statements are equivalent*:

(1) A *is* P^γ*-embedded in* X;

(2) A *is* P^γ*(point-countable)-embedded in* X;

(3) *for every sum-complete collection* $\{f_\alpha : \alpha \in \Omega\}$ *on* A *with* $|\Omega| \leq \gamma$, *there exists a sum-complete collection* $\{g_\alpha : \alpha \in \Omega\}$ *on* X *such that* $g_\alpha|A = f_\alpha$ *for every* $\alpha \in \Omega$.

Proof: Since any sum-complete collection of functions (hence any partition of unity) is point-countable, (1) $\Leftrightarrow$ (2) follows from [3], and (2) $\Leftrightarrow$ (3) follows from Lemma 3.2. This completes the proof. □

Let P be point-countable. In the case $\gamma > \omega$, the conditions in Lemma 3.1 are different from those in Lemma 3.2. For instance, see the examples in [16]. In the case $\gamma = \omega$, all of the conditions in Lemmas 3.1 and 3.2 are equivalent to A being C-embedded in X.

Remark 3.6. Clearly, uniformly local finiteness implies local finiteness, the latter implies point-finiteness, which in turn implies point-countableness. Theorems 3.4 and 3.5 show that the two extremes only coincide with each other for the notion of $P^\gamma(\mathsf{P})$-embedding. On the other hand, we have the following: every uniformly locally finite partition (locally finite partition, point-finite partition, or point-countable partition (i.e., partition)) of unity, with cardinality at most γ, on a subspace A of a space X can be extended to a partition of unity on X if and only if A is P^γ-embedded in X.

Next, we give a result that three extension properties (hence, all of the conditions in Theorems 3.3, 3.4 and 3.5) are equivalent. It should be noted that the additional P-space assumption is required only on the subspace A, not on the whole space X. A space is said to be a *P-space* if every cozero-set is closed.

Theorem 3.7. *Let X be a space and A a subspace of X. Assume that A is a P-space. Then, the following statements are equivalent*:
(1) *A is P^γ-embedded in X*;
(2) *A is P^γ(locally-finite)-embedded in X*;
(3) *A is P^γ(point-finite)-embedded in X.*

To prove Theorem 3.7, we first present a lemma.

Lemma 3.8. *For a space X, the following statements are equivalent*:
(1) *X is a P-space*;

(2) *if* $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ *is a point-finite collection of subsets of* X *and* $\bigcup_{\alpha \in \delta} U_\alpha$ *is a cozero-set of* X *for every* $\delta \subset \Omega$*, then* $\mathcal{U}$ *is locally finite in* X;

(3) *every countable point-finite cozero-set collection of* X *is locally finite in* X*.*

Proof: (1) $\Rightarrow$ (2): Let $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ be a point-finite collection of subsets of X and let $\bigcup_{\alpha \in \delta} U_\alpha$ be a cozero-set of X for every $\delta \subset \Omega$. For every $x \in X$, the set $\bigcup\{U_\alpha : x \notin U_\alpha\}$ is a cozero-set of X, hence it is a closed set. So, $\mathcal{U}$ is locally finite at x. Hence (2) holds.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Assume (3) and let U be a cozero-set of X. Let $f : X \to [0,1]$ be a continuous function such that $U = f^{-1}((0,1])$. Since $\{f^{-1}((1/(n+2), 1/(n-1)) : n \in \mathbb{N}\}$ is a point-finite cozero-set collection of X, this is locally finite. Hence, $\{f^{-1}([1/(n+1), 1/n]) : n \in \mathbb{N}\}$ is also locally finite collection of X. It follows that

$$U = f^{-1}((0,1]) = \bigcup_{n \in \mathbb{N}} f^{-1}([1/(n+1), 1/n])$$

is a closed subset of X. So, (1) holds. This completes the proof. □

Proof of Theorem 3.7: (1) $\Rightarrow$ (2): Since every locally finite cozero-set cover of a P-space is uniformly locally finite, this follows from Corollaries 3.3 and 3.4.

(2) $\Rightarrow$ (3): Let $\{f_\alpha : \alpha \in \Omega\}$ be a point-finite partition of unity on A with $|\Omega| \leq \gamma$. Then, it is clear that $\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\}$ is a point-finite cozero-set collection of A. Fix $\delta \subset \gamma$ arbitrarily. Since $\{f_\alpha : \alpha \in \Omega\}$ is a partition of unity of A, it follows that $\{f_\alpha^{-1}((1/(n+1), 1]) : \alpha \in \delta\}$ is locally finite for every $n \in \mathbb{N}$. Hence

$$\bigcup_{\alpha \in \delta} f_\alpha^{-1}((0,1]) = \bigcup_{n \in \mathbb{N}} \bigcup_{\alpha \in \delta} f_\alpha^{-1}((1/(n+1), 1])$$

is a cozero-set of A. By the equivalence of (1) and (2) of Lemma 3.8, $\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\}$ is locally finite in A. Hence, by (2), $\{f_\alpha : \alpha \in \Omega\}$ can be extended to a locally finite (hence, point-finite) partition of unity on X. So, (3) holds.

(3) $\Rightarrow$ (1): This was proved in [3]. This completes the proof. □

Dydak asked in [3] whether P(point-finite)-embedded subspace A of a space X is P(locally-finite)-embedded in X. If the answer to this question is negative for a closed subspace A, the whole space X must be a Dowker space, i.e., a normal space but not countably paracompact (cf. [13], [15]). Recall that Rudin's Dowker space is a P-space.

From these points of view, we give the following result which follows from Theorem 3.7 directly. The following can also be proved indirectly by combining some results in [3], [9], [13], [14] and [15].

Corollary 3.9. *Every closed subspace of Rudin's Dowker space is P(point-finite)-embedded and P(locally-finite)-embedded.*

We conclude this paper answering a question of Frantz in [5] related to controlling extensions of partitions of unity. It is proved in [5] that:

Theorem 3.10. (Frantz [5, Theorem 7]). *Let A be a closed subspace of a normal space X, and let $\{f_1, \dots, f_n\}$ be a partition of unity on A subordinated to an open cover $\{U_1, \dots, U_n\}$ of A. If $\{V_1, \dots, V_n\}$ is an open cover of X such that $V_i \cap A = U_i$ for each i, then there exists a partition of unity $\{g_1, \dots, g_n\}$ on X subordinated to $\{V_1, \dots, V_n\}$ such that $g_i|A = f_i$ for each i.*

And a question is asked in [5, Remark p. 68]:

Question (Frantz [5]). *Does Theorem 7 hold for an infinite partition of unity?*

But the answer is negative in a sense. Indeed, let X be a Dowker space. Then, there exists an increasing cover $\{V_i : i \in \mathbb{N}\}$ of non-empty open sets of X that does not have a locally finite open refinement. Let a be a point in V_1 and $A = \{a\}$. Consider constant functions $f_1 : A \to \{1\}$ and $f_i : A \to \{0\}$ $(i \geq 2)$, and let $U_i = A$ $(i \in \mathbb{N})$. Then, the open cover $\{V_i : i \in \mathbb{N}\}$ does not have a partition of unity subordinated to itself (see [4, 5.1.8]).

If we require the extended cover $\{V_\alpha : \alpha \in \Omega\}$ of X to be locally finite in the above question, we have a positive answer as follows:

Proposition 3.11. *Let A be a closed subspace of a normal space X, and let $\{f_\alpha : \alpha \in \Omega\}$ be a partition of unity on A subordinated to a locally finite open cover $\{U_\alpha : \alpha \in \Omega\}$ of A. If $\{V_\alpha : \alpha \in \Omega\}$ is a locally finite open cover of X such that $V_\alpha \cap A = U_\alpha$ for each $\alpha \in \Omega$, then there exists a partition of unity $\{g_\alpha : \alpha \in \Omega\}$ on X subordinated to $\{V_\alpha : \alpha \in \Omega\}$ such that $g_\alpha|A = f_\alpha$ for each $\alpha \in \Omega$.*

Proof: Let $\{V_\alpha : \alpha \in \Omega\}$ be a locally finite open cover of X such that $V_\alpha \cap A = U_\alpha$ for each $\alpha \in \Omega$. Since $\{f_\alpha^{-1}((0,1]) \cup (V_\alpha - A) : \alpha \in \Omega\}$ is a locally finite open cover of a normal space X, there exists a locally finite cozero-set cover $\{W_\alpha : \alpha \in \Omega\}$ of X such that

$$W_\alpha \subset f_\alpha^{-1}((0,1]) \cup (V_\alpha - A)$$

for each $\alpha \in \Omega$ (cf. [1, Theorems 10.10 and 11.7]). Since X is normal and A is closed in X, for every $\alpha \in \Omega$, there exists a cozero-set W'_α in X such that

$$W'_\alpha \cap A = f_\alpha^{-1}((0,1]) \text{ and } W'_\alpha \subset V_\alpha.$$

Then, $\{W_\alpha \cup W'_\alpha : \alpha \in \Omega\}$ is a locally finite cozero-set cover of X satisfying

$$(W_\alpha \cup W'_\alpha) \cap A = f_\alpha^{-1}((0,1])$$

for each $\alpha \in \Omega$. By Theorem 1.1 or Theorem 1.2, for every $\alpha \in \Omega$, there exists a continuous function $h_\alpha : X \to [0,1]$ such that

$$h_\alpha^{-1}((0,1]) = W_\alpha \cup W'_\alpha \text{ and } h_\alpha | A = f_\alpha.$$

Let $h : X \to \mathbb{R}$ be a function defined by $h = \sum_{\alpha\in\Omega} h_\alpha$; then h is positive-valued and continuous. Put $g_\alpha = h_\alpha / h$ for each $\alpha \in \Omega$. Then, $\{g_\alpha : \alpha \in \Omega\}$ is the required partition of unity. This completes the proof. □

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