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COMPLETE HOMOGENEOUS LOTS

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ABSTRACT. We call a linearly ordered topological space (LOTS) homogeneous when it is order isomorphic to any nonempty open convex subset of itself (HLOTS). Homogeneity is preserved by completion. Of special interest are the complete examples (CHLOTS). Any dense subset of the reals, \mathbf{R} , which is invariant under positive rational affine motions, e.g. \mathbf{Q} , is a HLOTS with completion the CHLOTS \mathbf{R} . At first glance it is hard to think of other examples of CHLOTS and indeed \mathbf{R} is the only separable CHLOTS. We construct topologically distinct examples organized according to a natural notion of size in a well-ordered tower of height Ω , the first uncountable ordinal. By using some results of Hart and van Mill we construct 2^c such towers.

INTRODUCTION

When you first meet the Cantor Set $C \subset [0, 1]$ it does not appear to be homogeneous. Aside from the maximum and minimum there is the countable family of endpoint pairs, each of which forms a gap $x^- < x^+$ such that the intersection $C \cap [x^+, \infty)$ is clopen in C . Then there is the uncountable residuum of points whose very existence is not obvious until the bijection from the set of zero/one sequences to C is revealed. This bijection is in fact a homeomorphism of C with a topological group. The automorphism group $H(C)$ contains all the translations of the compact group structure on C from which topological homogeneity is clear.

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Our original intuition comes from the order structure on C inherited from \mathbf{R} . Indeed, if you restrict to $H_+(C)$, the subgroup of order preserving automorphisms, then there are five equivalence classes with respect to the action: $\{max\}$, $\{min\}$, the set of left endpoints $\{x^-\}$, the set of right endpoints $\{x^+\}$, and the remaining residual subset. If you allow order reversing automorphisms, using $H_\pm(C)$ which contains $H_+(C)$ as a subgroup of index two, then the max and min pair up and the left and right endpoints are equivalent, leading to three classes. Ignoring the anomalous endpoints we focus on the distinction between the gap pairs and the rest.

There is an interesting construction called the *Alexandrov-Sorgenfrey Double Arrow*

$$\mathbf{R}' \stackrel{def}{=} \mathbf{R} \times \{-1, +1\} \quad (0.1)$$

in which we denote by t^- the pair $(t, -1)$ and similarly $t^+ = (t, +1)$. On \mathbf{R}' we introduce the lexicographic ordering and use the associated order topology. For every $t \in \mathbf{R}$ $t^- < t^+$ is a gap pair in \mathbf{R}' and so every point is either a left or a right endpoint. Every closed, bounded subset of \mathbf{R}' is compact and so the space is locally compact and σ -compact. The family of clopen intervals $\mathcal{B} = \{[t^+, s^-] : t < s \text{ in } \mathbf{R}\}$ is uncountable and so \mathbf{R}' is not metrizable. Since \mathcal{B} is a basis for the topology the space is zero-dimensional. The space \mathbf{R}' is a famous example in part because the subset $\{t^- : t \in \mathbf{R}\}$ is order isomorphic with \mathbf{R} and so the order topology is the usual one on \mathbf{R} . However, the subspace topology induced from \mathbf{R}' is neither metrizable, nor locally compact. It is the topology on \mathbf{R} with basis the right-closed, left-open intervals.

We denote by $\bullet\mathbf{R}'\bullet$ the two point compactification obtained by attaching a minimum m and a maximum M to \mathbf{R}' . We call this space the *Extended Cantor Set*. For every $t < s$ in \mathbf{R} there is an order preserving homeomorphism

$$f : \bullet\mathbf{R}'\bullet \rightarrow [t^+, s^-]. \quad (0.2)$$

It easily follows that the group $H_\pm(\mathbf{R}')$ acts transitively on \mathbf{R}' .

This paper began with the idle question: If the Extended Cantor Set is like C then what is like \mathbf{R} ? There should be a linearly ordered space X with the order topology, that is, a LOTS, which is connected, which contains the Extended Cantor Set and which is homogeneous in the sense that for all $a < b$ in X there exists an order preserving homeomorphism

$$f : X \rightarrow (a, b). \quad (0.3)$$

Our first thought was $\mathbf{R} \times J$ with $J = [-1, +1] \subset \mathbf{R}$ with the lexicographic ordering. This LOTS is connected but it is not homogeneous. However, equipped with the lexicographic ordering the countably infinite product

$$\mathbf{R}_\omega =_{def} \mathbf{R} \times J \times J \times \dots \tag{0.4}$$

is a connected and homogeneous LOTS, which we call a CHLOTS. Define for $t \in J$

$$j(t^-) = (t, -1, -1, \dots) \text{ and } j(t^+) = (t, +1, +1, \dots). \tag{0.5}$$

Then $j : [(-1)^+, (+1)^-] \rightarrow \mathbf{R}_\omega$ is an order preserving, topological embedding onto a closed subset. Thus, \mathbf{R}_ω naturally contains the Extended Cantor Set.

How many CHLOTS are there? At first, we could not think of any others but it turns out that there are lots of CHLOTS. We first discovered that as in (0.4) we obtain a CHLOTS when the analogous product is indexed not just by the first infinite ordinal ω but by any countable, *tail-like* ordinal, i.e an ordinal α such that

$$\beta < \alpha \quad \Rightarrow \quad \beta + \alpha = \alpha \tag{0.6}$$

where $+$ denotes ordinal addition. An ordinal is tail-like exactly when it is an ordinal power of ω .

There is a rough notion of *size* for LOTS. In comparing two LOTS X_1 and X_2 we say that X_1 *injects into* X_2 if there exists a, not necessarily continuous, injective order preserving map from X_1 into X_2 . If there is a surjective order preserving map from X_2 to X_1 then X_1 injects into X_2 and when X_1 and X_2 are CHLOTS the converse is true as well. We say that X_2 is *bigger* than X_1 if X_1 injects into X_2 but not vice-versa. For example, any CHLOTS which is not \mathbf{R} itself is bigger than \mathbf{R} .

Beginning with any CHLOTS F , e.g. $F = \mathbf{R}$, there is a tower of CHLOTS F_α indexed by the tail-like ordinals $\alpha < \Omega$, the first uncountable ordinal. Furthermore, if $\alpha > \beta$ then F_α is bigger than F_β . By using a tree construction it is possible to extend the construction to get the double tower which is indexed by the ordinal product Ω^2 with each CHLOTS strictly smaller than its successors.

We had completed this work when we were directed to the paper of Hart and van Mill (1985) whose work is complementary to ours. They construct an uncountable family of distinct CHLOTS no two of which have comparable size but all of which are bigger than \mathbf{R} but smaller than \mathbf{R}_ω . So their class of examples extends horizontally where ours proceed vertically.

In the end we are left with more questions. Our arguments and those of Hart and van Mill for distinguishing between different CHLOTS are all rather ad hoc. It is hard for us to imagine how classifying these objects might proceed from the zoo of examples which have turned up.

We would like to thank Richard Wilson for some helpful discussions as we began this work.

1. LOTS AND ORDINALS

For a totally ordered set X we will use the usual notation (a, b) for the *open interval* and $[a, b]$ for the *closed interval* with *endpoints* $a \leq b$ in X and we will write (a, ∞) and $(-\infty, b)$ for unbounded open intervals. An interval is called *nontrivial* when it contains more than one point. A subset A of X is *bounded* when $A \subset [a, b]$ for some $a, b \in X$. So X itself is bounded iff it has a maximum and a minimum (hereafter *max* and *min*). Somewhat abusively, we will call X *unbounded* when it has *neither max* nor *min*.

A linearly ordered topological space (hereafter a LOTS) is a totally ordered set equipped with the order topology. That is, the set of open intervals is a base for the topology. The topology is Hausdorff and the order and topology properties are closely related. A LOTS X is called *order complete* (hereafter *complete*) when every bounded subset A has a supremum and an infimum (denoted *sup* A and *inf* A), or, equivalently, when every closed bounded interval $[a, b]$ is compact. In particular, X is compact iff it is complete and bounded. X is called *order dense* if between any two points of X there lie other points of X . X is connected iff it is complete and order dense, in which case, every subinterval is connected. If X is not order dense then there exists a *gap pair*, $a < b$ in X with $(a, b) = \emptyset$. Then a is called the *left endpoint* and b is called the *right endpoint* of the pair. By convention the *max* of X , if it exists, is a left endpoint and the *min* is a right endpoint. Thus, $a \in X$ is a left (or right) endpoint iff the closed interval $(-\infty, a]$ (resp. $[a, \infty)$) is open. A point is topologically isolated, i.e. open, iff it is both a left and a right endpoint.

A subset A of a LOTS X is a *convex* subset if $x_1 < x_2 < x_3$ and $x_1, x_3 \in A$ imply $x_2 \in A$. Intervals are convex subsets and if X is complete then every convex subset is an interval.

A *Dedekind cut* in X is a partition of X into a pair of nonempty disjoint sets (A_1, A_2) such that $x_1 < x_2$ and $x_2 \in A_1$ imply $x_1 \in A_1$ (and so $x_1 \in A_2$ implies $x_2 \in A_2$). A *hole between x_1 and x_2* is a Dedekind cut (A_1, A_2) with $x_1 \in A_1$, $x_2 \in A_2$ such that A_1 and A_2 are clopen. For example, if $x_1 < x_2$ is a gap pair then $((-\infty, x_1], [x_2, \infty))$ is a hole between x_1 and x_2 . On the other hand, while the LOTS Q of rational numbers is order dense, every irrational number creates a hole in Q . We say that a LOTS X has *dense holes* if there is a hole between every pair $x_1 < x_2$ in X . Thus Q and the Cantor Set have dense holes.

The *reverse* of a LOTS X , denoted X^* , is the set equipped with the reverse order. Clearly, the intervals, and so the topology, for X and X^* are the same.

A function $f : X_1 \rightarrow X_2$ between LOTS is an *order map* if it is order preserving, i.e. $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$, while f is called an *order* map* if it is order reversing, i.e. $f : X_1^* \rightarrow X_2$ is an order map or, equivalently, if $f : X_1 \rightarrow X_2^*$ is an order map. An injective (or surjective) order map is called an *order injection* (resp. an *order surjection*). A bijective order map is called an *order isomorphism*. This is the isomorphism concept for the category of LOTS with order maps. We say that order isomorphic LOTS X_1, X_2 have the same *order type* and we write $X_1 \cong X_2$.

While an order isomorphism is a homeomorphism, an order map need not be continuous. In particular, if X_1 is a subset of a LOTS X then with the restricted order X_1 is itself a LOTS. However, the LOTS topology on X_1 might be strictly coarser than the relative topology induced from X in which case the inclusion map is not continuous. We will call a map $f : X_1 \rightarrow X_2$ an *order embedding* if it is a continuous order injection, or equivalently, it is an order map which is a topological embedding, i.e. $f : X_1 \rightarrow f(X_1)$ is a homeomorphism with the topology on $f(X_1)$ induced from X_2 .

Proposition 1.1. *Let $f : X_1 \rightarrow X_2$ be an order map with X_2 order dense.*

- (a) *If the image of f is a dense subset of a convex set in X_2 , i.e. if the closure of $f(X_1)$ in X_2 is convex, then f is continuous.*
- (b) *Assume that the image of f is dense in X_2 , e.g. f is surjective, and that X_2 has no max or min. The map f is continuous, X_1 has no max or min and the preimage of every bounded subset of X_2 is a bounded subset of X_1 .*

If, in addition, X_1 is complete, then f is topologically proper, i.e. the preimage of every compact subset of X_2 is a compact subset of X_1 .

- (c) Let A be a subset of X_1 . If A is bounded in X_1 then the image $f(A)$ is bounded in X_2 . If f is continuous and $x = \inf A$ (or $= \sup A$) then $f(x) = \inf f(A)$ (resp. $= \sup f(A)$) in X_2 .

Proof. (a) If $f(x) \in (a_2, b_2)$ and $f(x)$ is neither $\max f(X_1)$ nor $\min f(X_1)$ then there exist a_3, b_3 in X_1 such that

$$a_2 < f(a_3) < f(x) < f(b_3) < b_2$$

and so (1.1)

$$x \in (a_3, b_3) \subset f^{-1}((a_2, b_2)).$$

Thus, the latter is a neighborhood of x in X_1 . The cases where $f(x)$ is the \max or \min use easy variations of the above argument.

(b) Continuity follows from (a). If M is an upper bound for $A \subset X_1$ then $f(M)$ is an upper bound for $f(A)$. In particular, if $M = \max X_1$ then $f(M) = \max X_2$ since f has a dense image and X_2 is order dense. So if X_2 has no \max or \min then X_1 does not. Furthermore, if B is bounded above in X_2 then because X_2 has no \max , there exists $a \in X_1$ such that $y < f(a)$ for all $y \in B$. Hence, $x < a$ for all $x \in f^{-1}(B)$. If B is compact then it is closed and bounded in X_2 so $f^{-1}(B)$ is closed as well as bounded. Thus, it is compact if X_1 is complete.

(c) If x is a lower bound for A then $f(x)$ is a lower bound for $f(A)$. If $f(x) < a_2$ and a_2 is a lower bound for $f(A)$ then $f(x)$ is not in the closure of $f(A)$ and so by continuity x is not $\inf A$. \square

As with **R**, continuity of a map implies order properties when the domain is connected.

Lemma 1.2. *If X_1 and X_2 are LOTS with X_1 connected and $f : X_1 \rightarrow X_2$ is a continuous map, then the image is a convex subset of X_2 . If, in addition, f is injective then f is either order preserving or order reversing.*

Proof. The image of f is a connected subset of X_2 by continuity. Any connected subset of a LOTS is convex.

Assume now that f is injective. If $f(x_1) < f(x_2) < f(x_3)$ then the image of the open interval between x_1 and x_3 is connected and

so contains $f(x_2)$. Since f is injective, x_2 must lie in this interval. That is, either $x_1 < x_2 < x_3$ or $x_1 > x_2 > x_3$. Thus, on each triple of points in X_1 f either preserves or reverses order. If f preserves the order of some pair a, b in X_1 then it preserves the order of every triple which includes a, b and so of every pair which includes either a or b . So it preserves every triple which includes a and so preserves every pair. The remaining possibility is that f reverses every pair. \square

Of special interest are the ordinals. As usual we let $0 = \emptyset$ and define the ordinal α to be the set of ordinals smaller than α , with the ordering by set inclusion. The *successor* $\alpha + 1$ of α is $\alpha \cup \{\alpha\}$. If $A \subset \alpha$ is nonempty then *inf* A is the first element of A , i.e. the minimum and *sup* A is the union. Any well-ordered set has the order type of an ordinal. If $A \subset \alpha$ then there is a unique order isomorphism of A onto an ordinal $\beta \leq \alpha$.

For ordinal results we follow Rosenstein(1982) and Jech(1980). The arithmetic of ordinals is defined inductively to be continuous in the β variable.

$$\begin{aligned} \alpha + 0 &= \alpha & \text{and} & & \alpha + (\beta + 1) &= (\alpha + \beta) + 1. \\ \alpha \cdot 0 &= 0 & \text{and} & & \alpha \cdot (\beta + 1) &= (\alpha \cdot \beta) + \alpha. \\ \alpha^0 &= 1 & \text{and} & & \alpha^{(\beta+1)} &= (\alpha^\beta) \cdot \alpha. \end{aligned} \tag{1.2}$$

In particular, if α and β are countable ordinals then the results of all of these operations are countable ordinals. We will use the usual order type sloppiness, writing $A + B$ for well-ordered sets A and B to mean the ordinal which is the sum of the ordinals having order types A and B .

If α is an ordinal and $\beta < \alpha$ then the *tail*

$$\begin{aligned} \alpha \setminus \beta &= \{i : \beta \leq i < \alpha\} \subset \alpha, \text{ so that} \\ \beta + (\alpha \setminus \beta) &= \alpha \end{aligned} \tag{1.3}$$

α is called *tail-like* if it is positive and all of the tails of α have order type α . This is clearly equivalent to $\beta + \alpha = \alpha$ for all $\beta < \alpha$, and to $\beta_1 + \beta_2 < \alpha$ for all $\beta_1, \beta_2 < \alpha$.

For ω , the first infinite ordinal, $1 = \omega^0$ is tail-like and, inductively, the identities for $\epsilon < \beta$ and $N < \omega$:

$$\begin{aligned}\omega^\epsilon + \omega^\beta &= \omega^\epsilon \cdot (1 + \omega^{\beta \setminus \epsilon}) = \omega^\epsilon \cdot \omega^{\beta \setminus \epsilon} = \omega^\beta, \\ \omega^\beta \cdot N + \omega^{\beta+1} &= \omega^\beta \cdot (N + \omega) = \omega^\beta \cdot \omega = \omega^{\beta+1}\end{aligned}\tag{1.4}$$

imply that ω^β is tail-like for any ordinal β . The *Cantor Normal Form Theorem* (see Rosenstein(1982) Thm. 3.46) says that any positive ordinal α can be written uniquely as the sum

$$\begin{aligned}\alpha &= \omega^{\beta_1} + \dots + \omega^{\beta_N} \\ \text{with } \beta_1 &\geq \dots \geq \beta_N\end{aligned}\tag{1.5}$$

which shows that any tail-like ordinal is of the form ω^β .

We denote by Ω the first uncountable ordinal.

If α is a positive ordinal and $\{X_i : i \in \alpha\}$ is an α indexed family of nonempty LOTS then we define the *order space product* to be the set $\prod_{i \in \alpha} X_i$ with the lexicographic ordering. That is, for $x \neq y$ in the product

$$x < y \iff x_\beta < y_\beta \text{ with } \beta = \min\{j : x_j \neq y_j\}.\tag{1.6}$$

If $\alpha = 2$, we write $X_0 \times X_1$ for the product. When $X_i = X$ for all i then we obtain X^α , the space of functions from α to X , as a LOTS. Observe that the LOTS topology is *not* the product topology.

The ordinal conventions for products and powers given by (1.2) disagree with these new definitions. The ordinal product $\beta \cdot \alpha$ is the order space product $\alpha \times \beta$. The order space power 2^ω is the uncountable space of all zero/one valued sequences. The ordinal 2^ω is the limit of the finite ordinals 2^N and so is just ω .

If $\{X_i : i \in \alpha\}$ is an ordinal indexed family of nonempty LOTS and $0 < \beta \leq \alpha$ then we can write Π_β for the subproduct $\prod_{i \in \beta} X_i$ and if $0 < \epsilon \leq \beta \leq \alpha$ then we denote by

$$\pi_\beta^\epsilon : \Pi_\beta \rightarrow \Pi_\epsilon\tag{1.7}$$

the projection map obtained by forgetting the coordinates in $\beta \setminus \epsilon$. Since ϵ is an initial segment of β it is clear that π_β^ϵ is an order surjection. However, it need not be continuous. The first coordinate projection $\omega \times \omega \rightarrow \omega$ does not have closed point inverses.

Proposition 1.3. *Let $X = \prod_{i \in \alpha} X_i$ be an order space product of LOTS.*

- (a) *If each X_i is order dense then X is order dense. In that case, each projection π_β^ϵ for $0 < \epsilon \leq \beta \leq \alpha$ is continuous.*
- (b) *If each X_i for $i > 0$ is bounded then each projection π_β^ϵ is continuous. If, in addition, each X_i is complete then X is complete.*

Proof. (a) If $x < y$ in X then with $\beta = \min\{j : x_j \neq y_j\}$ we can choose $z_\beta \in X_\beta$ such that $x_\beta < z_\beta < y_\beta$. Define $z_j = x_j$ for all $j \neq \beta$. Then $x < z < y$ in X . The projections are continuous by Proposition 1.1a.

(b) Given $a < b$ in Π_ϵ with $\epsilon > 0$ define $a+$ and $b-$ in Π_β by:

$$\begin{aligned} (a+)_i &= \begin{cases} a_i & i \in \epsilon \\ \max X_i & i \in \beta \setminus \epsilon \end{cases} \\ (b-)_i &= \begin{cases} b_i & i \in \epsilon \\ \min X_i & i \in \beta \setminus \epsilon \end{cases} \end{aligned} \tag{1.8}$$

Note that $i \in \beta \setminus \epsilon$ is positive and so X_i is bounded. Clearly,

$$(\pi_\beta^\epsilon)^{-1}((a, b)) = (a+, b-). \tag{1.9}$$

Hence, π_β^ϵ is continuous.

If $A \subset X$ is contained in $[a, b]$ then by replacing X_0 by $[a_0, b_0]$ we can assume that all of the X_i 's are bounded. We prove by induction on β that if all of the X_i 's are compact then $A \subset \Pi_\beta$ has an *inf*. With similar results for *sup*, completeness follows.

If $\beta = 1$ then $\Pi_\beta = X_0$ which is compact.

Now assume that the result holds for all $\epsilon < \beta$ and let $x^\epsilon = \inf \pi_\beta^\epsilon(A)$ in Π_ϵ . By Proposition 1.1c it is clear that $\delta \leq \epsilon$ implies

$$\pi_\epsilon^\delta(x^\epsilon) = x^\delta. \tag{1.10}$$

Case i: If β is a limit ordinal then define x^β so that

$$x_i^\beta = x_i^\epsilon \quad \text{for } i < \epsilon < \beta, \tag{1.11}$$

which is well-defined by (1.10). Clearly, $x^\beta = \inf A$ in this case.

Case ii: If $\beta = \epsilon + 1$ then there are two possibilities. The set $\{x \in \Pi_\beta : \pi_\epsilon^\beta(x) = x^\epsilon\}$ is order isomorphic with X_ϵ which is compact. If this set meets A then the *inf* in this set is *inf* A .

If this set does not meet A then we obtain $\inf A = x^\beta$ by defining $x_i^\beta = x_i^\epsilon$ for $i < \epsilon < \beta$ and,

$$x_\epsilon^\beta = \max X_\epsilon. \quad (1.12)$$

□

Any LOTS X can be regarded as a subset of a smallest complete LOTS \hat{X} called its *completion*, see Rosenstein (1982) Theorem 2.32. We will only need \hat{X} in the case when X is order dense and unbounded. In that case, Dedekind cuts are used to obtain \hat{X} from X in the same way that \mathbf{R} is obtained from the rationals \mathbf{Q} . The completion \hat{X} can be characterized as an unbounded, connected LOTS which contains X as a dense subset.

Let Y be a connected LOTS and $f : X \rightarrow Y$ be an order map with $f(X)$ dense in Y . By Proposition 1.1a f is continuous and we define $\hat{f} : \hat{X} \rightarrow Y$ by

$$\hat{f}(x) =_{\text{def}} \sup \{f(t) : t \in (-\infty, x] \cap X\} = \inf \{f(t) : t \in [x, \infty) \cap X\}. \quad (1.13)$$

It is easy to see that \hat{f} is an order map and so it is continuous by Proposition 1.1a. The image is therefore connected. It follows that \hat{f} is surjective and if f is an order injection then \hat{f} is an order isomorphism.

A bounded convex subset J in a complete space is an interval with endpoints $\inf J$ and $\sup J$. With $X \subset \hat{X}$ as above, a bounded convex $J \subset X$ is equal to $\hat{J} \cap X$ where $z \in \hat{J}$ iff there exist $x_1, x_2 \in J$ such that $x_1 \leq z \leq x_2$. It follows that a convex set in X is the intersection of an interval of \hat{X} with X .

We conclude this section with some countability conditions.

A topological space X is *separable* if it has a countable dense subset.

A subset A of a LOTS X is *cofinal* if for any $x \in X$ there exists $a \in A$ such that $x \leq a$. If $M = \max X$ exists then A is cofinal iff $M \in A$. If X has no \max then any dense subset of X is cofinal. A is *coinitial* if it is cofinal for the reverse X^* . X is called *σ -bounded* if it admits countable cofinal and coinitial subsets. If X is complete then it is σ -bounded iff it is σ -compact.

A point x in a LOTS X has a countable neighborhood base iff the interval $(-\infty, x)$ has a countable cofinal subset and the interval $(x, +\infty)$ has a countable coinital subset. Thus, X is first countable iff for every point $x \in X$, x is either a right endpoint or the limit of an increasing sequence, and x is either a left endpoint or the limit of a decreasing sequence. Equivalently, X is first countable iff every bounded interval in X is a σ -bounded LOTS in its own right.

We say that a LOTS X has *countable type* if every convex subset of X is a σ -bounded LOTS. It clearly suffices that every open, convex subset of X be σ -bounded. Obviously, X is separable or of countable type iff its reverse X^* satisfies the corresponding property.

Lemma 1.4.

- (a) *A LOTS X is of countable type iff there does not exist an injective order map from Ω or Ω^* into X . If X is of countable type then it is first countable and σ -bounded.*
- (b) *Let $f : X \rightarrow X_1$ be an order map. If f is injective and X_1 is of countable type then X is of countable type. If f is surjective and X is of countable type then X_1 is of countable type.*

Proof. (a) If $f : \Omega \rightarrow X$ is an order injection then $J = \{x : x < f(i) \text{ for some } i \in \Omega\}$ is a nonempty, open convex subset of X and if A is a countable subset of J then we can choose for each $a \in A$, $i(a) \in \Omega$ such that $a < f(i(a))$. Let $j \in \Omega$ with $j > \sup \{i(a) : a \in A\}$. Then $f(j) > a$ for all $a \in A$ and $f(j) < f(j+1)$ so that $f(j) \in J$. Thus, A is not cofinal in J .

Conversely, if J is a nonempty, open convex subset of X with no countable cofinal subset then we can construct an order injection $f : \Omega \rightarrow J$ inductively by choosing $f(j) \in J$ larger than all the $f(i)$ previously chosen for $i < j$.

Similarly, every nonempty, open convex subset of X has countable coinital subsets iff there does not exist an order injection of the reverse Ω^* into X .

Since X itself and every interval is a convex set, countable type implies σ -boundedness and first countability.

(b) If Ω injects into X and f is injective then Ω injects into X_1 . If f is a surjection then we can define an order injection $g : X_1 \rightarrow X$ by choosing $g(x)$ to be an arbitrary element of $f^{-1}(x)$. \square

Proposition 1.5. *Let X be an order dense, unbounded LOTS. Let \hat{X} be the completion of X and let $a < b$ be points in X .*

- (a) *If A is a countable LOTS then there exists an order injection $f : A \rightarrow (a, b)$. If X is countable and A is order dense then f can be chosen to be an order isomorphism.*
- (b) *If X is complete then there exists a continuous order surjection $f : [a, b] \rightarrow J$ where $J = [-1, +1] \subset \mathbf{R}$. If X is also separable then f can be chosen to be an isomorphism.*
- (c) *If α is a positive ordinal and $f : \alpha + 1 \rightarrow X$ is an order embedding with $f(0) = a$ and $f(\alpha) = b$ then*

$$[a, b] = \bigcup \{ [f(i), f(i+1)) : i \in \alpha \} \quad (1.14)$$

partitioning $[a, b]$ by an α indexed family of intervals.

- (d) *If X is first countable and α is a positive, countable ordinal then there exists an order embedding $f : \alpha + 1 \rightarrow X$ with $f(0) = a$ and $f(\alpha) = b$.*
- (e) *The following conditions are equivalent.*
 - (1) *X is of countable type.*
 - (2) *\hat{X} is of countable type.*
 - (3) *\hat{X} is first countable and σ -bounded.*
 - (4) *There does not exist an order embedding of Ω or Ω^* into \hat{X} .*
- (f) *If $f : \Omega \rightarrow X$ is an order map or an order* map and X is of countable type then f is eventually constant. That is, there exists $\alpha \in \Omega$ such that $f(i) = f(\alpha)$ for all $i \in \Omega$ with $i \geq \alpha$.*

Proof. (a) This is a standard inductive argument using a counting of the points of A . If X , too, is countable then one counts X as well and proceeds back and forth between A and X to build the isomorphism.

(b) With A the set of rationals in J , use (a) to get an order injection $g : A \rightarrow [a, b]$ such that $g(-1) = a$ and $g(1) = b$. Define for $x \in [a, b]$

$$f(x) = \sup g^{-1}([a, x]) = \inf g^{-1}([x, b]). \quad (1.15)$$

If X is separable, choose g with a dense image in $[a, b]$.

(c) If $x \in [a, b)$ then $b = f(\alpha) > x$. Let $\beta = \min \{i \in \alpha + 1 : f(i) > x\}$. Since $a = f(0) \leq x$, β is positive and by continuity of f , β is not a limit ordinal. Hence, $\beta = i + 1$ for some $i \in \alpha$ and $x \in [f(i), f(i + 1))$.

(d) We construct the embedding f by induction on α . If $\alpha = 1$ let $f(0) = a$ and $f(1) = b$. Now assume the result is true for all $\beta < \alpha$.

Case i: If $\alpha = \beta + 1$ then choose \tilde{b} so that $a < \tilde{b} < b$. By inductive hypothesis there exists an order embedding $\tilde{f} : \beta + 1 \rightarrow [a, \tilde{b}]$ with $\tilde{f}(0) = a$ and $\tilde{f}(\beta) = \tilde{b}$. Extend \tilde{f} to f by $f(\alpha) = b$.

Case ii: If α is a limit ordinal then because it is countable, there exists an increasing cofinal sequence $\{\beta_n\}$ in α . Because X is first countable and order dense there exists an increasing sequence $\{x_n\}$ in (a, b) with limit b . By inductive hypothesis there exists an order embedding of the interval $(\beta_n, \beta_{n+1}]$ in α to $(x_n, x_{n+1}]$ with β_{n+1} mapped to x_{n+1} . Put these together and map 0 to a and α to b to get f .

(e) We repeatedly use the characterization of Lemma 1.4a.

(1) \Leftrightarrow (2) If \hat{X} is of countable type then X is by Lemma 1.4b. On the other hand, if $f : \Omega \rightarrow \hat{X}$ is an order injection then we can choose for each $i \in \Omega$ a point $g(i) \in X \cap (f(i), f(i + 1))$ to define an order injection $g : \Omega \rightarrow X$. Use a similar construction if f is an order reversing injection.

(2) \Leftrightarrow (3) Every convex set in \hat{X} is an interval.

(2) \Rightarrow (4) Apply Lemma 1.4a.

(4) \Rightarrow (2) If \hat{X} is not of countable type then by Lemma 1.4a there exists an injective map $\tilde{f} : \Omega \rightarrow \hat{X}$ which is either order preserving or order reversing. Without loss of generality assume that \tilde{f} is an order map. Define $f : \Omega \rightarrow X$ by

$$f(\beta) = \begin{cases} \sup \{\tilde{f}(i) : i < \beta\} & \text{if } \beta \text{ is a limit ordinal} \\ \tilde{f}(\beta) & \text{otherwise.} \end{cases} \tag{1.16}$$

It is easy to check that f is a continuous, injective order map.

(f) Assuming that f is not eventually constant, we will define an order map $q : \Omega \rightarrow \Omega$ so that $f \circ q$ is injective. We consider the case when f is order preserving. Define $q(0) = 0$. If q has been defined for all $j < i$ then let $i^* = \sup \{q(i) : i < j\}$ and define $q(i) = \min \{k \in \Omega : f(k) > f(i^*)\}$. This set is nonempty because f is not eventually constant. Clearly, $f \circ q$ is strictly increasing and so is an order injection. By Lemma 1.4 X is not of countable type. \square

Proposition 1.6. *Let $\{X_i : i \in \alpha\}$ be a family of nonempty LOTS indexed by a positive ordinal α . If α is countable then the order space product $\prod_{i \in I} X_i$ is of countable type iff each X_i is of countable type. If α is uncountable and each X_i is nontrivial then $\prod_{i \in \alpha} X_i$ is not of countable type.*

Proof. Notice first that each factor X_i and subproduct $\prod_{i \in \beta} X_i$ for $\beta < \alpha$ can be order injected into $\prod_{i \in \alpha} X_i$ and so they are of countable type when the product is.

Now assume that α is a countable ordinal. We prove by induction that the product is of countable type when the factors are. If $\alpha = 1$ then the product Π is X_0 and so the product is of countable type if X_0 is.

Now assume, inductively, the result for all $\beta < \alpha$ and let $f : \Omega \rightarrow \prod_{i \in \alpha} X_i$ be an order map. We will prove that f is not injective. For each $\beta < \alpha$ we can apply the inductive hypothesis and Lemma 1.4f to see that each $\pi_\alpha^\beta \circ f : \Omega \rightarrow \prod_{i \in \beta} X_i$ is eventually constant, i.e. there exists $\epsilon(\beta)$ such that $\pi_\alpha^\beta \circ f$ is constant on the tail $\Omega \setminus \epsilon(\beta)$.

Case i: Assume $\alpha = \beta + 1$; then the restriction of f to the tail $\Omega \setminus \epsilon(\beta)$ is constant on the coordinates $i < \beta$ and $j \mapsto f(j)_\beta$ is an order map to the LOTS X_β which is of countable type. Since $\Omega \setminus \epsilon(\beta) \cong \Omega$ it follows from Lemma 1.4f that f is eventually constant on the β coordinate as well.

Case ii: Assume now that α is a limit ordinal. Because α is a countable ordinal $\epsilon(\alpha) =_{def} \sup \{\epsilon(\beta) : \beta < \alpha\}$ is a countable ordinal. Clearly, f is constant on the tail $\Omega \setminus \epsilon(\alpha)$.

Together with a similar argument for Ω^* , this shows that $\prod_{i \in \alpha} X_i$ is of countable type.

This completes the induction for the case when α is assumed to be countable.

Finally, suppose that each X_i is nontrivial. There exist $a, b \in \prod_{i \in \alpha} X_i$ such that $a_i < b_i$ for all $i \in \alpha$. If we define $f : \alpha + 1 \rightarrow \prod_{i \in \alpha} X_i$ by

$$f(\beta)_i = \begin{cases} b_i & i < \beta \\ a_i & \beta \leq i < \alpha. \end{cases} \quad (1.17)$$

then f is an order injection and if α is uncountable then it restricts to an order injection of Ω into Π . \square

2. COMPLETE HOMOGENEOUS LOTS

If a group G acts on a set S then we call s_1 and s_2 in S G -equivalent if $g(s_1) = s_2$ for some $g \in G$. We say that G acts transitively on S when all points are G equivalent.

For a topological space X we let $H(X)$ denote the automorphism group of X , i.e. the group of homeomorphisms from X to itself. We call X topologically homogeneous if $H(X)$ acts transitively on X . If X is a LOTS then any order automorphism of X , i.e. order isomorphism from X to itself, is a homeomorphism. We denote by $H_+(X)$ the subgroup of order automorphisms and by $H_\pm(X)$ the subgroup of bijections which either preserve or reverse order. If X admits an order reversing homeomorphism then we call X a symmetric LOTS in which case $H_+(X)$ is a subgroup of $H_\pm(X)$ of index 2. Otherwise, $H_+(X) = H_\pm(X)$.

A LOTS X is called \pm transitive if $H_\pm(X)$ acts transitively on X and transitive if $H_+(X)$ acts transitively on X . X is doubly transitive if $H_+(X)$ acts transitively, via the diagonal action, on $\{(x_1, x_2) : x_1 < x_2\} \subset X \times X$.

We call X a homogeneous LOTS, written HLOTS, if X is a nontrivial LOTS and it is order isomorphic with every nonempty, open, convex subset of itself. If a HLOTS is complete then we call it a CHLOTS, a complete homogeneous LOTS. Otherwise, we call it an IHLOTS, an incomplete homogeneous LOTS.

The motivating example of an IHLOTS is the set of rationals \mathbf{Q} with completion the CHLOTS \mathbf{R} .

Proposition 2.1. *Let X be a nontrivial LOTS.*

- (a) X is a HLOTS iff X is unbounded, is of countable type, and any two nonempty, bounded open intervals in X are order isomorphic.
- (b) If X is a HLOTS then it is order dense, σ -bounded, first countable, transitive and doubly transitive.
- (c) The completion \hat{X} of a HLOTS X is a CHLOTS. A HLOTS X is a CHLOTS iff $X = \hat{X}$. If X is a CHLOTS then it is connected and σ -compact.
- (d) A HLOTS X is an IHLOTS iff it is a proper subset of \hat{X} . If X is an IHLOTS then it has dense holes. Furthermore, $\hat{X} \setminus X$ is then an IHLOTS which is dense in \hat{X} , and so the completion of $\hat{X} \setminus X$ is \hat{X} .

Proof. (a),(b) For the duration of the proof of these parts call X a HLOTS' if it is unbounded and any pair of nonempty, open, bounded intervals in X are order isomorphic.

An isolated point p of a LOTS X is a nonempty, open, convex subset of X and if p is neither the *max* nor the *min* of X then $\{p\}$ is a bounded, open interval. Since X is nontrivial it is not isomorphic to $\{p\}$. Thus, if X is a HLOTS or a HLOTS' then it has no isolated points and so is infinite. Now assume that X is a HLOTS or a HLOTS' and $x \in X \setminus \{max, min\}$. Choose $a, b \in X$ with $a < x < b$. Because x is not isolated, either (x, b) is nonempty and has no *min* or (a, x) is nonempty and has no *max*. Assume the first. Then no bounded, nonempty open interval in X has a *min* and in the HLOTS case no nonempty, open convex set, including X , has a *min*. If $x_1 < x_2$ were a gap pair in X and $c > x_2$ (or $c = \infty$ in the HLOTS case), then (x_1, c) would have a *min*. Thus, there are no gap pairs and so X is order dense. Hence, (a, x) is nonempty and has no *max*. We have proved that a HLOTS is a HLOTS' and that a HLOTS' is order dense.

If X is a HLOTS then by Proposition 1.5 we can choose an order injection f from the LOTS \mathbf{Z} of integers into X and define

$$I \stackrel{def}{=} \{x \in X : f(i) < x < f(j) \text{ for some } i, j \in \mathbf{Z}\}. \quad (2.1)$$

Clearly, I is a nonempty, open, convex subset of X with a countable cofinal and coinital subset. Hence, every nonempty, open, convex subset of X has a countable cofinal and coinital subset, i.e. X is of countable type and so is first countable and σ -bounded.

If $a < b$ and $c < d$ in a HLOTS X then we can put together isomorphisms

$$(-\infty, a) \cong (-\infty, c), \quad (a, b) \cong (c, d), \quad (b, \infty) \cong (d, \infty) \quad (2.2)$$

to get an element of $H_+(X)$ which maps the pair a, b to the pair c, d . Hence a HLOTS is doubly transitive. Since X is unbounded, it is transitive.

It remains to show that if X is a HLOTS' of countable type then X is a HLOTS. Since we have shown that a HLOTS' is order dense, countable type implies that any nonempty, open, convex set J admits an order injection $f : \mathbf{Z} \rightarrow J$ with image cofinal and coinital.

If $\tilde{f} : \mathbf{Z} \rightarrow \tilde{J}$ is a similar order injection to another nonempty, open, convex set, then isomorphisms $(f(i), f(i + 1)) \cong (\tilde{f}(i), \tilde{f}(i + 1))$ extend to the endpoints and we can put them together to get an order isomorphism from J to \tilde{J} . Hence, X is a HLOTS.

(c), (d) X is complete iff $X = \hat{X}$. Completeness together with order dense implies connectedness and together with σ -boundedness implies σ -compactness. In general, if X is a HLOTS then by (a) and (b) it is order dense, doubly transitive and of countable type. If J is a nonempty, open, convex subset of \hat{X} then $J \cap X$ is a nonempty, open, convex subset of X with completion J . If $f : X \rightarrow J \cap X$ is an order isomorphism then the completion $\hat{f} : \hat{X} \rightarrow J$ is an order isomorphism. Hence, \hat{X} is a HLOTS and so is a CHLOTS.

Now assume that X is a proper subset of \hat{X} with $z \in \hat{X} \setminus X$. Choose $a, b \in X$ so that $a < z < b$. For any $c < d$ in X there exists $f \in H_+(X)$ mapping the pair a, b to c, d . The completion $\hat{f} \in H_+(\hat{X})$ maps z to a point of $\hat{X} \setminus X$ between c and d . Thus, X has dense holes. That is, $\hat{X} \setminus X$ is dense in \hat{X} and so it is order dense with completion \hat{X} . Now if \tilde{J} is a nonempty, open, convex subset of $\hat{X} \setminus X$ then there exists an open, convex subset J of \hat{X} such that $\tilde{J} = J \cap (\hat{X} \setminus X)$. The completion $\hat{f} : \hat{X} \cong J$ of the isomorphism $f : X \rightarrow J \cap X$ restricts to an isomorphism $\hat{X} \setminus X \cong \tilde{J}$. Thus, $\hat{X} \setminus X$ is a HLOTS, indeed an IHLOTS since X is nonempty. \square

For any LOTS X we define the *Alexandrov-Sorgenfrey double arrow* of X , hereafter the *AS double* of X , to be the order space product

$$X' = X \times \{-1, +1\}, \tag{2.3}$$

regarding $\{-1, +1\}$ as a two point LOTS. For x in X we denote by x^\pm the pair $(x, \pm 1)$ and we define the first coordinate projection map

$$\begin{aligned} \pi' : X' &\rightarrow X \\ \pi'(x^\pm) &= x. \end{aligned} \tag{2.4}$$

Clearly, we have for $a < b$ in X :

$$\begin{aligned} (\pi')^{-1}((a, b)) &= (a^+, b^-) \\ (\pi')^{-1}([a, b]) &= [a^-, b^+]. \end{aligned} \tag{2.5}$$

In particular, π' is a continuous, surjective order map.

For each $x \in X$, $x^- < x^+$ is a gap pair in X' so that x^- is a left endpoint and x^+ is a right endpoint. x^- is also a right endpoint, and so is an isolated point of X' , iff x is a right endpoint in X , i.e. $[x, \infty)$ is open. For if $[x, \infty)$ is open then by (2.5), $\{x^-\} = (-\infty, x^+) \cap [x^-, \infty)$ is open, while if $A \subset X \setminus \{x\}$ with $x = \sup A$ then $x^- = \sup (\pi')^{-1}(A)$. Similarly, x^+ is isolated iff x is a left endpoint in X . In particular, if X has a $\max = M$ (or a $\min = m$) then M^+ (resp. m^-) is an isolated point in X' .

If $f : X_1 \rightarrow X_2$ is an order injection then we define

$$\begin{aligned} f' : X'_1 &\rightarrow X'_2 \\ f'(x^\pm) &= f(x)^\pm. \end{aligned} \tag{2.6}$$

Clearly, f' is the unique order injection such that the diagram

$$\begin{array}{ccc} X'_1 & \xrightarrow{f'} & X'_2 \\ \pi' \downarrow & & \downarrow \pi' \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

commutes. If f is continuous then f' is. If f is not injective then the map defined by (2.6) does not preserve order.

If $r : X_1 \rightarrow X_2$ is an injective order* map then we define the injective order* map

$$\begin{aligned} r^* : X'_1 &\rightarrow X'_2 \\ r^*(x^\pm) &= r(x)^\mp. \end{aligned} \tag{2.7}$$

If f (or r) is bijective, then f' (resp. r^*) is.

Lemma 2.2. *Let X be an order dense, unbounded LOTS.*

- (a) *Let X_1 be a LOTS and $g : X' \rightarrow X'_1$ be a homeomorphism. The LOTS X_1 is unbounded and order dense. If g is an order isomorphism then there exists $f : X \rightarrow X_1$ an order isomorphism such that $g = f'$. If g is order reversing then there exists $r : X \rightarrow X_1$ an order reversing homeomorphism such that $g = r^*$.*
- (b) *Assume that X is complete and so is connected. The bounded clopen subintervals of X' are compact sets which form a basis for X' , and so X' is zero-dimensional. If C is any nonempty, bounded, clopen subset of X' then there is a unique finite sequence $a_1 < b_1 < a_2 \dots < b_n$ in X such that*

$$C = \bigcup_{i=1}^n [a_i^+, b_i^-]. \tag{2.8}$$

Proof. (a) X' has no isolated points and so X'_1 has none. Hence, X_1 has no left or right endpoints. If g is either order preserving or order reversing then gap pairs are mapped to gap pairs. Hence g induces a bijection from X to X_1 which preserves or reverses order according to which g does.

(b) Since X is connected and unbounded, any bounded clopen interval of X' is of the form $[a^+, b^-]$ with $a < b$ in X . These form a basis for X' . Since C is open it is a union of such subintervals and since C is compact it is a finite union of them. If two such intervals overlap, or if the *max* of one and the *min* of another form a gap pair, then the union of the two is a clopen interval. Combining in this way we obtain C as the finite, disjoint union of clopen intervals with points of $X' \setminus C$ between any two successive intervals, i.e. (2.8) holds. Furthermore, the intervals $[a_i, b_i]$ are the components of the image $\pi'(C)$ in X . Uniqueness follows. \square

If X is a complete, unbounded LOTS, then its *two-point compactification*, denoted $\bullet X \bullet$, is the LOTS obtained by attaching a *max* and a *min*

$$\bullet X \bullet = \{m\} \cup X \cup \{M\}, \tag{2.9}$$

with the obvious ordering.

Proposition 2.3. *If F is a CHLOTS and C is a nonempty, bounded, clopen subset of F' then*

$$C \cong \bullet F' \bullet. \tag{2.10}$$

Proof. Let $a_1 < b_1 < a_2 \dots < b_n$ be the sequence in F such that (2.8) holds. Let $f_i : [a_i, b_i] \cong [a_i, a_{i+1}]$ for $i = 1, \dots, n-1$. Each f'_i restricts to an isomorphism $[a_i^+, b_i^-] \cong [a_i^+, a_{i+1}^-]$. Put them together to get an isomorphism:

$$C \cong \left(\bigcup_{i=1}^{n-1} [a_i^+, a_{i+1}^-] \right) \cup [a_n^+, b_n^-] = [a_1^+, b_n^-]. \tag{2.11}$$

If $g : F \rightarrow (a_1, b_n)$ is an order isomorphism then so is

$$g' : F' \rightarrow (a_1, b_n)' = (a_1^+, b_n^-). \tag{2.12}$$

Attaching the endpoints we obtain an isomorphism $\bullet F' \bullet \cong [a_1^+, b_n^-]$. □

We now describe the *Tower of HLOTS* which can be built upon a HLOTS X .

In each HLOTS X we pick out a distinguished nontrivial, closed, bounded subinterval J whose endpoints we will label ± 1 . Thus, $J = [-1, +1]$ and its interior $J^\circ = (-1, +1)$. Because X is homogeneous it is order isomorphic to J° .

For every positive ordinal α we define the subset of the order space product X^α

$$X_\alpha = \{x \in X^\alpha : x_i \in J \text{ for all } 0 < i < \alpha\}. \tag{2.13}$$

Thus, X_α is the order space product indexed by α with the first factor X and the remaining factors copies of J . In particular,

$$\begin{aligned} X_1 &= X \\ X_2 &= X \times J \\ X_\omega &= X \times J \times J \times \dots \end{aligned} \tag{2.14}$$

For $0 < \beta \leq \alpha$ we have projections $\pi_\alpha^\beta : X_\alpha \rightarrow X_\beta$. Identifying X_1 with X we have the special case $\pi_\alpha : X_\alpha \rightarrow X$, the projection to the first coordinate.

Following (1.8) we define for $z \in X_\beta$ with $0 < \beta < \alpha$ the points $z+$ and $z-$ in X_α by

$$(z\pm)_i = \begin{cases} z_i & i < \beta \\ \pm 1 & \beta \leq i < \alpha. \end{cases} \tag{2.15}$$

As in (1.9) we have for $a < b$ in X_β

$$\begin{aligned} (\pi_\alpha^\beta)^{-1}((a, b)) &= (a+, b-) \\ (\pi_\alpha^\beta)^{-1}([a, b]) &= [a-, b+]. \end{aligned} \tag{2.16}$$

Using (2.15) we define for $0 < \beta < \alpha$

$$\begin{aligned} j_\beta^\alpha : (X_\beta)' &\rightarrow X_\alpha \\ j_\beta^\alpha(z^\pm) &= z \pm. \end{aligned} \tag{2.17}$$

Proposition 2.4. *Let X be a HLOTS with distinguished subinterval $J = [-1, +1]$ and let α be a positive ordinal.*

- (a) X_α is unbounded, order dense and σ -bounded.
- (b) If X is an IHLOTS then X_α has dense holes.
- (c) If X is a CHLOTS then X_α is connected and σ -compact.
- (d) If α is countable then X_α is of countable type. If α is uncountable then X_α is not even first countable.
- (e) If $0 < \beta < \alpha$ then π_α^β is a continuous order surjection and j_β^α is an order embedding onto a closed subset of X_α .
- (f) If $a < b$ in X then $(a+, b-) \subset X_\alpha$ is order isomorphic with X_α itself.

Proof. (a) X_α is order dense by Proposition 1.3a which also shows that the π_α^β 's are continuous order surjections. Since X is unbounded and σ -bounded, X_α is by Proposition 1.1b applied to the order surjection $\pi_\alpha : X_\alpha \rightarrow X$.

(b) If $z < w$ in X_α let $\beta = \min\{j : z_j \neq w_j\}$ so that $z_\beta < w_\beta$ and $z_i = w_i$ for $i < \beta$. Since X has dense holes by Proposition 2.1d there exists a clopen subset A of X such that $z_\beta \in A, w_\beta \notin A$, and $x \in A \Rightarrow (-\infty, x) \subset A$. Define

$$\tilde{A} = \{x \in X_\alpha : x < z\} \cup \{x \in X_\alpha : x_i = z_i \text{ for } i < \beta \text{ and } x_\beta \in A\}. \tag{2.18}$$

It is clear that \tilde{A} defines a hole in X_α between z and w .

(c) If X is complete then J is compact and so X_α is complete by Proposition 1.3b. As X_α is order dense and σ -bounded it is connected and σ -compact.

(d) Since X and J are of countable type the result follows from Proposition 1.6 when α is countable. When α is uncountable, define a, b by

$$a_i = -1 \quad \text{and} \quad b_i = +1 \quad \text{for } i < \alpha. \tag{2.19}$$

In that case, $f : \alpha+1 \rightarrow X_\alpha$ defined by (1.17) is an order embedding and so $f(\Omega)$ is not the limit of an increasing sequence. Thus, X_α is not first countable.

(e) The complement of the image of j_β^α is the union of the collection of open intervals $\{(z-, z+) : z \in X_\beta\}$. To prove continuity let $a, b \in X_\alpha \setminus j_\beta^\alpha(X_\beta)$.

$$(j_\beta^\alpha)^{-1}((a, b)) = ((\pi_\alpha^\beta(a))-, (\pi_\alpha^\beta(b))+), \tag{2.20}$$

from which continuity follows.

(f) If $\tilde{f} : X \rightarrow (a, b)$ is an order isomorphism then

$$f(x)_i = \begin{cases} \tilde{f}(x_0) & i = 0 \\ x_i & 0 < i < \alpha \end{cases} \tag{2.21}$$

defines the required isomorphism $f : X_\alpha \rightarrow (a+, b-)$. □

Theorem 2.5. *Let α be a countable, tail-like ordinal. If X is an IHLOTS then X_α is an IHLOTS. If X is a CHLOTS then X_α is a CHLOTS.*

Proof. By Proposition 2.4 X_α has dense holes if X is an IHLOTS and it is complete if X is a CHLOTS. Also, X_α is unbounded and since α is countable, X_α has countable type. If $\alpha = 1$ then $X_\alpha = X$ which is a HLOTS. Thus, we can assume that $\alpha > 1$ and so that it is a limit ordinal.

Choose $0 \in J^\circ$ so that $-1 < 0 < +1$. Define $\tilde{a} < \tilde{c} < \tilde{b}$ in X_α by

$$\begin{aligned} \tilde{a}_0 &= -1 & \tilde{c}_0 &= 0 & \tilde{b}_0 &= +1 \\ \tilde{a}_i &= +1 & \tilde{c}_i &= 0 & \tilde{b}_i &= -1 \quad \text{for } 0 < i < \alpha. \end{aligned} \tag{2.22}$$

Given an arbitrary pair $a < b$ in X_α it suffices by Proposition 2.1a to prove $(a, b) \cong (\tilde{a}, \tilde{b})$.

Let $\beta = \min \{j : a_j \neq b_j\}$ so that $a_\beta < b_\beta$ and $a_j = b_j$ for $j < \beta$. Because X is order dense we can choose $c \in X_\alpha$ such that

$$\begin{aligned} a_i &= c_i = b_i & \text{for } i < \beta \\ a_\beta &< c_\beta < b_\beta & \text{for } i = \beta \\ c_i &= 0 & \text{for } \beta < i < \alpha. \end{aligned} \tag{2.23}$$

We will construct an order isomorphism $[c, b] \cong [\tilde{c}, \tilde{b}]$. Applying a similar argument (or the same argument to X^*) we obtain an isomorphism $(a, c] \cong (\tilde{a}, \tilde{c}]$. Putting them together we get $(a, b) \cong (\tilde{a}, \tilde{b})$ as required.

Now define

$$\begin{aligned} K &= \{\beta\} \cup \{k : \beta < k < \alpha \text{ and } b_k > -1 \text{ in } J\} \\ K'' &= K \cup \{\alpha\}. \end{aligned} \tag{2.24}$$

Thus, K'' is a subset of $\alpha + 1$ and so is a countable well-ordered set. We define $f : K'' \rightarrow X_\alpha$ by

$$\begin{aligned}
 f(\beta) &= c \\
 f(k)_i &= \begin{cases} b_i & i < k \\ -1 & k \leq i < \alpha \end{cases} & (2.25) \\
 &\text{for } \beta < k \leq \alpha.
 \end{aligned}$$

In particular, $f(\alpha) = b$. For $k \in K''$ let k'' denote its successor in the well-ordered set K'' . For $k \in K'' \setminus \{\beta\}$ we have

$$\begin{aligned}
 f(k)_i &= f(k'')_i = b_i & \text{for } i < k \\
 -1 &= f(k)_k < f(k'')_k = b_k \\
 f(k)_i &= -1 = b_i = f(k'')_i & \text{for } ki < k'' \\
 f(k)_i &= -1 = f(k'')_i & \text{for } k'' \leq i < \alpha.
 \end{aligned} \tag{2.26}$$

Thus, f is an order injection.

If k is a limit element of K and $x \in X_\alpha$ with $x < f(k)$ then for some $j < k$, $x_j < f(k)_j$. So either $j \leq \beta$ or $j \in K$. In either case, there exists some $\tilde{k} < k$ in K such that $x < f(\tilde{k})$. It follows that f is continuous and so Proposition 1.5c implies that

$$[c, b) = \bigcup_{k \in K} [f(k), f(k'')). \tag{2.27}$$

By Proposition 1.5d there exists an order embedding $\tilde{f}_0 : K'' \rightarrow [0, +1]$ with $\tilde{f}_0(\beta) = 0$ and $\tilde{f}_0(\alpha) = +1$. Now define $\tilde{f} : K'' \rightarrow X_\alpha$ by

$$\begin{aligned}
 \tilde{f}(k)_0 &= \tilde{f}_0(k) & \text{for } k \in K'' \\
 \tilde{f}(\beta)_i &= 0 & \text{for } 0 < i < \alpha \\
 \tilde{f}(k)_i &= -1 & \text{for } 0 < i < \alpha \text{ and } k \in K'' \setminus \{\beta\}.
 \end{aligned} \tag{2.28}$$

Clearly, \tilde{f} is an order embedding and again continuity implies

$$[\tilde{c}, \tilde{b}) = \bigcup_{k \in K} [\tilde{f}(k), \tilde{f}(k'')). \tag{2.29}$$

Notice that $\tilde{f}(\beta) = \tilde{c}$ and $\tilde{f}(\alpha) = \tilde{b}$.

Because X is a HLOTS we can choose for each $k \in K$ an order isomorphism between intervals in J :

$$q_k : [f(k)_k, f(k'')_k] = [-1, b_k] \rightarrow [\tilde{f}(k)_0, \tilde{f}(k'')_0] = [\tilde{f}_0(k), \tilde{f}_0(k'')]. \quad (2.30)$$

Because α is tail-like there exists for each $k \in K$ a unique order isomorphism $\tau^k : \alpha \rightarrow \alpha \setminus k = \{\epsilon : k \leq \epsilon < \alpha\}$. Define for each $k \in K$ the map between intervals of X_α

$$Q_k : [f(k), f(k'')] \rightarrow [\tilde{f}(k), \tilde{f}(k'')] \\ Q_k(x)_i = \begin{cases} q_k(x_k) & \text{for } i = 0 \\ x_{\tau^k(i)} & \text{for } 0 < i < \alpha. \end{cases} \quad (2.31)$$

Notice that by (2.26) $x_i = b_i$ for all $i < k$ when $x \in [f(k), f(k'')]$. It follows that each Q_k is an order isomorphism.

Putting together these isomorphisms we obtain the required isomorphism $[c, b] \cong [\tilde{c}, \tilde{b}]$. \square

Remark. If F is a CHLOTS and α is a positive ordinal such that F_α is a HLOTS, and so is of countable type, then α is countable by Proposition 1.5. It is not hard to show that α must be tail-like as well.

In distinguishing between CHLOTS we define a rough order of size.

Definition 2.6. For LOTS X and X_1 we say that X *injects into* X_1 if there exists an order injection $g : X \rightarrow X_1$. We say that X_1 is *bigger than* X if X injects into X_1 but not the reverse. On the other hand, we say that X *has the same size as* X_1 when each injects into the other. Finally, we say that X *has size between* X_1 and X_2 when X_1 injects into X which injects into X_2 .

This is the usual crude partial ordering used to compare order types. We will now see that for CHLOTS F and F_1 , F injects into F_1 iff there exists an order surjection $f : F_1 \rightarrow F$. By Proposition 1.1a such a surjection is always continuous. On the other hand, there exists a continuous order injection, i.e. an order embedding, $g : F \rightarrow F_1$ iff $F \cong F_1$.

Lemma 2.7.

- (a) *If $f : X_1 \rightarrow X$ is an order surjection of LOTS then there exists a map $g : X \rightarrow X_1$ such that $f \circ g = 1_X$. Any such map g is an order injection.*

- (b) Let $g : X \rightarrow X_1$ be an order map. Assume that X is order dense and D is a dense subset of X . If the restriction $g|_D$ is injective, then g is an order injection, i.e. it is injective on all of X .
- (c) Assume that X is unbounded, order dense and that X_1 is complete. If X injects into X_1 then the completion \hat{X} injects into X_1 .
- (d) Assume that F and F_1 are CHLOTS and that F injects into F_1 . There then exists a, necessarily continuous, order surjection from F_1 onto F . Furthermore, for any ordinal α F_α injects into $(F_1)_\alpha$.
- (e) Assume that F and F_1 are CHLOTS. If there exists a continuous, nonconstant order map from F_1 to F then F injects into F_1 . If there exists an order embedding of F into F_1 then $F \cong F_1$.

Proof. (a) We can define $g(x)$ by choosing any element of $f^{-1}(x)$. Such choices exactly define the functions g such that $f \circ g = 1_X$. In that case, if $g(x_1) \leq g(x_2)$ then $x_1 = f(g(x_1)) \leq f(g(x_2)) = x_2$. Contrapositively, $x_1 > x_2$ implies $g(x_1) > g(x_2)$.

(b) If $a < b$ in X with $g(a) = g(b)$ then $(a, b) \cap D$ is an infinite subset of D which is mapped to $g(a)$.

(c) If $g : X \rightarrow X_1$ is an order injection then extend g to \hat{X} by

$$g(z) = \sup g(X \cap (-\infty, z]) \tag{2.32}$$

This is clearly order preserving and it is injective by (b).

(d) Let $g : F \rightarrow F_1$ be an order injection.

Define $J = \{y \in F_1 : \text{For some } x_1, x_2 \in F \quad g(x_1) < y < g(x_2)\}$. J is an open, convex subset of F_1 . Because F has no *max* or *min*, $g(F)$ is a subset of J which is both cofinal and cointial in J . Because F_1 is a HLOTS there exists an order isomorphism $q : J \rightarrow F_1$. By replacing g by $q \circ g$, we can assume that $g(F)$ is cofinal and cointial in F_1 .

Define for $x \in F$ the closed interval J_x by

$$F_1 \setminus J_x = (\bigcup\{(-\infty, g(a)) : a < x\}) \cup (\bigcup\{(g(b), \infty) : b > x\}). \tag{2.33}$$

The interval J_x can be trivial but it is nonempty because F_1 is connected.

If $x_1 < x_2$ in F then because F is order dense we can choose a, b such that $x_1 < b < a < x_2$. Since $g(b) < g(a)$

$$(-\infty, g(a)) \cup (g(b), \infty) = F_1 \quad (2.34)$$

and so $y_1 \in J_{x_1}$ and $y_2 \in J_{x_2}$ imply $y_1 < y_2$. In particular, $J_{x_1} \cap J_{x_2} = \emptyset$.

If $y \in F_1$ equals $g(x)$ then $y \in J_x$. If $y \notin g(F)$ then because the image is cofinal and cointial, the pair $g^{-1}((-\infty, y)), g^{-1}((y, \infty))$ is a partition of F by nonempty convex sets. By completeness we can define $x = \inf g^{-1}((y, \infty))$. If $g(b) < y$ then b is a lower bound for $g^{-1}((y, \infty))$ and so $b \leq x$. Contrapositively, $b > x$ implies $g(b) > y$. If $g(a) > y$ then $a \in g^{-1}((y, \infty))$ and so $x \leq a$. Contrapositively, $a < x$ implies $g(a) \leq y$. Thus, $y \in J_x$.

It follows that $\{J_x : x \in F\}$ is an F indexed family of nonempty, closed intervals with union F_1 . So mapping J_x to x defines an order surjection $f : F_1 \rightarrow F$ which is continuous by Proposition 1.1a. If $x \in F$ then $g(x) \in J_x$ implies $f(g(x)) = x$.

If α is a positive ordinal then choose the distinguished interval $J_1 \subset F_1$ so that $g(J) \subset J_1$. The obvious product of copies of g defines an order injection from F_α into $(F_1)_\alpha$.

(e) If $f : F_1 \rightarrow F$ is a continuous order map and $c < d$ in the image $f(F_1)$ then let $a = \sup f^{-1}(c)$ and $b = \inf f^{-1}(d)$. The interval $[a, b]$ is compact and connected and so by continuity of f the image is as well. Since f is an order map $f([a, b])$ is a compact, connected subset of $[c, d]$ which contains c and d . Hence, $f([a, b]) = [c, d]$ and so, by definition of a and b , $f((a, b)) = (c, d)$. Let $h_1 : F_1 \rightarrow (a, b)$ and $h : (c, d) \rightarrow F$ be order isomorphisms. Then $f_1 = h \circ f \circ h_1$ is an order surjection of F_1 onto F and so F injects into F_1 by (a).

If, in addition, f is injective then the restriction to (a, b) is an order isomorphism to (c, d) and so f_1 is an isomorphism. \square

We will now show that for a CHLOTS F and positive ordinals $\alpha > \beta$ it is always true that F_α is bigger than F_β .

Definition 2.8. We call a LOTS X *order simple* if X' is bigger than X where X' is the AS double of X .

The map π' of (2.4) is an order surjection of X' onto X and so Lemma 2.7a implies that X injects into X' . So X is order simple when X' does not inject into X .

Lemma 2.9.

- (a) *If X is a separable, uncountable, order dense LOTS then X is order simple.*
- (b) *If X_1 and X_2 are LOTS of the same size then X_1 is order simple iff X_2 is.*
- (c) *If X is order simple then the reverse X^* is order simple.*
- (d) *If X is order simple then there does not exist an injective order* map from X' into X .*

Proof. (a) If $f : X'_1 \rightarrow X$ is an order injection for any LOTS X_1 then $\{(f(z-), f(z+)) : z \in X_1\}$ is a family of open intervals in X and each is nonempty if X is order dense. If $z_1 < z_2$ in X_1 then $f(z_1+) < f(z_2-)$ so that the intervals are pairwise disjoint. If X is separable then X_1 must be countable.

(b) If $f : X_2 \rightarrow X_1$ is an order injection then from (2.6) we obtain the order injection $f' : X'_2 \rightarrow X'_1$. So if X_2 injects into X_1 then X'_2 injects into X'_1 . Thus, if X_1 and X_2 have the same size and X'_1 has the same size as X_1 then X'_2 has the same size as well.

(c) It is clear that

$$(X^*)' = (X')^*. \tag{2.35}$$

So any order injection of X' into X is an order injection of $(X^*)'$ into X^* .

(d) Define the map

$$\begin{aligned} q : X' &\rightarrow (X')' \\ q(x^\pm) &= (x^\pm)^\pm. \end{aligned} \tag{2.36}$$

This is an order embedding of X' onto the set whose complement is the open set of isolated points $\{(x^+) ^-, (x^-) ^+\} : x \in X$. Now if $g : X' \rightarrow X$ is an order reversing injection then we use (2.7) to define an order injection from X' to X as the composition:

$$X' \xrightarrow{q} (X')' \xrightarrow{g^*} X' \xrightarrow{g} X. \quad \square$$

It is easy to check that $\mathbf{Z}' \cong \mathbf{Z}$ and so \mathbf{Z} is not order simple. By Proposition 1.5a \mathbf{Q} is not order simple. There exist connected LOTS which are not order simple as well. However, such examples are not compact.

Lemma 2.10. (The Shift Lemma) *Let X be a complete LOTS with $\min = m$. If $f : X' \rightarrow X$ is an order injection then for all $x \in X$*

$$f(x^+) > x. \quad (2.37)$$

In particular, X has no \max .

Proof. Define $S = \{a \in X : f(x^+) > x \text{ for all } x \leq a\}$. Hence, $a \in S$ implies $(-\infty, a] \subset S$. Since $m = \min X$

$$m \leq f(m^-) < f(m^+) \quad (2.38)$$

and so $m \in S$. It suffices to show that S is unbounded as this implies $S = X$.

Assume S is bounded and let $z = \sup S$.

First we show that $z \in S$. If not then $a < z$ for all $a \in S$ and so $a^+ < z^-$. Hence

$$a < f(a^+) < f(z^-) < f(z^+). \quad (2.39)$$

Thus, $f(z^-)$ is an upper bound for S . Since $z = \sup S$

$$z \leq f(z^-) < f(z^+). \quad (2.40)$$

Furthermore, $z = \sup S$ implies that all $x < z$ are elements of S . It follows that $z \in S$ after all.

Now let $y = f(z^+) > z$. If $z < x \leq y$ then $z^+ < x^+$ and so

$$x \leq y = f(z^+) < f(x^+). \quad (2.41)$$

Thus, $y \in S$. Since $y > z$, this contradicts the assumption that $z = \sup S$.

X has no \max because S is unbounded. Also, $x = \max$ could not satisfy (2.37). \square

Corollary 2.11. *A LOTS X is order simple if it satisfies one of the following conditions:*

- (1) X is compact.
- (2) X is a CHLOTS.
- (3) $X \cong F_\alpha$ where F is any CHLOTS and α is any positive ordinal.

Proof. (1) The Shift Lemma implies that if a complete LOTS X is not order simple and has a \min then it has no \max . In particular, a compact LOTS is order simple.

(2) This is just case (3) with $\alpha = 1$.

(3) For the closed interval J in F , the subset J^α of F_α is compact and so is order simple. If $a < b$ in J then Proposition 2.4f implies that F_α is order isomorphic with the subset $(a+, b-)$ of J^α . Hence, F_α is order simple by Lemma 2.9b. \square

Theorem 2.12. *Assume that F is a CHLOTS. If $\alpha > \beta$ are positive ordinals then F_α is bigger than F_β and F_α is not homeomorphic to F_β .*

Proof. If $f : F_\alpha \rightarrow F_\beta$ is an injective map then the composite $f \circ j_\beta^\alpha : F_\beta' \rightarrow F_\beta$ is an injection which is order preserving or order reversing if f is. Because F_β is order simple by Corollary 2.11, f cannot be order preserving and by Lemma 2.9d it cannot be order reversing either. If f were a homeomorphism then by Lemma 1.2 it would be either order preserving or reversing. \square

For any CHLOTS F the associated *Cantor Space for F* , denoted $C(F)$, is the two point compactification

$$C(F) =_{def} \bullet F' \bullet \tag{2.42}$$

where F' is the AS double. For any positive ordinal α we define

$$C_\alpha(F) = \bullet (F_\alpha)' \bullet \tag{2.43}$$

so that $C_1(F) = C(F)$. Observe that for $a < b$ in F the isomorphism $f : F_\alpha \rightarrow (a+, b-)$ of Proposition 2.4f induces the isomorphism $f' : (F_\alpha)' \rightarrow ((a+)^+, (b-)^-)$ which extends to the two-point compactification to show

$$C_\alpha(F) \cong [(a+)^+, (b-)^-]. \tag{2.44}$$

Theorem 2.13. *Assume that F is a CHLOTS. If $\alpha > \beta$ are positive ordinals then $C_\alpha(F)$ is bigger than F_α which is bigger than $C_\beta(F)$.*

Proof. Because F_α is order simple, there is no order injection from $(F_\alpha)'$ into it. C_α projects onto F_α and contains $(F_\alpha)'$. Hence, C_α is bigger than F_α .

By (2.43) C_β is the same size as $(F_\beta)'$. Now choose $a < b$ in $J^\circ \subset F$. Clearly, we have

$$\begin{aligned} \{-1, +1\} \times \{-1, +1\} &\cong \{-1, a, b, +1\} && \text{and so} \\ ((F_\beta)')' &\cong F_\beta \times \{-1, a, b, +1\} \subset F_{\beta+1}. \end{aligned} \tag{2.45}$$

Hence, there is an order injection from $(C_\beta)'$ into $F_{\beta+1}$ which injects into F_α . If F_α were to inject into C_β then C_β would not be order simple, contradicting Corollary 2.11. \square

Remark. A homeomorphism between Cantor Spaces of CHLOTS need not be order preserving or reversing even locally and we cannot prove that $C_\alpha(F)$ and $C_\beta(F)$ are topologically distinct. The original Extended Cantor Set $C(\mathbf{R})$ is separable. In general, $C(F)$ is separable iff F is and so iff $F \cong \mathbf{R}$. We have been unable to distinguish topologically among any other Cantor Spaces $C(F)$. Also, we are curious which zero-dimensional, compact Hausdorff spaces are homeomorphic to some Cantor Space $C(F)$. Such a space is first countable, but nonmetrizable, but at the moment that is all we can say. For example, is the ordinary topological product of two CHLOTS Cantor Spaces homeomorphic to some CHLOTS Cantor Space?

The size ordering of the Tower of CHLOTS over a given CHLOTS F is order isomorphic with Ω by the map

$$i \in \Omega \quad \mapsto \quad F_{\omega^i}. \quad (2.46)$$

It is possible to extend the towers further. By using tree constructions, see Todorčević (1984), one can build a tower of CHLOTS indexed by Ω^2 such that $i > j$ in Ω^2 implies F_i is bigger than F_j . We turn now, instead, to a consideration of towers of height Ω which meet only at their common base.

Let X be an IHLOTS with completion $\hat{X} = F$. We might expect that the CHLOTS which is the completion of the IHLOTS X_α would be F_α or possibly F_β with some shift in the ordinal index. We will now see that this is usually not the case.

When α is understood, we will write $\pi : X_\alpha \rightarrow X$ for the projection map $\pi_\alpha : X_\alpha \rightarrow X$ and $\hat{\pi} : \widehat{X_\alpha} \rightarrow F$ for the order surjection which is its completion. Recall from Proposition 2.1d that the complement $Y = F \setminus X$ is an IHLOTS with completion F .

Let J be a bounded, closed interval with endpoints $a \leq b$ in an unbounded, order dense LOTS. $J = [a, b]$ is *trivial* when $a = b$. Otherwise, J is infinite and the interior J° is the open interval (a, b) which is dense in J .

Lemma 2.14. *Let X be an IHLOTS with completion F and complementary IHLOTS $Y = F \setminus X$. Let α be an ordinal with $\alpha \geq 2$.*

(a) *If $y \in F$ then*

$$P(y) =_{def} \hat{\pi}^{-1}(y) \tag{2.47}$$

is a compact subinterval of \widehat{X}_α . If $y \in Y$ then $P(y)$ is trivial, i.e. it is a singleton. If $y \in X$ then $P(y) \cap X_\alpha$ is infinite and so $P(y)$ is nontrivial.

(b) *Assume that J is a compact interval in \widehat{X}_α . The projection $\hat{\pi}(J)$ is a compact interval in F and*

$$y \in (\hat{\pi}(J))^\circ \Rightarrow P(y) \subset J^\circ. \tag{2.48}$$

Furthermore,

$$\hat{\pi}(J) \text{ is nontrivial} \Leftrightarrow J \text{ is nontrivial and } \hat{\pi}(J) \cap Y \neq \emptyset. \tag{2.49}$$

(c) *If J_1 and J_2 are disjoint compact intervals in \widehat{X}_α then the projections $\hat{\pi}(J_1)$ and $\hat{\pi}(J_2)$ have disjoint interiors in F .*

Proof. (a), (b) By Proposition 1.1a, $\hat{\pi}$ is continuous and topologically proper. Hence, the image of a compact, connected set is a compact, connected set and the preimage of a compact, convex set is a compact convex set. In an unbounded, connected LOTS a subset J is a compact, connected set iff it is a compact, convex set iff it is a closed, bounded interval.

Suppose $J = [a_1, a_2]$ and $x_k = \hat{\pi}(a_k)$ for $k = 1, 2$ so that $\hat{\pi}(J) = [x_1, x_2]$. If $x_1 < y < x_2$ and $b \in P(y)$ then $a_1 < b < a_2$ which proves (2.48).

X_α is dense in its completion and is mapped to X by π which is the restriction of $\hat{\pi}$ to X_α . If $y \in Y$, $P(y) \cap X_\alpha = \emptyset$ and so $P(y)$ has empty interior. That is, $P(y)$ is trivial. On the other hand, if $y \in X$ then $\alpha > 1$ implies that $(\pi)^{-1}(y) = P(y) \cap X_\alpha$ is infinite.

If $\hat{\pi}(J)$ is nontrivial then its interior meets the dense set Y and of course J is nontrivial. On the other hand, if $y \in Y \cap \hat{\pi}(J)$ then $\hat{\pi}(J)$ trivial would imply $J \subset P(y)$ which is trivial since $y \in Y$. Contrapositively, J nontrivial implies $\hat{\pi}(J)$ is nontrivial. Thus, the equivalence of (2.49) holds.

(c) If $y \in (\hat{\pi}(J_1))^\circ \cap (\hat{\pi}(J_2))^\circ$ then (2.48) implies that $P(y) \subset J_1^\circ \cap J_2^\circ$. □

Theorem 2.15. *Let X be an IHLOTS with completion the CHLOTS F and complementary IHLOTS $Y = F \setminus X$. Let α be an ordinal with $\alpha > 1$. In order that \widehat{X}_α be order isomorphic to F_β for β any countable, limit ordinal it is necessary that X be of first category in F , i.e. Y contains a dense G_δ subset of F .*

Proof. Assume that $f : \widehat{X}_\alpha \rightarrow F_\beta$ is an order surjection, which is continuous and topologically proper by Proposition 1.1a. For $y \in Y$ the interval $P(y) \subset \widehat{X}_\alpha$ is trivial and so is its image $f(P(y)) \subset F_\beta$; we will write $f(y)$ for this interval and the unique point contained in it. Applying the projection $\pi_\beta^i : F_\beta \rightarrow F_i$, defined for $i \leq \beta$ we obtain the point $\pi_\beta^i(f(y)) \in F_i$.

Define for each $y \in Y$ and $i < \beta$

$$Q(y, i) =_{def} (\pi_\beta^i \circ f)^{-1}(\pi_\beta^i(f(y))) \subset \widehat{X}_\alpha. \tag{2.50}$$

For any $z \in F_i$ the preimage $(\pi_\beta^i)^{-1}(z)$ is a nontrivial, compact subinterval of F_β . Since f is surjective, each $Q(y, i)$ is a nontrivial, compact subinterval of \widehat{X}_α . Since $y \in \hat{\pi}(Q(y, i))$, (2.49) implies that $(\hat{\pi}(Q(y, i)))^\circ$ is nonempty.

Clearly, for $y_1, y_2 \in Y$ and $i < \beta$

$$\begin{aligned} Q(y_1, i) \cap Q(y_2, i) \neq \emptyset &\implies \pi_\beta^i(f(y_1)) = \pi_\beta^i(f(y_2)) \\ &\implies Q(y_1, i) = Q(y_2, i). \end{aligned} \tag{2.51}$$

For each $i < \beta$ let

$$\begin{aligned} \mathcal{Q}_i &=_{def} \{\hat{\pi}(Q(y, i)) : y \in Y\} \\ O_i &=_{def} \bigcup \{(\hat{\pi}(Q(y, i)))^\circ : y \in Y\}. \end{aligned} \tag{2.52}$$

Since the intervals in \mathcal{Q}_i are nontrivial, the open set O_i is dense in $\bigcup \mathcal{Q}_i$ which contains Y . Since Y is dense in F it follows that for each $i < \beta$ the open set O_i is dense in F . Since F is locally compact and β is countable the set

$$D = \bigcap \{O_i : i < \beta\} \tag{2.53}$$

is a G_δ which is dense in F by the Baire Category Theorem.

Note that if $x \in D$ then for each $i < \beta$ there exists $y \in Y$ such that $x \in (\hat{\pi}(Q(y, i)))^\circ$ and so by (2.48) $P(x) \subset Q(y, i)$ which implies

$$\pi_\beta^i(f(z)) = \pi_\beta^i(f(y)) \quad \text{for all } z \in P(x). \tag{2.54}$$

It follows that $x \in D$ implies that f is constant on the interval $P(x)$. So if f is injective as well as surjective the interval $P(x)$ is trivial for every $x \in D$. By Lemma 2.14a this requires $x \in Y$.

To summarize, if f is an order isomorphism then Y contains the dense, G_δ set D . □

We will conclude by considering examples which are constructed from IHLOTS in \mathbf{R} . First we will see that there are many such.

Proposition 2.16. *Let \mathcal{G} be the countable group of positive, affine transformations of \mathbf{R} with rational coefficients, i.e. of the form $t \mapsto at + b$ with $a, b \in \mathbf{Q}$ and $a > 0$. If X is a nonempty, proper subset of \mathbf{R} which is invariant with respect to the action of \mathcal{G} then X is an IHLOTS. In particular, any proper subfield of \mathbf{R} is an IHLOTS as is its complement.*

Proof. First, assume that \mathbf{Q} meets X and so is contained in X . If $a < b$ and $c < d$ in \mathbf{Q} then some element of \mathcal{G} maps the pair a, b to c, d and this map restricts to an order isomorphism on X . If J is any nonempty, open convex set in X then we can choose $f : \mathbf{Z} \rightarrow J \cap \mathbf{Q}$ and $g : \mathbf{Z} \rightarrow \mathbf{Q}$ with cofinal and coinitial images. Put together the elements of \mathcal{G} which relate $(f(i), f(i + 1)) \cong (g(i), g(i + 1))$.

If \mathbf{Q} is disjoint from X , apply the result to the complement and use Proposition 2.1d. □

Now we examine those F_σ subsets of \mathbf{R} which are HLOTS. In addition to \mathbf{R} itself, any countable, dense subset of \mathbf{R} is by Proposition 1.5a order isomorphic to \mathbf{Q} and the completion of the isomorphism is an order isomorphism on \mathbf{R} . While \mathbf{Q}_ω is not countable, it is separable and so we have:

$$\widehat{\mathbf{Q}} \cong \mathbf{R} \cong \widehat{\mathbf{Q}_\omega}. \tag{2.55}$$

There is one other order equivalence class of F_σ IHLOTS in \mathbf{R} .

We call a subset A of \mathbf{R} a *Mycielski set* if it is a countable union of Cantor sets. By the Baire Category Theorem the complement of a Mycielski set is a dense, G_δ subset of \mathbf{R} . It can be proved that a dense Mycielski set is an IHLOTS with completion \mathbf{R} and that any two dense Mycielski sets are order isomorphic. Furthermore, if X is a dense Mycielski set then it can be proved that

$$\widehat{X}_\omega \cong \mathbf{R}_\omega. \tag{2.56}$$

Thus, the rationals and the dense Mycielski sets illustrate that the exceptional cases allowed by Theorem 2.15 can occur. The more typical case is given by:

Proposition 2.17. *Let X be a dense subset of \mathbf{R} which is an IHLOTS. If X contains a Cantor set then for every positive ordinal α the HLOTS \mathbf{R}_α , X_α and \widehat{X}_α all have the same size. If, in addition, X is not of first category, i.e. X is not a subset of a Mycielski set, then for α any infinite, tail-like ordinal, the CHLOTS \widehat{X}_α and \mathbf{R}_α have the same size but are not order isomorphic. For example, $X = \mathbf{I}$, the irrationals in \mathbf{R} , is the dense, G_δ IHLOTS complementary to \mathbf{Q} . So for α any infinite, tail-like ordinal, the CHLOTS $\widehat{\mathbf{I}}_\alpha$ and \mathbf{R}_α have the same size but are not order isomorphic.*

Proof. Removing the *min* and the left endpoints from a Cantor set we obtain a LOTS which is order isomorphic to \mathbf{R} and so \mathbf{R} injects into any LOTS which contains a Cantor set. So if $X \subset \mathbf{R}$ contains a Cantor set it is the same size as \mathbf{R} . The results for α then follow from Lemma 2.7c,d. If X is not of first category then X_α is not isomorphic to \mathbf{R}_α by Theorem 2.15.

Since \mathbf{I} is of second category and contains Cantor sets the results apply to it. \square

Given these examples of nonisomorphic CHLOTS with the same size the following result, whose proof we will omit, has some interest.

Theorem 2.18. *Assume that F_1 and F_2 are CHLOTS*

- (a) *If F_1 and F_2 have the same size and if α is a countable, tail-like ordinal with $\alpha \geq \omega^\omega$, then*

$$(F_1)_\alpha \cong (F_2)_\alpha. \quad (2.57)$$

- (b) *If for some positive, countable ordinal β , the size of F_1 lies between F_2 and $(F_2)_\beta$ then the isomorphism of (2.57) holds provided that α is a sufficiently large countable, tail-like ordinal.*

We conclude by describing a result of Hart and van Mill. They call $Y \subset \mathbf{R}$ a *Bi-Bernstein subset*, hereafter a *BB set*, if neither Y nor its complement X contains a Cantor Set or, equivalently, if

both Y and X meet every Cantor Set in \mathbf{R} . From the symmetry of the definition we see that the complement of a BB set is a BB set.

We denote by \mathfrak{c} the cardinal number of \mathbf{R} and so of every Cantor set. Recall that any uncountable G_δ subset of \mathbf{R} admits a complete metric and so contains a nonempty, perfect subset which, in turn, contains a Cantor set.

Lemma 2.19. *Let Y be a BB set.*

- (a) *If A is any uncountable, G_δ subset of \mathbf{R} then $Y \cap A$ has cardinality \mathfrak{c} . In particular, if A is a dense G_δ subset of \mathbf{R} then $Y \cap A$ and $(\mathbf{R} \setminus Y) \cap A$ are dense in \mathbf{R} .*
- (b) *If $f : Y \rightarrow X$ is a order map with X any LOTS, then the image $f(Y)$ is countable or has cardinality \mathfrak{c} .*

Proof. (a) Let C be a Cantor subset of A . There exists a homeomorphism $s : C \times C \rightarrow C$ where $C \times C$ has the usual product topology, ignoring the order structure. For each $x \in C$, $s(C \times \{x\})$ is a Cantor set which meets Y . As x varies over C we obtain a pairwise disjoint family of cardinality \mathfrak{c} which consists of nonempty subsets of $Y \cap A$. In particular, if A is a dense G_δ and I is a nonempty open interval then both Y and its complement meet $A \cap I$.

(b) Let $B = \{x \in X : f^{-1}(x) \text{ contains more than one point}\}$. For $x \in B$, $f^{-1}(x)$ is a nontrivial interval in Y and so B is countable because Y is separable. For each $x \in B$ let $I(x)$ be the smallest closed interval in \mathbf{R} which contains $f^{-1}(x)$, i.e. the convex hull of the closure in \mathbf{R} . Let A_1 be the countable set of endpoints of the intervals $I(x)$ and let A be the complement in \mathbf{R} of union of the intervals $I(x)$. Thus, A is a G_δ subset of \mathbf{R} . If A is countable then $f(Y) = B \cup f(A_1 \cap Y) \cup f(A \cap Y)$ is countable. If A is uncountable, then by (a) $A \cap Y$ has cardinality \mathfrak{c} . Since f is injective on $A \cap Y$ the image has cardinality \mathfrak{c} . \square

Definition 2.20. Let $V \subset \mathbf{R}$ and \mathcal{H} be a nonempty set of subsets of \mathbf{R} . We say that \mathcal{H} is a *Hart-van Mill collection with base set V* when the following conditions hold.

- (i) $\mathbf{Q} \subset V$.
- (ii) Each $Y \in \mathcal{H} \cup \{V\}$ is a BB set, which is \mathcal{G} invariant, where \mathcal{G} be the countable group of positive, affine transformations of \mathbf{R} with rational coefficients.
- (iii) The elements of $\mathcal{H} \cup \{V\}$ are pairwise disjoint.

- (iv) If $Y \in \mathcal{H}$ then $-Y =_{def} \{-x : x \in Y\}$ is an element of \mathcal{H} distinct from Y .
- (v) If $f : \mathbf{R} \rightarrow \mathbf{R}$ is an order map and $Y \in \mathcal{H}$ is such that the the cardinality of $f(Y) \setminus Y$ is \mathbf{c} then the cardinality of $f(Y) \cap V$ is \mathbf{c} .

Remark. If $f : Y \rightarrow \mathbf{R}$ is an order map with $Y \in \mathcal{H}$ then we can extend f to \mathbf{R} by defining $f(t) = \sup f((-\infty, t] \cap Y)$. The resulting order map on \mathbf{R} has the same image as f on Y . So we can apply condition (v) even when f is only defined on Y .

If $Y \in \mathcal{H}$ then we let $X(Y) = \mathbf{R} \setminus Y$. More generally, if \mathcal{J} is a nonempty subset of \mathcal{H} we let $X(\mathcal{J}) = \mathbf{R} \setminus \cup \mathcal{J}$. Each $X(\mathcal{J})$ is an IHLOTS containing \mathbf{Q} with completion \mathbf{R} . We use $[-1, +1]$ as the distinguished interval in each.

The amazing result of Hart and van Mill (1985) is the following.

Theorem 2.21.

- (a) *There exists a Hart-van Mill collection \mathcal{H} of cardinality \mathbf{c} .*
- (b) *Let \mathcal{H} be a Hart-van Mill collection with base set V . Let \mathcal{J} and \mathcal{J}_1 be subsets of \mathcal{H} with associated complementary IHLOTS X and X_1 , respectively. If $\mathcal{J} \subset \mathcal{J}_1$ then $X_1 \subset X$ and so for every ordinal α , $\widehat{(X_1)_\alpha}$ injects into \widehat{X}_α . If \mathcal{J} is not a subset of \mathcal{J}_1 then $\widehat{(X_1)_\omega}$ does not inject into \widehat{X}_ω . So if neither \mathcal{J} nor \mathcal{J}_1 includes the other then the CHLOTS \widehat{X}_ω and $\widehat{(X_1)_\omega}$ are not comparable with respect to size, i.e. neither injects into the other. In particular, if for some $Y \in \mathcal{J}$ we have $-Y \notin \mathcal{J}$ then the CHLOTS \widehat{X}_ω is not even comparable in size with its reverse CHLOTS.*

Proof. The construction of the collection is Theorem 2.0 in Hart and van Mill (1985) although an easy bit of adjustment is required to get condition (iv). In their Theorem 4.4, where \widehat{X}_ω is denoted L_Y , the argument is provided which can be adapted to show that if \mathcal{J} is not a subset of \mathcal{J}_1 then the only continuous order maps from \widehat{X}_ω to $\widehat{(X_1)_\omega}$ are constants. Lemma 2.7e then implies that $\widehat{(X_1)_\omega}$ does not inject into \widehat{X}_ω . For $\mathcal{J} \subset \mathcal{H}$ the reverse HLOTS for $X(\mathcal{J})$ is $X(\tilde{\mathcal{J}})$ where $\tilde{\mathcal{J}} = \{-Y : Y \in \mathcal{J}\}$. \square

Remark. For every IHLOTS $X \subset \mathbf{R}$ the CHLOTS $F = \widehat{X}_\omega$ has size between \mathbf{R} and \mathbf{R}_ω . It follows from Theorem 2.18b that with α a sufficiently large tail-like ordinal (in fact $\alpha \geq \omega^{\omega \cdot 2}$ suffices)

$$F_\alpha \cong \mathbf{R}_\alpha. \tag{2.58}$$

In particular, for $F = \widehat{X(Y)}_\omega$ with $Y \in \mathcal{H}$, F is not symmetric but F_α is symmetric.

Use the Axiom of Choice to select a subset \mathcal{H}_+ of \mathcal{H} so that for all $Y \in \mathcal{H}$ exactly one member of the pair $\{Y, -Y\}$ lies in \mathcal{H}_+ and let \mathcal{H}_- be the complement. For every $\mathcal{A} \subset \mathcal{H}_+$ define $\mathcal{J}(\mathcal{A}) = \mathcal{A} \cup (\mathcal{H}_+ \setminus \{-Y : Y \in \mathcal{A}\})$. If $\mathcal{A}_1 \neq \mathcal{A}_2$ then neither of the two sets $\mathcal{J}(\mathcal{A}_1), \mathcal{J}(\mathcal{A}_2)$ contains the other. So from the Hart-van Mill Theorem 2.21 we obtain a cardinality 2^c family of CHLOTS each with size between \mathbf{R} and \mathbf{R}_ω no two of which are comparable with respect to size.

Our final result combines the arguments of Hart and van Mill with Theorem 2.15.

We require a bit more detail about the completion \widehat{X}_β for an IHLOTS $X \subset \mathbf{R}$ than was needed for Theorem 2.15. For convenience we will assume that $-1, +1 \in X$ and that $[-1, +1] \cap X$ is the distinguished interval chosen for X . For $0 < i < \beta$ if $w \in \widehat{X}_\beta$ is such that $\hat{\pi}_\beta^i(w) \in X_i$ then we define

$$J(w, i) = (\hat{\pi}_\beta^i)^{-1}(\hat{\pi}_\beta^i(w)). \tag{2.59}$$

$J(w, i)$ is a nontrivial interval in \widehat{X}_β which therefore contains $J(w, i) \cap X_\beta$ as a dense subset. Define the order surjection

$$\pi_{w,i} : J(w, i) \cap X_\beta \rightarrow [-1, +1] \cap X \quad \text{by} \quad \pi_{w,i}(r) = r_i \tag{2.60}$$

and let $\hat{\pi}_{w,i} : J(w, i) \rightarrow [-1, +1]$ be the order surjection which is its completion.

Theorem 2.22. *Assume that \mathcal{H} is a nonempty Hart-van Mill collection of subsets of \mathbf{R} . For distinct, nonempty subsets $\mathcal{J}_1, \mathcal{J}_2$ of \mathcal{H} let $X_1 = X(\mathcal{J}_1)$ and $X_2 = X(\mathcal{J}_2)$ be the associated IHLOTS. If α and β are limit ordinals then $\widehat{(X_1)}_\alpha$ is not order isomorphic to $\widehat{(X_2)}_\beta$.*

Proof. Since the two subsets are distinct, we can assume that $Y \in \mathcal{J}_1 \setminus \mathcal{J}_2$. Thus, Y and the base set V are BB sets with

$$Y \subset X_2 \setminus X_1 \quad \text{and} \quad V \subset X_1 \cap X_2. \quad (2.61)$$

Assume that $f : \widehat{(X_1)_\alpha} \rightarrow \widehat{(X_2)_\beta}$ is an order isomorphism. We will obtain a contradiction by proving that f is not injective.

As in Theorem 2.15 the interval $P(y)$ is trivial for all $y \in \mathbf{R} \setminus X_1$, and in particular for $y \in Y$. We denote by $f(y)$ the single point in the image $f(P(y)) \subset (X_2)_\beta$.

For each $i < \beta$ define

$$Y_i = \{y \in Y : \hat{\pi}_\beta^i(f(y)) \in (X_2)_i\} \quad (2.62)$$

and for each $y \in Y_i$ let

$$Q(y, i) =_{def} (\hat{\pi}_\beta^i \circ f)^{-1}(\hat{\pi}_\beta^i(f(y))) = f^{-1}(J(f(y), i)) \subset \widehat{(X_1)_\alpha}, \quad (2.63)$$

where the second equation uses the notation of (2.60) above.

As in Theorem 2.15 each $Q(y, i)$ is a compact, nontrivial interval in $\widehat{(X_1)_\alpha}$ and $\hat{\pi}(Q(y, i))$ is a compact interval in \mathbf{R} which contains y . Since Y is a subset of the complement of X_1 (2.49) implies as in Theorem 2.15 that the open interval $(\hat{\pi}(Q(y, i)))^\circ$ is nonempty.

For each $i < \beta$ let

$$\begin{aligned} \mathcal{Q}_i &=_{def} \{\hat{\pi}(Q(y, i)) : y \in Y_i\}, \\ \mathcal{O}_i &=_{def} \bigcup \{(\hat{\pi}(Q(y, i)))^\circ : y \in Y_i\}. \end{aligned} \quad (2.64)$$

By Lemma 2.14c the intervals in \mathcal{Q}_i are nonoverlapping. Since \mathbf{R} is separable, \mathcal{Q}_i is countable. It follows that the set $\hat{\pi}_\beta^i(f(Y_i))$ is countable.

Notice that if $z \in Y \cap (\hat{\pi}(Q(y, i)))^\circ$ then by (2.48) $P(z) \subset Q(y, i)$ and so $\hat{\pi}_\beta^i(f(z)) = \hat{\pi}_\beta^i(f(y)) \in (X_2)_i$. Thus, $z \in Y_i$. Thus, we have

$$Y_i \cup \mathcal{O}_i \subset \bigcup \mathcal{Q}_i \quad \text{and} \quad Y \cap \mathcal{O}_i \subset Y_i. \quad (2.65)$$

Thus, if Y_i is dense in \mathbf{R} then $\bigcup \mathcal{Q}_i$ is dense. Since the open set \mathcal{O}_i is dense in $\bigcup \mathcal{Q}_i$ it follows that \mathcal{O}_i is dense in \mathbf{R} when Y_i is. Since Y is dense, $Y \cap \mathcal{O}_i$ is dense in the open set \mathcal{O}_i . So, conversely, if \mathcal{O}_i is dense in \mathbf{R} then Y_i is by (2.65).

We prove by induction on $i < \beta$ that \mathcal{O}_i is dense in \mathbf{R} , or equivalently, that Y_i is dense, for all i .

Case i: For the initial step with $i = 1$ we define the order map $\tilde{f} : Y \rightarrow \mathbf{R}$ as follows. For $z \in Y$ the interval $P(z) \subset \widehat{(X_1)_\alpha}$ is trivial and we have denoted by $f(z)$ the point in its image under f . Let $\tilde{f}(z) = \hat{\pi}_\beta^1(f(z))$. Notice that $\tilde{f}(Y_1) = \hat{\pi}_\beta^1(f(Y_1))$ and as we observed above, this set is countable. By definition of Y_1 this set equals $X_2 \cap \tilde{f}(Y)$. So (2.61) implies that $\tilde{f}(Y) \cap (V \cup Y)$ is countable. So condition (v) of Definition 2.20, adjusted according to the Remark thereafter, implies that the cardinality of $\tilde{f}(Y) \setminus Y$ is less than \mathfrak{c} . Since $\tilde{f}(Y) \cap Y$ is countable, we can apply Lemma 2.19b to get that the entire image $\tilde{f}(Y)$ is countable. Now f is assumed injective and if $x \in \mathbf{R} \setminus X_2$ then $P(x) = (\hat{\pi}_\beta^1)^{-1}$ is a singleton. Hence, \tilde{f} is injective on $Z_\emptyset =_{def} \{z \in Y : \tilde{f}(z) \notin X_2\}$ and so Z_\emptyset is countable. So $Y_1 = Y \cap (\mathbf{R} \setminus Z_\emptyset)$ is dense in \mathbf{R} by the Baire Category Theorem and Lemma 2.19a.

With $i = j + 1$ we fix y in the set Y_j and prove that $Y_i \cap (\hat{\pi}(Q(y, j)))^\circ$ is dense in $(\hat{\pi}(Q(y, j)))^\circ$. It follows that $Y_i \cap O_j$ is dense in O_j which, by induction hypothesis, is dense in \mathbf{R} . So Y_i is dense in \mathbf{R} in this case as well.

To analyze $Y_i \cap (\hat{\pi}(Q(y, j)))^\circ$ we use a variation of the initial step argument. Let $[a, b]$ be the closed interval $\hat{\pi}(Q(y, j))$, so that $(a, b) = (\hat{\pi}(Q(y, j)))^\circ$. Let $r : \mathbf{R} \rightarrow [a, b]$ be the canonical retraction which maps $(-\infty, a]$ to a and $[b, \infty)$ to b . For $z \in Y \cap [a, b]$ the interval $P(z)$ is trivial and $f(z)$ is the point in its image under f , which lies in $J(f(y), j) \subset \widehat{(X_2)_\beta}$, see (2.59). We let $\tilde{f} = \hat{\pi}_{f(y), j} \circ f \circ r : Y \rightarrow \mathbf{R}$. Notice that for $z \in Y \cap (a, b)$ (2.65) implies that $z \in Y_j$ and so $z \in Y_i$ iff $\tilde{f}(z) \in X_2$. Since $\hat{\pi}_\beta^i(f(Y_i))$ is countable we have that $\tilde{f}(Y) \cap X_2$ is countable. As before, condition (v) of Definition 2.20 and Lemma 2.19b imply that $\tilde{f}(Y)$ is countable and so $\tilde{f}(Y \cap (a, b)) \setminus X_2$ is countable. If $z \in Y \cap K$ with $\tilde{f}(z) \notin X_2$ then $f(z)$ is the unique point in $(\hat{\pi}_\beta^i)^{-1}(\hat{\pi}_\beta^i(f(z)))$. So, as before, $Z_{y, j} =_{def} \{z \in Y \cap (a, b) : \tilde{f}(z) \notin X_2\}$ is a countable set. Finally, the Baire Category Theorem and Lemma 2.19a imply that $Y_i \cap (a, b) = Y \cap ((a, b) \setminus Z_{y, j})$ is dense in (a, b) .

Case ii: With i a limit ordinal notice that (2.65) implies

$$Y_i = \bigcap \{Y_j : j < i\} \supset (\bigcap \{O_j : j < i\}) \cap Y. \tag{2.66}$$

Because β is countable, $\cap\{O_j : j < i\}$ is a G_δ which is dense by induction hypothesis and the Baire Category Theorem and so the intersection with Y is dense by Lemma 2.19a. Thus, Y_i is dense in this case as well.

Having completed the induction we see, as in Case ii, that $(\cap\{O_j : j < \beta\}) \cap X_1$ is dense in \mathbf{R} . As in the final portion of the proof of Theorem 2.15, if y is in this set then $P(y)$ is nontrivial and f is constant on $P(y)$. Hence, f is not injective, contradicting our previous assumption. \square

Using a Hart-van Mill collection of cardinality \mathfrak{c} we obtain a collection of cardinality $2^{\mathfrak{c}}$ consisting of CHLOTS towers of height Ω which do not intersect.

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