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**AN APPROACH TO EFFECTIVE FUNCTIONALS
ON THE REAL NUMBERS VIA FILTER SPACES**

FRÉDÉRIC DE JAEGER

ABSTRACT. We consider the hierarchy of filter spaces built on the real numbers using the product and the function space constructors. We prove that the induced canonical base on each object of this hierarchy is decidable. As a corollary, we deduce that the category of computable elements in this hierarchy is Cartesian closed.

INTRODUCTION

Hyland [3] studied the continuous functionals over the integers (see [5, 4]). He described those functionals using a more geometric approach. Because continuous functionals should be closed under the formation of product and function spaces, it's necessary to look for a Cartesian closed category. So we can't use the category of topological spaces. Hyland used the category of filter spaces, introduced by Kuratowski [6]. A filter space is a generalization of the notion of a topological space. Basically, on a topological space, it is possible to define when a *filter* converges to a point (see for example [1]). Then, a filter space is simply a set of points with the information that some filters may converge to some points. The main difference is that given a topological space, the set of filters converging to a point x has a minimal element namely the neighborhood filter of x . In a general filter space, we have to drop this property. But instead, we gain the Cartesian closeness of this new category.

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Hyland considered the hierarchy of filter spaces built on the integers using the product and the function space constructors. This can be seen as the Cartesian closed category generated by the set of integers with the discrete topology in the category of filter spaces. He equipped those filter spaces with enumerable bases, and proved that inclusion between basic sets is decidable. Then he can deduce a nicer characterization of Kleene's functionals.

In this paper, we consider the same hierarchy but built on the set of real numbers with the Euclidean topology. We prove the same decidability result but the proof is completely different. In the case of the natural numbers, the topologies are zero dimensional, which gives a basis of clopen sets and makes the proofs easier. In our situation, we don't have such strong separation properties. Instead, we have very geometrical notions such as *convexity* and a vector space structure that are essential to the success of our proofs.

In the first part of this work, we will introduce filter spaces with bases and recall basic results that will be necessary for the later. We thus introduce our object of study: the Cartesian closed category of filter spaces generated by the real numbers. In the second part we prove the decidability result on this category. In the third part, we give some consequences of this result such as the Cartesian closedness of the effective version of this category.

1. EFFECTIVE FILTER SPACES

We recall here basic definitions about effective filter spaces that can be found in [3]. We assume some classical knowledge about ordered sets such as the one we can find in the first chapter of [2]. We introduce the notation we will use in the next sections.

1.1. Notations

We will note here:

- Δ_n : the set of integers k such that the k th digit of n in its binary expansion is 1. This is the standard way to enumerate bijectively and constructively the set of finite subsets of \mathbb{N} .
- $\langle \mathbf{m}, \mathbf{n} \rangle$: the standard pairing function. It's a recursive bijection from $\mathbb{N} \times \mathbb{N}$ to \mathbb{N} .

Given an ordered set X and a $A \subseteq X$, we will write $\uparrow A := \{y \in X \mid \exists x \in A \ x \leq y\}$, $\downarrow x := \{y \in X \mid \exists x \in A \ x \geq y\}$. By extension, if $x \in X$, $\uparrow x := \uparrow\{x\}$ and $\downarrow x := \downarrow\{x\}$. We will say that a subset A of X is *upper closed* if $\uparrow A = A$.

With any functions $\rho : X \rightarrow \mathbb{R}$ and any property $P[y]$ of a real number y , we will write (like probabilists do) $\{P[\rho]\}$ the set $\{x \in X \mid P[\rho(x)]\}$. For instance, $\{\rho > 0\}$ is $\rho^{-1}(]0, +\infty[)$.

Definition 1.1. A *filter* \mathcal{F} on a set X is a non-empty set of \mathcal{F} on a set X is a non-empty set of non-empty subsets of X such that $(\mathcal{P}(X), \subseteq)$ and stable under finite intersections.

Definition 1.2. A *filter base* Φ on X is a set of non-empty subsets of X such that $\uparrow\Phi$ is a filter on X . We will denote by $[\Phi]$ the filter $\uparrow\Phi$ generated by Φ .

1.2. Filter spaces

Definition 1.3. A filter structure F on a set X is an operation that associates to each point $x \in X$ a collection $F(x)$ of filters on X such that

1. $F(x)$ is itself a filter on the set of all filters on X , ordered by inclusion.
2. $F(x)$ contains the principal ultrafilter $[\{x\}]$.

A *filter space* $\mathcal{X} = (X, F)$ is a set X equipped with a filter structure F . The relation $\Phi \in F(x)$ can be read as “ Φ converges to x ” and will be denoted by $\Phi \downarrow x$. We extend this convention to filter bases.

A function $f : X \rightarrow Y$, where (X, F) and (Y, G) are filter spaces, is *continuous* if for all $x \in X$, whenever $\Phi \downarrow x$, then $f(\Phi) \downarrow f(x)$.

Definition 1.4. The category whose objects are filter spaces and morphisms are continuous functions of filter spaces is written **FIL**.

Given a topological space $\mathcal{T} = (T, \Omega T)$, there is a canonical filter structure on T that we denote by **F \mathcal{T}** : we say that $\Phi \downarrow x$ for $x \in T$ if Φ is finer than the neighborhood filter of x in \mathcal{T} . Conversely, suppose we have a filter space $\mathcal{X} = (X, F)$, then we say that a subset $O \subseteq X$ is open if for all point $x \in O$, whenever $\Phi \downarrow x$, then $O \in [\Phi]$. This defines a topology on X that we will denote by **T \mathcal{X}** .

Property 1.5. **F** and **T** are functors; they form an adjunction such that **T** is left adjoint to **F**.

A set $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *base* of $\mathcal{X} = (X, F)$ iff for all $x \in X$ and for all $\Phi \downarrow x$, then $(\Phi \cap \mathcal{B}) \downarrow x$. An *effective filter space* is a filter space $\mathcal{X} = (X, F)$ with an enumerable base $\mathcal{B} = (U_n)$. We will denote it by $\mathfrak{X} = (X, F, (U_n))$. Given a set $I \subseteq \mathcal{P}(\mathbb{N})$ of indices, we will say that I is a *representation* of a filter ϕ iff $[\{U_n \mid n \in I\}] = \phi$. Sometimes, we will denote by $[I]_{\mathfrak{X}}$ the filter represented by I .

Remark 1.6. Even if $\mathcal{X} = \mathbf{F}(\mathcal{T})$, the base of \mathcal{X} can be made of any kind of subsets of X . In particular, the members of the base are not necessarily open. And in fact, in our case we will choose bases made of closed sets.

A base doesn't contain all the information needed to recover the filter structure from it; we need to know that some filter bases built with sets in \mathcal{B} converge to some points.

Given two effective filter spaces $\mathfrak{X} = (X, F, (U_n))$ and $\mathfrak{Y} = (Y, G, (V_n))$, we can define the product of \mathfrak{X} and \mathfrak{Y} to be $(X \times Y, F \times G, (U_n \times V_m)_{\langle n,m \rangle})$. The filter structure $F \times G$ on $X \times Y$ is defined by saying that $\Phi \downarrow (x, y)$ iff $p_1(\Phi) \downarrow x$ and $p_2(\Phi) \downarrow y$, where p_1 and p_2 are respectively the first and second projections. We leave to the reader the verification that $(U_n \times V_m)_{\langle n,m \rangle}$ is a base of this filter structure. We will denote by $\mathfrak{X} \times \mathfrak{Y}$ this effective filter space.

Moreover, we can define a canonical filter structure on $[X, Y]$, the set of continuous functions from (X, F) to (Y, G) . We say that a filter Θ converges to a function f iff for all $x \in X$, whenever $\phi \downarrow x$, then $\Theta(\phi) \downarrow f(x)$. $\Theta(\phi)$ is the filter base $\{U(V) \mid U \in \Theta, V \in \phi\}$ where $U(V) := \cup_{f \in U} f(V)$. We shall denote by $F \rightarrow G$ this filter structure and by $\mathcal{X} \rightarrow \mathcal{Y}$ the associated filter spaces. Now, we can define a base on $\mathcal{X} \rightarrow \mathcal{Y}$:

For $A \subset X$ and $B \subseteq Y$, we define

$$[A, B] := \{f : X \rightarrow Y \mid f \text{ is continuous and } f(A) \subseteq B\}.$$

Then we define W_n , the base of $\mathcal{X} \rightarrow \mathcal{Y}$, to be

$$W_n = \bigcap_{\langle i,j \rangle \in \Delta_n} [U_i, V_j].$$

In the the following, we will often express W_n as follows

$$W_n = \bigcap_{\alpha \in \Delta_n} [U_{\alpha_1}, V_{\alpha_2}].$$

So, α_1 is a short hand for $\pi_1(\alpha)$ and α_2 for $\pi_2(\alpha)$ where for all n , $\langle \pi_1(n), \pi_2(n) \rangle = n$. Sometimes, we will even write $[U_\alpha, V_\alpha]$, instead of $[U_{\alpha_1}, V_{\alpha_2}]$, when there's no risk of confusion. We refer to Hyland [3] for the proof that it is actually a base of $\mathcal{X} \rightarrow \mathcal{Y}$. We will denote by $\mathfrak{X} \rightarrow \mathfrak{Y}$ the filter space $\mathcal{X} \rightarrow \mathcal{Y}$ with base (W_n) .

With this canonical product and this canonical function space, we have that:

Property 1.7. *FIL is a Cartesian closed category.*

Now, we are going to introduce our objects of study and the notations we will need later. We consider \mathbb{R} with the filter structure coming from the Euclidean topology. We choose a base $I_{(n,m)} := [a_n, b_m]$ consisting of closed intervals with rational endpoints for a suitable constructive enumeration of the rational numbers. In this way, we are able to decide when $I_n \subseteq \cup I_{m_i}$ or $I_n = \emptyset$. Moreover, we assume that a_n and b_n enumerate disjoint dense subsets of \mathbb{Q} . It is only a technical trick to simplify the proofs. Basically, it allows us to say that I_n is either empty or infinite and that this property can be inherited at all types (see Lemma A.4). This saves us from degenerated cases that occur in the proofs when I_n can be a singleton.

We now define certain formal expressions called *types* by induction: (1) ι is a type (called a *ground*) type. (2) If σ and τ are types then the expression $\sigma \times \tau$ is a type (called a *product type*) and (3) $\sigma \rightarrow \tau$ is a type (called a *function type*). For example, $(\iota \rightarrow \iota) \rightarrow \iota$ is a type. The intuition is that this is the type of functionals from $C(\mathbb{R}) = (\mathbb{R} \rightarrow \mathbb{R})$ to \mathbb{R} . We make this intuition precise, working in the category of filter spaces, as follows.

We write $\mathbb{R}_\iota := \mathbb{R}$, and we denote by \mathfrak{R}_ι the effective filter space of the real numbers with the canonical base (I_n) . Then we define inductively \mathfrak{R}_σ by $\mathfrak{R}_{\sigma \times \sigma'} := \mathfrak{R}_\sigma \times \mathfrak{R}_{\sigma'}$ and $\mathfrak{R}_{\sigma \rightarrow \sigma'} := \mathfrak{R}_\sigma \rightarrow \mathfrak{R}_{\sigma'}$. We will write \mathbb{R}_σ the ground set of \mathfrak{R}_σ . The *length* of a type σ is defined inductively by

$$\begin{aligned} \text{length}(\iota) &= 1 \\ \text{length}(\sigma \times \sigma') &= \text{length}(\sigma) + \text{length}(\sigma') \\ \text{length}(\sigma \rightarrow \sigma') &= \text{length}(\sigma) + \text{length}(\sigma') \end{aligned}$$

Property 1.8. We have the following properties for $\mathfrak{R}_\sigma = (\mathbb{R}_\sigma, F_\sigma, (U_n))$.

1. U_n is closed in the induced topology.
2. $(U_n)_{n \in \mathbb{N}}$ is closed under finite intersections and there is a computable function $\psi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that

$$U_n \cap U_m = U_{\psi(n,m)}.$$

3. There is a recursive operator $\wedge : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ such that $i \in \hat{I}$ iff there exist i_1, \dots, i_n in I such that $\bigcap_{k=1}^n U_{i_k} = U_i$.

Proof.

1. See Hyland [3].
2. Induction over the type.
3. Consequence of 2.

□

Suppose we have a family of basic subsets of \mathfrak{R}_σ , say $(U_n)_{n \in I}$. We will say it is *complete* iff it does not contain the empty set and whenever i and j are in I , then there is $k \in I$ such that $U_k = U_i \cap U_j$; we will write $k = i \wedge j$. Roughly speaking, this means that this family is stable under intersection.

It will be convenient later to write $i \preceq j$ to mean that $U_i \subseteq U_j$. We will write $i \perp j$ when $U_i \cap U_j = \emptyset$.

Now, we remark the following trivial property for $A, A' \subseteq X$ and $B, B' \subseteq Y$.

- (1) $[A, B] \cap [A', B'] \subseteq [A \cap A', B \cap B']$.
- (2) $[A, B] \cap [A, B'] = [A, B \cap B']$.

So when we consider a basic set W_n of $\mathfrak{R}_{\tau \rightarrow \tau'}$, we know that

$$(3) \quad W_n = \bigcap_{\alpha \in \Delta_n} [U_\alpha, V_\alpha].$$

Then, we can always add new sets of the form $[U_\alpha \cap U_\beta, V_\alpha \cap V_\beta]$ for $\alpha, \beta \in \Delta_n$ to the intersection of equation (3) without changing W_n . So, by enriching Δ_n if necessary, we may always assume that whenever we have $\alpha, \beta \in \Delta_n$, then if $U_\alpha \cap U_\beta \neq \emptyset$, we can find a $\gamma \in \Delta_n$ such that $U_\alpha \cap U_\beta = U_\gamma$. Moreover, using equation (2), we can replace any pair of $\alpha, \beta \in \Delta_n$ such that $U_\alpha = U_\beta$ by a unique index γ such that $U_\gamma = U_\alpha$ and $V_\gamma = V_\alpha \cap V_\beta$ without changing W_n . Finally, $[\emptyset, A] = [X, Y]$ for any A , so any $\alpha \in \Delta_n$

such that $U_\alpha = \emptyset$ is useless and can be removed from Δ_n . We will say that W_n is in *canonical form* when we have applied all those reductions. In this case, we remark that the family $(U_{\alpha_1})_{\alpha \in \Delta_n}$ is complete. It's easy to verify that when W_n is in canonical form, then $U_\alpha \subseteq U_\beta \implies V_\alpha \subseteq V_\beta$.

1.3. A vector space structure

We can define a structure of vector space on the set \mathbb{R}_σ by induction over the type starting with the canonical vector structure on \mathbb{R} . If $\sigma = \tau \rightarrow \tau'$, we define operations on σ pointwise using operations on τ' . We have the following property which can be proved easily by induction over the type structure.

Property 1.9. $+_\sigma : \mathbb{R}_\sigma \times \mathbb{R}_\sigma \rightarrow \mathbb{R}_\sigma$ and $.\sigma : \mathbb{R} \times \mathbb{R}_\sigma \rightarrow \mathbb{R}_\sigma$ are continuous maps for the canonical filter structures.

2. DECIDABILITY PROPERTIES

In this section, we prove all the decidability results for the base of the filter space in the Cartesian closed category $(\mathfrak{R}_\sigma)_\sigma$. The proof is by induction on the length of the type. We will need many technical lemmas that will be proved in the appendix. Some of them use the main result but without any circularities because we use them at a lower length.

2.1. Main results

Theorem 2.1. Let $\mathfrak{R}_\sigma = (\mathbb{R}_\sigma, F, (W_n))$. Then the base W_n satisfies the following condition.

1. $W_n = \emptyset$ is a decidable predicate.
2. $W_n \subseteq \bigcup_{i=1}^p W_{n_i}$ is decidable in all the variables n, p, n_i .

We are going to prove this theorem by an induction over the type structure. We will need the following lemma, which is trivially established by induction on σ :

Lemma 2.2. W_n is a convex set.

We now assume that the theorem has been proven for all the types of length lower than N , and let σ a type of length $N + 1$.

Suppose $\sigma = \tau \times \tau'$, then $W_{\langle n, m \rangle} = U_n \times V_m$ for (U_n) and (V_n) the respective bases of \mathfrak{R}_τ and $\mathfrak{R}_{\tau'}$. Thus $W_{\langle n, m \rangle} = \emptyset \iff U_n = \emptyset$

or $V_m = \emptyset$. So we can apply the induction hypothesis and part 1 of the theorem 2.1 is proved.

For part 2, we remark that

$$\begin{aligned} \bigcup_{i=1}^p A_i \times B_i &= \bigcup_{i=1}^p (A_i \times Y \cap X \times B_i) \\ &= \bigcap_{I \subseteq [1,p]} \left(\left(\bigcup_{i \in I} A_i \times Y \right) \cup \left(X \times \bigcup_{j \in I^c} B_j \right) \right) \end{aligned}$$

so that $U_n \times V_m \subseteq \bigcup_{i=1}^p U_{n_i} \times V_{m_i}$ is equivalent to

$$\forall I \subseteq [1,p] \quad U_n \subseteq \bigcup_{i \in I} U_{n_i} \text{ or } V_m \subseteq \bigcup_{j \in I^c} V_{m_j},$$

and using the induction hypothesis, this is equivalent to a finite conjunction of decidable predicates so it is decidable.

Now, suppose that $\sigma = \tau \rightarrow \tau'$. Then by construction

$$W_n = \bigcap_{\alpha \in \Delta_n} [U_{\alpha_1}, V_{\alpha_2}].$$

Property 2.3.

$$(4) \quad W_n \neq \emptyset \iff \left(\forall I \subseteq \Delta_n \bigcap_{\alpha \in I} U_{\alpha_1} \neq \emptyset \implies \bigcap_{\alpha \in I} V_{\alpha_2} \neq \emptyset \right).$$

Proof. \implies is trivial. It is only a set theoretical argument.

\impliedby : Suppose that the right side is true, we are going to construct a function $f \in W_n$.

As discussed before, we can assume without loss of generality that W_n in canonical form. So let's assume that W_n is in canonical form. In this case, the right hand side of (4) is equivalent to $\forall \alpha U_{\alpha_2} \neq \emptyset \implies V_{\alpha_1} \neq \emptyset$.

Now, for each α , choose $y_\alpha \in V_{\alpha_2}$ (which we know to be not empty) and choose any $y_0 \in Y$. We want to construct continuous masses $m_\alpha : X \rightarrow \mathbb{R}^+$ such that f is a barycenter of (y_α) with positive masses (m_α) :

$$f(x) = \text{Bar} \left((y_0, m_0(x)), (y_\alpha, m_\alpha(x))_\alpha \right).$$

To build these mass functions, we will need *separating functions* as in the following lemma.

Lemma 2.4 (separating functions). *Let $(U_\alpha)_{\alpha \in \Delta_m}$ be a complete finite set of basic closed sets of \mathbb{R}_σ . Then, there exists a family of functions $\rho_\alpha : \mathbb{R}_\sigma \rightarrow [0, 1]$ such that*

1. for all α , $U_\alpha \subseteq \{\rho_\alpha = 1\}$,
2. for $\alpha \perp \beta$, we have $U_\beta \subseteq \{\rho_\alpha = 0\}$,
3. for all α et β , with $\alpha \not\leq \beta$ and $\alpha \not\perp \beta$ then $\{\rho_\alpha > 0\} \cap U_\beta \subseteq \{\rho_{\alpha \wedge \beta} = 1\}$.

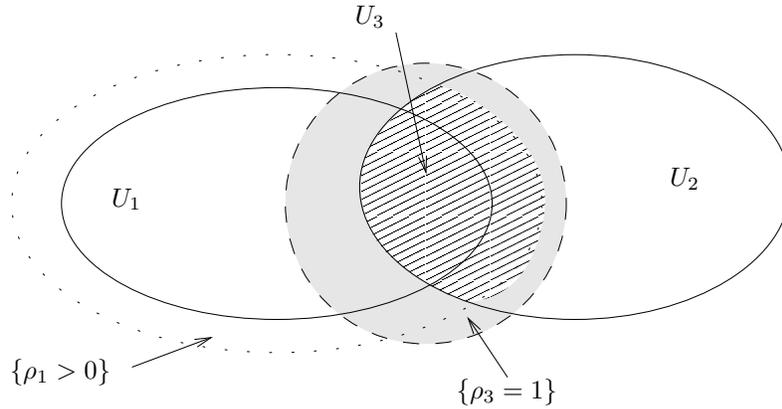


FIGURE 1. In this example, $U_3 = U_1 \cap U_2$, and we can check that $\{\rho_1 > 0\} \cap U_2$ (the dashed region) is a subset of $\{\rho_3 = 1\}$ (the gray region).

We apply this lemma to the collection of $(U_{\alpha_1})_{\alpha_1 \in \Delta_n}$. We choose m_α and m_0 as:

$$m_\alpha(x) = \rho_{\alpha_1}(x) \cdot \prod_{\beta_1 \prec \alpha_1} (1 - \rho_{\beta_1}(x)),$$

$$m_0(x) = 1 - \max_{\alpha \in \Delta_n} \rho_{\alpha_1}(x).$$

Property 2.5. *We have the following properties:*

1. $\rho_{\alpha_1}(x) > 0 \implies \exists \beta_1 \preceq \alpha_1$ such that $m_{\beta_1}(x) > 0$,
2. $x \in U_{\beta_1} \setminus U_{\alpha_1} \implies m_{\alpha_1}(x) = 0$.

Proof.

1. By induction on the order structure on the α_1 . The property is obviously true for minimal elements. Suppose $\rho_{\alpha_1}(x) > 0$ and $m_\alpha(x) = 0$ then there exists $\beta_1 \prec \alpha_1$ such that $\rho_{\beta_1}(x) = 1$. By induction, $\exists \gamma_1 \preceq \beta_1$ such that $m_\gamma(x) > 0$.
2. If $x \in U_{\beta_1} \setminus U_{\alpha_1}$, then $\alpha_1 \not\leq \beta_1$. If $\alpha_1 \perp \beta_1$, then condition 2 of Lemma 2.4 gives us that $\rho_{\alpha_1}(x) = 0$. If $\gamma_1 = \alpha_1 \wedge \beta_1$ and $\rho_{\alpha_1}(x) > 0$, then $\rho_{\alpha_1 \wedge \beta_1}(x) = 1$ so $m_\alpha(x) = 0$ by condition 3 of the same lemma.

□

With this properties, we have that if $x \in U_\alpha$, then $f(x)$ is a barycenter of the points y_β such that $\beta \preceq \alpha$ (all other coefficients are null). And we're sure that there exists at least one coefficient that is not null. So $f(x)$ is a barycenter of the points in $V_\beta \subseteq V_\alpha$ with positive coefficients. So $f(x) \in \text{Conv}(V_\alpha) = V_\alpha$ by Lemma 2.2. The continuity of this map come from Proposition 1.9. □

Now, we have to prove the second assertion of Theorem 2.1. We first investigate when $W_n \subseteq W_m$ for σ a functional type.

We have

Lemma 2.6. *$W_n \subseteq W_m$ if and only if $W_n = \emptyset$ or $\forall \beta \in \Delta_m, \exists I \subseteq \Delta_n$ such that $U_\beta \subseteq \cup_{\alpha \in I} U_\alpha$ and $\forall \alpha \in I, V_\alpha \subseteq V_\beta$.*

Proof. The fact that the right hand side implies the left hand side is a trivial set-theoretic argument. As before, the hard part is to show that those conditions are necessary. The scheme of the proof is the same. We suppose now that the right hand side condition is not satisfied. We are going to construct a function $f \in W_n$ such that $f \notin W_m$.

We know that $W_n \neq \emptyset$ and $\exists \beta \in \Delta_m$ such that whenever U_β is cover by a family of U_α for $\alpha \in \Delta_n$, then there exists at least one α in this family such that $V_\alpha \not\subseteq V_\beta$. We can reformulate this equivalently: Let $I_0 := \{\alpha \in \Delta_n \mid V_\alpha \subseteq V_\beta\}$; then

$$(5) \quad U_\beta \not\subseteq \bigcup_{\alpha \in I_0} U_\alpha.$$

Thus, there exists a point x_0 in U_β such that whenever $x_0 \in U_\alpha$ for $\alpha \in \Delta_n$, then $V_\alpha \not\subseteq V_\beta$.

Now, choose $f_0 \in W_n$. We want to perturbate f_0 into a new function f that will be still in W_n but not in W_m .

Let I_1 be the collection of all the $\alpha \in \Delta_n$ such that $x_0 \in U_\alpha$ (I_1 may be empty). Because $f_0 \in W_n$, $V_0 = \bigcap_{\alpha \in I_1} V_\alpha \ni f_0(x_0)$ is not empty. And the hypothesis implies that $V_0 \not\subseteq V_\beta$. Then choose $y_0 \in V_0 \setminus V_\beta$, and choose a function ϕ that separates x_0 from all the U_α such that $x_0 \notin U_\alpha$. We can find such a function because of the regularity property of the base (Lemma A.1). Now choose $f = (1 - \phi) \cdot f_0 + \phi \cdot y_0$. Then for any $\alpha \in \Delta_n$, if $\alpha \in I_1$, then for any $x \in U_\alpha$, $f(x) = \text{Bar}((f_0(x), (1 - \phi(x)), (y_0, \phi(x)))$ and $y_0 \in V_0 \subseteq V_\alpha$, so $f(x) \in V_\alpha$ by the convexity of V_α . Thus $f \in [U_\alpha, V_\alpha]$. Otherwise, if $\alpha \notin I_1$, then $f(x) = f_0(x)$ for all $x \in U_\alpha$ and we still have $f \in [U_\alpha, V_\alpha]$. So we conclude that $f \in W_n$. But $x_0 \in U_\beta$ and $f(x_0) = y_0 \notin V_\beta$, and hence, $f \notin W_m$. \square

For the general case, we prove in fact a result that resembles a strong disjunction property we can find in Hyland’s paper [3].

Lemma 2.7. *If (W_n) is the canonical base of $\mathfrak{R}_{\tau \rightarrow \tau'}$, then*

$$W_n \subseteq \bigcup_{k=1}^M W_{n_k}$$

implies that there exists $k \in \{1, \dots, M\}$ such that $W_n \subseteq W_{n_k}$.

Proof. Suppose this is not the case. First, we know that $W_n \neq \emptyset$. Then for every k , we have $W_n \not\subseteq W_{n_k}$. Now, we apply the construction of the previous lemma that will give us for all k , a point x_k and a function f_k that will be a counterexample to the inclusion $W_n \subseteq W_{n_k}$ that will fail on x_k (That is, x_k is in U_β for some $\beta \in \Delta_{n_k}$ and $f_k(x_k) \notin V_\beta$.) Now assuming that all the x_k are distinct, we may build a set of partitioning functions ϕ_k such that $\phi_k(x_l) = \delta_{k,l}$ and $\sum \phi_k = 1$ (Lemma A.2). Now set

$$f = \sum_{k=1}^M \phi_k \cdot f_k.$$

Then, $f \in W_n$ because of the convexity of W_n , but for all k , $f(x_k) = f_k(x_k)$, which implies that $f \notin W_{n_k}$. So, f is the counterexample we were looking for.

If we look carefully at the proof of Property 2.6, then we conclude that we have the liberty to choose the x_k in a set that is of the form

$$U_{\beta_1} \setminus \bigcup_{\alpha \in I_0} U_{\alpha_1}.$$

But in our situation, this set is always infinite because of Lemma A.5. So, we can choose the x_k all distinct, as required. \square

This finally reduces the problem of decidability of inclusion to a finite disjunction of simple decidable questions and this concludes the proof of Theorem 2.1.

3. CONSEQUENCES

In this section, we introduce computability notions on effective filter spaces. We introduce a subcategory of computable objects and establish its Cartesian closedness using our previous results.

3.1. Computable objects

In this subsection, we assume that $\mathfrak{X} = (X, F, (U_n))$ and $\mathfrak{Y} = (Y, G, (V_n))$ are effective filter spaces.

Definition 3.1. We say that an element $x \in X$ is *computable* if we can find a recursively enumerable set I such that $\{U_n \mid n \in I\} \downarrow x$. In this situation, we will write $I \downarrow_{\mathfrak{X}} x$ or even simply $I \downarrow x$ when there is no risk of confusion.

Following Weihrauch [8], we can define computable functions from X to Y .

Definition 3.2. $f : X \rightarrow Y$ is a *computable function* if and only if there exists a recursive operator $\Phi : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$ (see [7]) such that whenever $I \downarrow_{\mathfrak{X}} x$, then $\Phi(I) \downarrow_{\mathfrak{Y}} f(x)$.

Property 3.3. $\text{eval} : \mathfrak{Y}^{\mathfrak{X}} \times \mathfrak{X} \rightarrow \mathfrak{Y}$ is a *computable function*.

Proof. We need to build a recursive operator that will do the job of the evaluation map. Suppose we are given two enumerations K and I such that $K \downarrow_{\mathfrak{Y}^{\mathfrak{X}}} f$ and $I \downarrow_{\mathfrak{X}} x$. So there exist two filters Θ and ϕ such that $\Theta = [K]_{\mathfrak{Y}^{\mathfrak{X}}}$ and $\phi = [I]_{\mathfrak{X}}$. We want to construct $K \bullet I$ such that $K \bullet I \downarrow_{\mathfrak{Y}} f(x)$.

Let

$$K \bullet I := \{\alpha_2 \mid \exists n \in K, \exists m \in I \ U_m \subseteq U_{\alpha_1} \text{ and } \alpha \in \Delta_n\}^{\wedge}$$

where \wedge was defined in Proposition 1.8.

$(K, I) \rightarrow K \bullet I$ is computable because of general property of continuous operators ([7]), the fact that $U_n \subseteq U_m$ is a decidable predicate (see Theorem 2.1) and Lemma 1.8. We need to verify that $K \bullet I$ is a representation of $\Theta(\phi)$.

By assumption $\Theta(\phi) \downarrow f(x)$. First, if $l \in K \bullet I$ then there exist $n \in K$, $m \in I$, $\alpha \in \Delta_n$ such that $l = \alpha_2$. But $W_n(U_m) \subseteq [U_{\alpha_1}, V_{\alpha_2}](U_m) \subseteq V_{\alpha_2} = V_l$ because $U_m \subseteq U_{\alpha_1}$, and, by definition, $W_n(U_m) \in \Theta(\phi)$, so we have that $[K \bullet I]_{\mathfrak{F}} \leq \Theta(\phi)$.

Now choose $V \in \Theta(\phi)$. Then, there exists $n \in K$, $m \in I$ such that $V \supseteq W_n(U_m)$. Now we remark that

$$W_n(U_m) = \bigcap_{\alpha \in \Delta_n} [U_\alpha, V_\alpha](U_m).$$

So, applying the Lemma A.3, we get that

$$W_n(U_m) = \bigcap_{\substack{\alpha \in \Delta_n \\ U_m \subseteq U_\alpha}} V_\alpha = V_{l_1} \cap V_{l_2} \cap \dots \cap V_{l_m} = V_l$$

for a suitable $l \in K \bullet I$. So this proves that $\Theta(\phi) \leq [K \bullet I]_{\mathfrak{F}}$. Finally, we have that $[K \bullet I]_{\mathfrak{F}} = \Theta(\phi)$, so $K \bullet I \downarrow_{\mathfrak{F}} f(x)$. \square

Property 3.4. *Let $f : X \rightarrow Y$ be a computable function, then f is continuous function and is a computable element of $\mathfrak{F}^{\mathfrak{X}}$.*

Proof. According to [7], there exists a computable partial function ϕ “representing” Φ in the following sense:

$$\Phi(I) = \bigcup_{\Delta_n \subseteq I} \Delta_{\phi(n)}.$$

Then, let

$$K = \{\langle a_n, b_n \rangle \mid n \in \mathbb{N}\},$$

where a_n is an index of $\bigcap_{i \in \Delta_n} U_i$ and b_n is an index of $\bigcap_{j \in \Delta_{\phi(n)}} V_j$

So, if $\langle i, j \rangle \in K$, there exists n such that $U_i = \bigcap_{\alpha \in \Delta_n} U_\alpha$. Then choose any $x \in U_i$, any filter base $I_x \downarrow_{\mathfrak{X}} x$. It follows that $(I_x \cup \Delta_n) \downarrow_{\mathfrak{X}} x$. Thus, $\Phi(I_x \cup \Delta_n) \supseteq \Delta_{\phi(n)}$, and this filter base converges to $f(x)$. Because our basic sets are closed, we have that $f(x) \in \bigcap_{\beta \in \Delta_{\phi(n)}} V_\beta = V_j$. So, we have that $f(U_i) \subseteq V_j$, and hence that $f \in [U_i, V_j]$.

From here, we can easily deduce continuity of f : for any $x \in X$ and any $\phi \downarrow x$, there exists a filter base I such that $[I]_{\mathfrak{X}} \leq \phi$. For any $j_0 \in \Phi(I)$, we can find $\langle i, j \rangle \in K$ such that $i \in I$ and $V_j \subseteq V_{j_0}$, so we have that $f(U_i) \subseteq V_j \subseteq V_{j_0}$. Thus, $f(\phi) \geq [\Phi(I)]_{\mathfrak{Y}} \downarrow f(x)$, so $f(\phi) \downarrow f(x)$.

The verification that $K \downarrow_{\mathfrak{Y}^{\mathfrak{X}}} f$ is routine and is left to the reader. \square

Corollary 3.5. *$f : X \rightarrow Y$ is a computable function if and only if it is a computable element of $\mathfrak{X} \rightarrow \mathfrak{Y}$.*

Proof. Lemma 3.4 answers one part of the question. Suppose f is a computable element of $\mathfrak{Y}^{\mathfrak{X}}$. Then there exists a recursively enumerable K such that $K \downarrow_{\mathfrak{Y}^{\mathfrak{X}}} f$. Then $\Phi := \lambda I. K \bullet I$ witnesses the fact that f is computable. \square

Now define $\mathbf{FIL}_{\text{eff}}^{\mathbb{R}}$ as the category whose objects are the \mathfrak{R}_{σ} and morphism are computable functions. We have

Corollary 3.6. $\mathbf{FIL}_{\text{eff}}^{\mathbb{R}}$ is a sub-Cartesian closed category of \mathbf{FIL} .

CONCLUSION

In this paper, we proved by a direct method that the canonical base on \mathbb{R}_{σ} is decidable. Our proofs use some kind of normality properties on the space, which we are still unable to prove in general. In fact, if we have the result that the topology induced by the filter structure is normal, then we could avoid most of the technical lemmas. It's an interesting and still open question to deduce properties of the induced topology of a filter space.

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APPENDIX A. TECHNICAL LEMMAS

Lemma A.1 (basic regularity). *Choose $f \in \mathbb{R}_{\sigma}$ and W_n an element of the base such that $f \notin W_n$. Then there exists a continuous function $\rho : \mathbb{R}_{\sigma} \rightarrow \mathbb{R}$ such that $\rho(f) = 1$ and $W_n \subseteq \{\rho = 0\}$.*

Proof. By induction over the type σ . We start the induction using the fact that \mathbb{R} is a regular space. The difficult case is when $\sigma = \tau \rightarrow \tau'$. Then $W_n = \bigcap_{\alpha \in \Delta_n} [U_\alpha, V_\alpha]$. So $f \notin W_n$ means that $\exists \alpha \in \Delta_n$ such that $f(U_\alpha) \not\subseteq V_\alpha$. Thus, there exists $x_0 \in U_\alpha$ such that $f(x_0) \notin V_\alpha$. Then we use the induction hypothesis to build a function $\phi : \mathbb{R}_{\tau'} \rightarrow \mathbb{R}$ that separates $f(x_0)$ from V_α and we set $\rho = \lambda g. \phi(g(x_0))$. We leave the verification to the reader that this ρ has the desired properties. \square

Lemma A.2 (partition). *Choose n distinct elements $(x_i)_{1 \leq i \leq n}$ of \mathbb{R}_σ . Then there exist n continuous functions $(\rho_i)_{1 \leq i \leq n}$ from \mathbb{R}_σ to \mathbb{R} such that $\rho_i(x_i) = 1$, $\rho_i(x_j) = 0$ when $i \neq j$ and $\sum \rho_i$ is the constant function 1.*

Proof. By induction on σ , it's easy to build functions ψ_i that verify $\psi_i(x_j) = 0$ or 1 wether $i \neq j$ or not and the codomain of each ψ_i is $[0, 1]$. Then, let $\psi_n(x) = 1 - \max_{1 \leq i \leq n-1} \psi_i(x)$. Now, for any x , there exists at least one i such that $\psi_i(x) \neq 0$. Then choose

$$\rho_i(x) := \frac{\psi_i(x)}{\sum_1^n \psi_j(x)}. \quad \square$$

Lemma A.3. *Let (U_n) the base of \mathbb{R}_σ and (V_n) the base of \mathbb{R}_τ . Then we have:*

$$[U_i, V_j](U_l) = \begin{cases} V_j & \text{when } U_l \subseteq U_i, \\ \mathbb{R}_\tau & \text{otherwise.} \end{cases}$$

Proof. The first case is easy because constant functions are continuous. For the second case, it's now a classical argument that uses the Lemma A.1. \square

Lemma A.4. *If W_n is a basic set of \mathfrak{A}_σ , then W_n is either empty or infinite.*

Proof. It's a proof by induction over σ . For $\sigma = \iota$, we choose the base in order to have this property. We only treat the case where $\sigma = \tau \rightarrow \tau'$. Then

$$W_n = \bigcap_{\alpha \in \Delta_n} [U_{\alpha_1}, V_{\alpha_2}].$$

If we look at the proof of 2.3, we remark that we could have built infinitely many solutions, simply because we have the freedom to choose any $y_\alpha \in V_{\alpha_2}$ which is infinite by the induction hypothesis. \square

Lemma A.5. *For (U_n) , the canonical base of \mathfrak{R}_σ we have:*

$$U_n \setminus \bigcup_{k=1}^M U_{m_k}$$

is either empty or infinite

Proof. Suppose it is not empty. Choose x in it. Then U_n is infinite according to Lemma A.4. If $\cup_{k=1}^M U_{m_k}$ is disjoint from U_n , then the result is clear. Otherwise, choose y in the intersection. Because of the convexity of U_n , the segment $[x, y]$ is contained in U_n . But, because of closedness, $[x, y] \cap \cup_{k=1}^M U_{m_k}$ is a closed subset of $[x, y]$ that doesn't contain x . So there exist $\epsilon > 0$ such that $I_\epsilon = [x, x + \epsilon(y - x)]$ is disjoint from $\cup_{k=1}^M U_{m_k}$. Therefore I_ϵ is infinite and is contained in $U_n \setminus \cup_{k=1}^M U_{m_k}$. \square

It remains to prove Lemma 2.4.

Proof. 2.4

Let $(U_\alpha)_{\alpha \in \Delta_n}$ be basic closed sets of $X = \mathbb{R}_\sigma$.

The proof consists in 3 steps:

1. we prove the lemma when X is \mathbb{R}^q ,
2. then when X is $X' \rightarrow Y'$,
3. then when X is $X' \times Y'$.

First step Suppose X is \mathbb{R}^q . We want to construct a set of functions ρ_α satisfying Lemma 2.4. We can do this very easily using the property of normality. If α is minimal, then we can construct a function $\bar{\rho}_\alpha$ such that $U_\alpha \subseteq \{\bar{\rho}_\alpha = 1\}$ and for all $\beta \perp \alpha$, $U_\beta \subseteq \{\bar{\rho}_\alpha = 0\}$. If α is not minimal, assume we have constructed $\bar{\rho}_\beta$ for all $\beta \prec \alpha$. Using normality one more time, we can construct a function $\bar{\rho}_\alpha$ such that $U_\alpha \subseteq \{\bar{\rho}_\alpha = 1\}$ and $\rho_\alpha = 0$ on the following closed set which is disjoint from U_α :

$$\left(\bigcup_{\beta \perp \alpha} U_\beta \right) \cup \left(\bigcup_{\substack{\beta \not\prec \alpha \\ \beta \not\perp \alpha}} U_\beta \cap \left\{ \bar{\rho}_{\alpha \wedge \beta} \leq \frac{1}{2} \right\} \right).$$

The assumption on β in the second union guarantees that $\alpha \wedge \beta$ exists and is strictly below α . So by hypothesis, $\bar{\rho}_{\alpha \wedge \beta}$ has been already constructed.

To get the ρ_α , we set

$$\rho_\alpha := \iota \circ \bar{\rho}_\alpha$$

where

$$\begin{aligned} \iota : [0, 1] &\rightarrow [0, 1] \\ x &\mapsto \begin{cases} 3x & \text{if } x \leq \frac{1}{3}, \\ 1 & \text{if } x \geq \frac{1}{3}. \end{cases} \end{aligned}$$

We leave the verification to the reader that these functions satisfy the desired properties.

Second step Now, we assume that $X = X' \rightarrow Y'$. Then each U_α for $\alpha \in \Delta_n$ is thus of the form

$$U_\alpha = \bigcap_{a \in \Delta_\alpha} [A_a, B_a]$$

where A_{a_1} and B_{a_2} are basic closed set of respectively X' and Y' (as earlier, we write $[A_a, B_a]$ instead of $[A_{\pi_1(a)}, B_{\pi_2(a)}]$). Now, we will use Roman indices when we are dealing with basic subsets or separating functions concerning X' or Y' , and we will use Greek indices for their *dual* concerning X . There should be no confusion. As before, we may assume that U_α is in canonical form.

Let $\Sigma_n = \cup_{\alpha \in \Delta_n} \Delta_\alpha$, consider the family $\mathcal{B} = \{B_a \mid a \in \Sigma_n\}$ and apply the induction hypothesis to this finite set of basic sets of Y' . We get the set $(\rho_a)_a$ of *separating* functions from Y' to \mathbb{R} satisfying the condition of Lemma 2.4 for the basic sets $(B_a)_{a \in \Sigma_n}$ (if this finite family is not complete, we add all the possible intersections of all the B_a to complete it).

Remark A.6. For α and β in Δ_n , notice that

$$(6) \quad \begin{aligned} U_\alpha \cap U_\beta &= \left(\bigcap_{a \in \Delta_\alpha} [A_a, B_a] \right) \cap \left(\bigcap_{b \in \Delta_\beta} [A_b, B_b] \right) \\ &\quad \cap \left(\bigcap_{(a,b) \in \Delta_\alpha \times \Delta_\beta} [A_a \cap A_b, B_a \cap B_b] \right) \end{aligned}$$

and that this is the canonical form for this intersection.

Now, assume that $U_\alpha \cap U_\beta = \emptyset$, then, we can use the induction hypothesis (because we are working at a type whose length is strictly smaller than σ) and have that there exists $(a, b) \in \Delta_\alpha \times \Delta_\beta$ such that $A_a \cap A_b \neq \emptyset$ and $B_a \cap B_b = \emptyset$. We will denote this by $a \perp b$ without any risk of confusion.

Thus, for all a and b in Σ_n such that $a \perp b$, we can choose an $x \in A_a \cap A_b$. Then, the function $\phi_{a,x} := \lambda f. \rho_a(f(x))$ will separate the two disjoint closed sets $[A_a, B_a]$ and $[A_b, B_b]$ in the sense that $[A_a, B_a] \subseteq \{\phi_{a,x} = 1\}$ and $[A_b, B_b] \subseteq \{\phi_{a,x} = 0\}$. Such an x is called a *witness* and for all such a and b in Σ_n we choose exactly one witness in $A_a \cap A_b$. So let \mathcal{W} , the set of all those witnesses. It is obviously finite.

For $\alpha \in \Delta_n$, we can define the *weight* of α to be the number of elements of $\downarrow \alpha$; and we will write it $|\alpha|$. Because Δ_n is finite, there is an M that dominate the weight of all the $\alpha \in \Delta_n$. The only property of the weight we are interested in is that if $\alpha \prec \beta$ then $|\alpha| < |\beta|$.

Given a function ϕ whose codomain is $[0, 1]$, we define

$$\phi^{\#n} := \iota_n \circ \phi,$$

where

$$\begin{aligned} \iota_n : [0, 1] &\rightarrow [0, 1] \\ x &\mapsto \begin{cases} 0 & \text{if } x \leq \frac{n}{M}, \\ Mx - n & \text{if } \frac{n}{M} \leq x \leq \frac{n+1}{M}, \\ 1 & \text{if } x \geq \frac{n+1}{M}. \end{cases} \end{aligned}$$

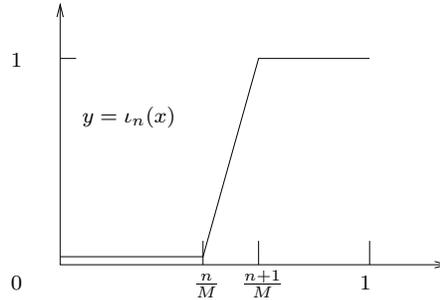


FIGURE 2. graph of ι_n .

Remark A.7. $\phi^{\#n+1}(x) > 0 \implies \phi^{\#n}(x) = 1$.

We define ρ_α as follow:

$$(7) \quad \rho_\alpha := \prod_{a \in \Delta_\alpha, x \in \mathcal{W} \cap A_a} \phi_{a,x}^{\#|\alpha|}.$$

Now we have to check the statements of the lemma. First, if $f \in U_\alpha$, then for all a in Δ_α and all witnesses $x \in A_a$ we have $f(x) \in B_a \implies \phi_{a,x}(f) = \rho_a(f(x)) = 1$ so $\rho_\alpha(f) = 1$. So the first statement is checked.

Now if $\beta \perp \alpha$ and $f \in U_\beta$, then, using Remark A.6, there exist $a \in \Delta_\alpha$ and $b \in \Delta_\beta$ such that $a \perp b$. So by construction, there exists a witness $x \in A_a \cap B_a$, such that $\phi_{a,x}$ appears in equation (7). Then, $\phi_{a,x}(f) = \rho_a(f(x)) = 0$ because $f(x) \in B_b$, $B_b \cap B_a = \emptyset$ and the separation properties of the family (ρ_a) . This implies $\rho_\alpha(f) = 0$. So the second statement is checked.

Now we assume that $\rho_\alpha(f) > 0$ and $f \in U_\beta$ for a $\beta \not\perp \alpha$ and $\beta \not\leq \alpha$. Then, let $\gamma = \alpha \wedge \beta$. We have $\gamma < \alpha$ and we have to check that $\rho_\gamma(f) = 1$. Choose $a \in \Delta_\gamma$ and a witness $x \in A_a$. Then, because $U_\gamma = U_\alpha \cap U_\beta$ and the Remark A.6, there are three possibilities:

1. $\mathbf{a} \in \Delta_\alpha$. So $\phi_{a,x}$ appears in the definition of ρ_α . But, because $\rho_\alpha(f) > 0$, we have that $\phi_{a,x}^{\#|\alpha|}(f) > 0$; and this implies $\phi_{a,x}^{\#|\gamma|}(f) = 1$ because of Remark A.7 and the fact that $|\gamma| < |\alpha|$.
2. $\mathbf{a} \in \Delta_\beta$. $f \in U_\beta \implies f(x) \in B_a \implies \rho_a(f(x)) = \phi_{a,x}(f) = 1 \implies \phi_{a,x}^{\#|\gamma|}(f) = 1$.
3. **none of the above.** This means that there exist $k \in \Delta_\alpha$ and $l \in \Delta_\beta$ such that $A_a = A_k \cap A_l$ and $B_a = B_k \cap B_l$. The witness x is in $A_k \supseteq A_a \ni x$. Thus, in the definition of ρ_α , $\phi_{k,x}$ is present. So there are several possibilities. The first one occurs when B_k and B_l are not comparable with respect to inclusion. Then B_a is strictly smaller than B_k and B_l . Because $\rho_\alpha(f) > 0$, we have that $\rho_k(f(x)) > 0$. Moreover, we have $f(x) \in B_l$ because $f \in U_\beta$. So applying Property 3 to the family ρ_a , we have that $\rho_a(f(x)) = 1$. This implies $\phi_{a,x}^{\#|\gamma|}(f) = 1$. The second possibility occurs when $B_k \subseteq B_l$, then $B_a = B_k$. In this case, $\rho_a = \rho_k$, so

$\rho_\alpha(f) > 0 \implies \phi_{a,x}^{\#\alpha|}(f) > 0 \implies \phi_{k,x}^{\#\alpha|}(f) > 0$ and, because $|\gamma| < |\alpha|$, we have $\phi_{a,x}^{\#\gamma|}(f) = 1$. The third possibility occurs when $B_l \subseteq B_k$. Then $B_a = B_l$. Because $f \in U_\beta$, we have $f(x) \in B_l = B_a$. So $\rho_a(f(x)) = 1 \implies \phi_{a,x}^{\#\gamma|}(f) = 1$.

In each situation, we can prove that $\phi_{a,x}^{\#\gamma|}(f) = 1$, so by definition of ρ_γ , we know that $\rho_\gamma(f) = 1$.

Third step Now we assume that $X = X' \times Y'$. Then we know that each U_α is of the form $U_\alpha = V_{\alpha_1} \times W_{\alpha_2}$. So we build Δ_m and Δ_p such that, whenever there exists $\alpha \in \Delta_n$, then $U_\alpha = V_{\alpha_1} \times W_{\alpha_2}$ for $(\alpha_1, \alpha_2) \in \Delta_m \times \Delta_p$. Then, we apply the induction hypothesis to build two sets of separating functions $(\rho'_{\alpha_1})_{\alpha_1 \in \Delta_m}$ and $(\rho''_{\alpha_2})_{\alpha_2 \in \Delta_p}$, and we define ρ_α with:

$$\rho_\alpha(x_1, x_2) = \rho'_{\alpha_1}^{\#\alpha|}(x_1) \rho''_{\alpha_2}^{\#\alpha|}(x_2).$$

We leave the verification to the reader that this family of functions satisfy the hypotheses of Lemma 2.4

This finally proves the lemma. \square

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ENS – DÉPARTEMENT D'INFORMATIQUE 45 RUE D'ULM, F-75230 PARIS
CEDEX 05, FRANCE

E-mail address: `dejaeger@di.ens.fr`