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SUBGROUPS OF PRODUCTS OF LOCALLY COMPACT GROUPS

DIKRAN DIKRANJAN AND SIDNEY A. MORRIS

Abstract. A topological group G is said to be *locally q-minimal* if there exists a neighbourhood V of the identity of G such that whenever H is a Hausdorff group and $f : G \to H$ is a continuous surjective homomorphism with $f(V)$ a neighbourhood of 1 in H, then f is open. Locally compact groups are locally q-minimal. It is shown that under certain circumstances complete locally q-minimal groups are locally compact. This occurs for subgroups of products of locally compact groups in two cases: a) for products of locally compact abelian groups; b) for connected subgroups of products of locally compact MAP groups. It is also shown that "MAP" cannot be removed.

1. Introduction

Throughout this note all topological groups are assumed to be Hausdorff, unless otherwise stated explcitly. Occasionally for emphasis we shall explicitly state that a certain space is Hausdorff. We denote by $\mathcal{V}_{(G,\tau)}(1)$ (or simply by $\mathcal{V}_\tau(1)$ or $\mathcal{V}_G(1)$ when no confusion is possible) the filter of neighbourhoods of 1 in a topological group (G, τ).

Minimal topological spaces have been extensively studied in the literature ([2]). Minimal topological groups were introduced by

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Choquet and Stephenson [7, 15] (see [3, 6] for recent advances in this field).

The following interesting generalization of minimality was recently introduced by Morris and Pestov [11]:

Definition 1.1. A topological group (G, τ) is said to be *locally minimal* if there exists a neighbourhood of the identity V such that whenever $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V has a non-empty σ-interior, then $\sigma = \tau$.

In the sequel we use the following equivalent form of local minimality: a topological group (G, τ) is locally minimal if and only if there exists a τ-neighbourhood of the identity V such that whenever $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V is a σ-neighbourhood of 1, then $\sigma = \tau$. We show below that all locally compact groups are locally minimal. While local compactness is preserved under taking quotient groups, this fails for minimality. The minimal groups which have all their Hausdorff quotients minimal are called *totally minimal* ([5]; some authors prefer the term *q-minimal*). Motivated by this idea, we introduce here a notion with similar properties with respect to *local* minimality.

Definition 1.2. A topological group G is said to be *locally q-minimal with respect to a neighbourhood* V of the identity of G if whenever H is a Hausdorff group and $f : G \to H$ is a continuous surjective homomorphism such that $f(V)$ is a neighbourhood of 1 in H, then f is open.

Often we say briefly G *is locally q-minimal* if there exists such a neighbourhood V. It is easy to see that a topological group G is locally q-minimal with respect to some neighbourhood V of 1 if and only if for every closed normal subgroup N of G the quotient group G/N is locally minimal with respect to the neighbourhood $(VN)/N$ of 1 in G/N. We give an example to distinguish local minimality and local q-minimality in 2.10 and we show that locally compact groups are actually locally q-minimal (cf. Lemma 2.8). In Proposition 2.7 we extend a result from [11] by showing that every subgroup of a Lie group is locally q-minimal.

Infinite products of locally compact (abelian) groups need not be locally compact. Closed subgroups of such products are necessarily complete.

Theorem 1.3. *A closed subgroup of a product of locally compact abelian groups is locally compact if and only if it is locally q-minimal.*

In other words, the complete locally q-minimal abelian groups that are closed subgroups of the products of locally compact abelian groups are precisely the LCA groups.

In the non-abelian case we can prove the following:

Theorem 1.4. *A closed connected subgroup of a product of locally compact MAP groups is locally compact if and only if it is locally q-minimal.*

Example 1.5. One cannot remove "MAP" in the above theorem since the group $G = SL_2(\mathbb{R})^\omega$ is totally minimal (hence, locally q-minimal) by a theorem of Stoyanov-Remus [13] (see also [6, Corollary 7.4.4]), but not locally compact.

We do not know whether local q-minimality can be replaced by the weaker assumption of local minimality in Theorems 1.3 and 1.4. We can see that this is true for products of locally compact abelian groups (instead of just subgroups of products).

Theorem 1.6. *For an infinite family $\{G_i\}_{i\in I}$ of locally compact abelian groups the following properties are equivalent:*

(a) *the product $\prod_{i\in I} G_i$ is locally minimal;*
(b) *the product $\prod_{i\in I} G_i$ is locally compact.*

The proofs of these three theorems are given in §3. In §2 we establish some useful properties of the locally (q-) minimal groups.

1.1. Notations and terminology

In what follows all group topologies are assumed to be Hausdorff. For a topological group G we denote by $\widetilde{G}$ the Raĭkov completion of G and by $c(G)$ the connected component of the identity of G.

We recall here some compactness-like conditions on a topological group G. A group G is said to be *precompact* (some authors prefer "totally bounded") if $\widetilde{G}$ is compact.

The subgroup generated by a subset X of a group G is denoted by $\langle X\rangle$, and $\langle x\rangle$ is the cyclic subgroup of G generated by an element $x \in G$.

We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive integers and prime numbers, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rational numbers, by $\mathbb{R}$ the real numbers, and by $\mathbb{T}$ the unit circle group which is identified with $\mathbb{R}/\mathbb{Z}$. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$. For a prime p the symbol $\mathbb{Z}(p^\infty)$ denotes the quasicyclic p-group and $\mathbb{Z}_p$ the p-adic integers.

The *torsion part* $t(G)$ of an abelian group G is the set $\{g \in G : ng = 0, \text{ for some } n \in \mathbb{N}\}$. Clearly, $t(G)$ is a subgroup of G. For a prime p, the *p-primary component* G_p of G is the subgroup of G that consists of all $x \in G$ satisfying $p^n x = 0$, for some positive integer n. The group G is said to be *divisible* if $nG = G$ for every $n \in \mathbb{N}$.

All unexplained topological terms can be found in [8]. For background on abelian groups, see [9] and [14].

2. Local minimality vs minimality

Here we establish properties of the locally q-minimal groups necessary for the proof of our main theorems. They may be of independent interest.

Lemma 2.1. *If the topological group G is locally q-minimal, then every quotient group of G is locally q-minimal.*

Proof. Let $V \in \mathcal{V}_G(1)$ witness local q-minimality of G. Consider a quotient homomorphism $f : G \to G/N$. Then $f(V)$ is a neighbourhood of 1 in G/N that witnesses local q-minimality of G/N. □

Now we give a canonical construction of an extension $\widetilde{\sigma}$ of a group topology σ defined on a closed central subgroup H of a group G.

Lemma 2.2. *Let G be a topological group with topology τ and let H be a closed central subgroup of G. Then every group topology σ on H with $\sigma \leq \tau|_H$ extends to a group topology $\widetilde{\sigma}$ of G such that $\widetilde{\sigma} \leq \tau$. Moreover,*

(a) if H is τ-open, then H is also $\widetilde{\sigma}$-open;

(b) if $\tau' \leq \tau$ is a group topology on G such that $\tau'|_H \leq \sigma$, then $\tau' \leq \widetilde{\sigma}$.

(c) if σ is Hausdorff, then $\widetilde{\sigma}$ is also Hausdorff;

(d) if for some neighbourhoods V_0, V of 1 in G, $H \cdot V_0$ is contained in V, then V is also a $\widetilde{\sigma}$-neighbourhood of 1 in G.

Proof. Let $\widetilde{\sigma}$ be the group topology on G with base of neighbourhoods of 1 consisting of all possible products $U \cdot V$, where U is a σ-neighbourhood of 1 in H and V is a τ-neighbourhood of 1. Obviously, $\widetilde{\sigma}|_H \leq \sigma$ and $\widetilde{\sigma} \leq \tau$. To show that $\widetilde{\sigma}$ extends σ take a neighbourhood U of 1 in (H, σ). Then there exists a neighbourhood W of 1 in (H, σ) such that $W^2 \subseteq U$. Now find a neighbourhood O of 1 in (G, τ) such that $O \cap H = W$. Then $W \cdot O \cap H \subseteq U$, so U is a neighbourhood of 1 in $(H, \widetilde{\sigma}|_H)$ too. This proves $\widetilde{\sigma}|_H = \sigma$.

(a) is obvious.

(b) Assume $\tau' \leq \tau$ is a group topology on G such that $\tau'|_H \leq \sigma$. Let B be a neighbourhood of 1 in (G, τ'). Then there exists a neighbourhood A of 1 in (G, τ') with $A^2 \subseteq B$. By our assumptions on τ' there exist $U \in \mathcal{V}_{(H,\sigma)}(1)$ and $O \in \mathcal{V}_{(G,\tau)}(1)$ with $U \subseteq H \cap A$ and $O \subseteq A$. Then $U \cdot O \subseteq B$, hence $B \in \mathcal{V}_{(G,\widetilde{\sigma})}(1)$.

(c) Since H is closed the intersection $\bigcap\{U \cdot V : U \in \mathcal{V}_{(H,\sigma)}(1), V \in \mathcal{V}_{(G,\tau)}(1)\}$ coincides with $\bigcap\{\overline{U} : U \in \mathcal{V}_{(H,\sigma)}(1)\} \subseteq H$. Since σ is Hausdorff, this intersection is $\{1\}$.

(d) This is obvious. □

Lemma 2.3 reveals the connection between "locally minimal" and "minimal" (in the sense that "sufficiently small" subgroups of a locally minimal group are minimal). This is an important step in our proof since minimality implies precompactness for abelian groups, due to a well known deep theorem of Prodanov and Stoyanov [12].

Lemma 2.3. *Let G be a locally minimal group with respect to a neighbourhood V of 1. If H is a closed central subgroup of G, such that $H \cdot V_0 \subseteq V$ for some neighbourhood V_0 of 1 in G, then H is minimal.*

Proof. Let τ denote the topology of G. To show that $(H, \tau|_H)$ is minimal pick a Hausdorff group topology $\sigma \leq \tau|_H$ on H. Then the extension $\widetilde{\sigma}$ of σ defined as in the previous lemma is coarser than τ and V is a $\widetilde{\sigma}$-neighbourhood of 1. Hence $\widetilde{\sigma} = \tau$ by the local minimality of τ with respect to V. Thus $\sigma = \widetilde{\sigma}|_H = \tau|_H$. This proves that $(H, \tau|_H)$ is minimal. □

The next lemma and corollary justify the name "locally minimal" (see also Example 2.6).

Lemma 2.4. *Let G be a topological group that admits an open subgroup H that is locally minimal as a topological subgroup of G. Then G is locally minimal.*

Proof. Let τ be the topology of G. Since H is open in G there exists a τ-open neighbourhood V of 1 in H witnessing local minimality of H. Let us see that V witnesses local minimality of G too. Indeed, assume $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V is σ-open. Since H is locally minimal and $V \in \mathcal{V}_{(H,\sigma|_H)}$, the identity $id_H : (H, \tau|_H) \to (H, \sigma|_H)$ is a homeomorphism. Since H is open in both τ and σ, we conclude that id_H is a local homeomorphism, thus it is a homeomorphism. □

Corollary 2.5. *Let G be a topological group that admits an open subgroup H that is minimal as a topological subgroup of G. Then G is locally minimal.*

It is tempting to conjecture that *all locally minimal groups can be obtained in this way.* The next example shows that this is not the case in general. We show below that this is true for the linear group topologies on arbitrary abelian groups (cf. Corollary 2.9).

Example 2.6. According to [11] every subgroup G of a Lie group L is locally minimal (we show in the next proposition that G is actually locally q-minimal).

(a) $\mathbb{R}$ is locally minimal (being a Lie group), but no open subgroup of $\mathbb{R}$ is minimal. More generally, a locally compact abelian group G has an open minimal subgroup if and only if G has an open compact subgroup (and consequently, G has no vector subgroups). Indeed, if H is an open minimal subgroup of G, then H is locally compact. By Stephenson's theorem [15] H must be compact. We show below that every locally compact group is locally minimal. Hence, this example shows how large the gap is between local minimality and the sufficient property given in Corollary 2.5 in the class of locally compact abelian groups.

(b) For a non-locally-compact example consider the subgroup $G = \mathbb{Z}(p^\infty)$ of $\mathbb{T}$. Then G is locally q-minimal by the next proposition, but G has no proper open subgroups and G itself is not minimal. Thus G has no open minimal subgroup. An analogous argument shows that any dense embedding of $\mathbb{Z}$ in $\mathbb{T}$ induces a locally q-minimal topology on $\mathbb{Z}$ without open minimal subgroups.

Proposition 2.7. *Every subgroup of a Lie group is locally q-minimal.*

Proof. Let G be a subgroup of a Lie group L. Since closed subgroups of a Lie group are Lie groups, we can assume that G is dense in L. To see that G is locally q-minimal, consider a quotient G/N of G, where N is a closed normal subgroup of G. In view of the result in [11] quoted in Example 2.6 it suffices to see that G/N is isomorphic to a subgroup of a quotient of L. Let $\overline{N}$ be the closure on N in L. Then $\overline{N}$ is a closed normal subgroup of L by the density of G in L. Since $N = \overline{N} \cap G$ is dense in $\overline{N}$, it follows from [6, Lemma 4.3.2] that the restriction $f|_G : G \to f(G)$ of the (open) quotient map $f : L \to L/\overline{N}$ is still open, where $f(G)$ carries the induced from L topology. This means that the subgroup $f(G)$ of the quotient group $L/\overline{N}$ has the quotient topology with respect to $f|_G$. Since $\ker f \cap G = N$, this proves that the quotient G/N is isomorphic to the subgroup $f(G)$ of $L/\overline{N}$. It remains to note that the quotient G/N is a isomorphic to a subgroup of a Lie group, hence G/N is locally minimal. $\square$

Lemma 2.8. *Every locally compact group is locally q-minimal.*

Proof. Let (G, τ) be a locally compact group and let $V \in \mathcal{V}_\tau(1)$ be a compact neighbourhood of 1. First we shall check that G is locally minimal with respect to V. Assume $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V a σ-neighbourhood of 1. Since V is τ-compact, the identity $id_V : (V, \tau|_V) \to (V, \sigma|_V)$ is a homeomorphism. We conclude as in Lemma 2.4. Now let G/N be a quotient group of G. Then $U = (V \cdot N)/N$ is a compact neighbourhood of 1 in G/N. Hence the above argument shows that G/N is locally minimal with respect to U. Then G is locally q-minimal with respect to V. $\square$

Corollary 2.9. *Let G be a linearly topologized abelian group. Then G is locally minimal if and only if G has an open minimal subgroup.*

Proof. Since by hypothesis $\mathcal{V}_G(1)$ has a base of open subgroups, we can assume without loss of generality that local minimality of G is witnessed by an open subgroup V of G. Then $V = V \cdot V$ so that, by Lemma 2.3, V is a minimal subgroup of G. $\square$

Example 2.10. Let c be a topological generator of the compact monothetic group $K = \prod_p \mathbb{Z}_p$. Consider the subgroups $N = \prod_p p\mathbb{Z}_p$ and $G = \langle c \rangle + N$ of K. Then G is dense and minimal by the Minimality Criterion [6], but for the closed subgroup N of K the quotient group $G/N \cong \langle c \rangle$ is not locally minimal as it has no open subgroup that is minimal (cf. Corollary 2.9).

3. Proof of the main theorems

The proof of Theorem 1.3 is based on the following lemma concerning a "varietal" property of LCA groups.

Lemma 3.1. *An LCA group G having an open compact subgroup is topologically isomorphic to a closed subgroup of a product $\mathbb{T}^\beta \times D$, where β is a cardinal and D is a discrete abelian group.*

Proof. Let G be a LCA group having a compact open subgroup of K. Then there exists an embedding $\nu : K \to \mathbb{T}^\beta$ for some cardinal β. Since $\mathbb{T}^\beta$ is divisible ν can be extended to a homomorphism $\widetilde{\nu} : G \to \mathbb{T}^\beta$ which is necessarily continuous since K is open in G and the restriction ν of $\widetilde{\nu}$ to K is continuous. Let $h : G \to G/K$ be the canonical homomorphism to the (discrete) quotient group $D = G/K$. Then the homomorphism $g = \langle \widetilde{\nu}, h \rangle : G \to \mathbb{T}^\beta \times D$ is continuous, as a cartesian product of two continuous homomorphisms. Moreover, $N = \ker \widetilde{\nu}$ intersects K in 0 since ν is injective. Therefore, g is a continuous monomorphism of G into $\mathbb{T}^\beta \times D$. Since its restriction to the open subgroup K coincides with the embedding $\nu : K \to \mathbb{T}^\beta$, we conclude that g is an embedding too. $\square$

For convenience we give the following corollary obtained by the simple observation that according to the lemma and the structure theory of LCA groups (cf. [10, 24.30]), every LCA group is a (closed) subgroup of some product $\mathbb{R}^n \times \mathbb{T}^\beta \times D$, where D is a discrete abelian group.

Corollary 3.2. *A closed subgroup of a product of locally compact abelian groups is isomorphic to a closed subgroup of a product $\mathbb{R}^\alpha \times \mathbb{T}^\beta \times \prod_{i \in I} D_i$, where α, β are cardinal numbers and D_i, $i \in I$, are discrete abelian groups.*

In other words, the class $\overline{S}C(LCA) = \overline{S}C(\mathbb{R}, \mathbb{T},$ all discrete abelian groups), where $\overline{S}$ and C respectively denote the formation of all closed subgroups and the formation of all (cartesian) products, while LCA here denotes the class of all locally compact abelian groups.

Proof of Theorem 1.3. Let G be a closed subgroup of a product of LCA groups. Then by Corollary 3.2 G can be considered as a closed subgroup of a group $K = \mathbb{R}^\alpha \times \mathbb{T}^\beta \times \prod_{i\in I} D_i$ for some cardinal numbers α, β and discrete abelian groups D_i, $i \in I$. Since local compactness implies local q-minimality (Lemma 2.8), we have to prove that G is LCA if it is locally q-minimal. Let V be a neighbourhood of 0 in G witnessing local q-minimality of G. We can assume without loss of generality that there exist finite sets $F \subset \alpha$, $F' \subseteq \beta$ a cofinite set $J \subseteq I$ and $\varepsilon > 0$ such that $V = G \cap W$, where

$$W = [(-\varepsilon, \varepsilon)^F \times \mathbb{R}^{\alpha\setminus F}] \times [O_\varepsilon \times \mathbb{T}^{\beta\setminus F'}] \times \prod_{i\in J} D_i,$$

with O_ε the ε-ball with center 0 and radius ε in $\mathbb{T}^{F'}$. Let $N_0 = \mathbb{R}^{\alpha\setminus F} \times \mathbb{T}^{\beta\setminus F'} \times \prod_{i\in J} D_i$. (We suppress any symbols for trivial factors such as $\{0\}^{I\setminus J}$.) Clearly, N_0 is a closed subgroup of K with $N_0 + W = W$. Then $N = G \cap N_0$ is a closed subgroup of G with $N + V \subseteq V$. By Lemma 2.3 the subgroup N of G is minimal, hence precompact by Prodanov–Stoyanov's theorem [12]. On the other hand, N is a closed subgroup of K and hence complete. This proves that the subgroup N is compact. Consider the quotient homomorphism $f : G \to K/N_0 \cong \mathbb{R}^F \times \mathbb{T}^{F'} \times D$, where $D = \prod_{i\in I\setminus J} D_i$. Since F, F' and $I \setminus J$ are finite, D is discrete and K/N_0 is locally compact. Let us see now that $f(V)$ is a neighbourhood of 0 in $f(G)$. Indeed, $U = f(W)\cap f(G)$ is clearly a neighbourhood of 0 in $f(G)$, so it suffices to prove that $U = f(V)$. The inclusion $f(V) \subseteq U$ is obvious. If $u = f(g) = f(w)$ for some $g \in G$ and $w \in W$, then $g-w \in N_0$, so that $g \in w+N_0 \subseteq W+N_0 = W$ and now $V = G\cap W$ implies $g \in V$. Hence $u \in f(V)$. By local q-minimality of G the restriction $f : G \to f(G)$ is open, i.e., $f(G) \cong G/N$. Now $f(G)$ is locally precompact as a subgroup of the locally compact group K/N_0, hence G must be locally precompact too, since N is compact. By the completeness of G we conclude that G is LCA.

Proof of Theorem 1.4. Assume that G is a closed connected subgroup of a product $\prod_{i\in I} L_i$, where each L_i is a locally compact MAP group. It is not restrictive to assume that each L_i is connected, otherwise one can replace L_i by the closure of the projection of G into L_i. By Freudenthal-Weyl theorem $L_i = \mathbb{R}^{n_i} \times K_i$, where K_i is a compact (connected) group. Let $\mathbb{U} = \prod_n \mathbb{U}(n)$, where $\mathbb{U}(n)$ is the group of unitary $n \times n$ matrices over $\mathbb{C}$. Then G is a closed subgroup of a product $K = \mathbb{R}^\alpha \times \mathbb{U}^\beta$. Arguing as in the above proof find finite sets $F \subseteq \alpha$, $F' \subseteq \beta$ and $\varepsilon > 0$ such that for $W = [(-\varepsilon, \varepsilon)^F \times \mathbb{R}^{\alpha\setminus F}] \times [O_\varepsilon \times \mathbb{U}^{\beta\setminus F'}]$, where O_ε is the ε-ball with center 1 and radius ε in $\mathbb{U}^{F'}$, $V = G \cap W$ witnesses local q-minimality of G. Let $N_0 = \mathbb{R}^{\alpha\setminus F} \leq K$ and note that N_0 is a closed central subgroup of K with $N_0 \cdot W = W$. Then $N = G \cap N_0$ is a closed central subgroup of G with $N \cdot V \leq V$. By Lemma 2.3 N is minimal. Being abelian, N is precompact. Since K is complete, N is complete too. Thus N is compact. As before, the quotient homomorphism $f : K \to K/N_0 \cong \mathbb{R}^F \times \mathbb{U}^\beta$ sends G to a closed subgroup $H = f(N)$ of K/N_0 and $f(V)$ is a neighbourhood of 1 in $f(G)$ (checked as before). This gives $H \cong G/N$ by the local q-minimality of G. Since K/N_0 is locally compact and G is complete, this entails (as in the above proof) that G is locally compact too. QED

Lemma 3.3. *Let $\{G_i\}_{i\in I}$ be a family of topological groups such that infinitely many groups G_i are not minimal. Then the product $\prod_{i\in I} G_i$ is not locally minimal.*

Proof. Let τ denote the Tychonov topology of $G = \prod_{i\in I} G_i$ and assume that some $V \in \mathcal{V}_\tau(1)$ witnesses local minimality. Then there exists a finite subset $J \subseteq I$ such that V contains $\ker p_J$, where p_J is the projection of G on the finite subproduct $\prod_{i\in J} G_i$. Now we have to build a topology $\sigma < \tau$ such that V is still a neighbourhood of 1 in (G, σ). To this end fix an index $i_0 \in I \setminus J$ such that the group G_{i_0} is not minimal. This is possible in view of our hypothesis. Let σ' be a topology on G_{i_0} strictly weaker than the original topology of G_{i_0}. Let σ denote the product topology on G where all groups G_i have their original topology except the group G_{i_0} that is equipped with σ'. Then V is a neighbourhood of 1 in σ and $\sigma < \tau$. $\square$

Proof of Theorem 1.6. (b) ⇒ (a) is trivial.

(a) ⇒ (b) Assume that the product is not locally compact. This means that infinitely many groups G_i are not compact. Since they are complete (being locally compact), by Prodanov-Stoyanov's theorem, infinitely many groups G_i are not minimal. Now by Lemma 3.3 the product $\prod_{i \in I} G_i$ is not locally minimal. QED

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Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze 206, 33100 Udine, Italy
E-mail address: dikranja@dimi.uniud.it

Head, School of Information Technology and Mathematical Sciences, University of Ballarat, P.O. Box 663, Ballarat, Victoria 3353, Australia
E-mail address: s.morris@ballarat.edu.au