Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

TOPOLOGY PROCEEDINGS
Volume 26, 2001–2002
Pages 627–649

COMPACT SEMIGROUPS AND SUITABLE SETS

J. HE, K.H. HOFMANN, S.M. MILLER, AND D.A. ROBBIE*

Abstract. A suitable set A in a topological semigroup S is a subset of S which contains no idempotents, any limit points of A in S are idempotents, and A, together with all idempotents of S, generates a dense subsemigroup of S. Following work of Hofmann and Morris, who showed that every compact Hausdorff topological group has such a suitable set, this paper extends that result to several classes of compact semigroups all of whose members satisfy $S^2 = S$. In particular all compact simple semigroups are shown to have a suitable set. Cartesian products of compact monoids each with a suitable set have suitable sets as do continuous homomorphic images of compact semigroups with suitable sets. It is shown that certain classes of $\mathcal{H}$-chain semigroups have suitable sets. The class of irreducible semigroups falls into two classes, where the members of one class always have a suitable set and in the other class a semigroup which contains no suitable set is constructed. It is shown that compactifications of subsemigroups of Lie groups tend to have suitable sets; these include the 'triangle semigroup' as a typical test case. If S is compact, connected, and $S^2 \neq S$, then S cannot have a suitable set.

2000 *Mathematics Subject Classification.* Primary 22A05, 54D20; Secondary 22A15.

Key words and phrases. Suitable sets, compact semigroups, $S^2 = S$, irreducible semigroups, Lie semigroups.

*The research was partially supported by a grant from the 2001 Melbourne Research Development Grant Scheme, University of Melbourne.

1. Introduction

The theory of compact semigroups was first developed in the early 1950's by Alexander Doniphan Wallace [16]. A *topological semigroup* is simply a Hausdorff space with a continuous associative multiplication. Compact groups, in particular, have been the subject of active research since the end of the 19th century. As a result they are now well understood. In stark contrast with compact groups, however, the myriad of possible forms of compact semigroups with identity makes a general structure theory all but impossible with the exception of a few common general features. For instance, the set $E(S)$ of idempotents of a compact semigroup S is never empty. If $E(S)$ is singleton and $S^2 = S$, then S is a group; on the other side if S is a compact space with a base point e, then the multiplication $st = e$ for all $s, t \in S$ makes it into a compact topological semigroup with a single idempotent. Of course there is an abundance of compact semigroups both commutative and non-commutative of non-trivial type that have more than one idempotent. The most important examples for this paper will be introduced as we proceed. Here is the crucial definition:

Definition 1.1. A *suitable set* A in a topological semigroup S is a subset of S which contains no idempotents, any limit points of A in S are idempotents, and A, together with all idempotents of S, generates a dense subsemigroup of S (i.e. $\overline{\langle A \cup E(S) \rangle} = S$).

Hofmann and Morris showed in [13] that every compact Hausdorff topological group has such a suitable set. Here we extend this result to various classes of compact semigroups S satisfying $S^2 = S$. No compact connected semigroup S with $S^2 \neq S$ has a suitable set as we shall show. An idempotent topological semigroup, and indeed any topological semigroup which is algebraically and topologically generated by its set $E(S)$ of idempotents has a suitable set, namely, the empty set. Each compact semigroup S has a unique minimal ideal which is completely simple; each completely simple compact semigroup will be seen to have a suitable set. In each compact connected topological monoid (semigroup with identity) S, by an application of Zorn's Lemma, we find a minimal compact connected subsemigroup T that joins the identity to the minimal ideal.

Definition 1.2. A topological monoid, S, is defined to be *irreducible*, if it is compact and connected, and if it contains no proper compact connected subsemigroup containing the identity and meeting the minimal ideal.

As the 'triangle semigroup' of 3.2 (see also 6.1) shows, irreducible subsemigroups of compact connected topological monoids need not be unique. We shall see that certain irreducible semigroups, recognizable by the structure of their set of idempotents, have suitable sets, and that others with a comparatively simple infinite set of idempotents have no suitable sets.

The concept of a suitable set generalised and grew from the existence in many topological groups of finite subsets which group generated a dense subgroup of the group. Since [13] many mathematicians have written papers on suitable sets in topological groups. Amongst these are Comfort, Dikranjan, Morris, Robbie, Svetlichny, Shakhmatov, Tkachenko, Tkachuk, Trigos-Arrieta and many others. See for example [5]. The present paper is, to the authors' knowledge, the first venture into suitable sets in topological semigroups as such.

We refer the reader to [2], [3], [11], [14], [16], and [7]; for the standard results on topological semigroups used in this paper. For general topology facts Engelking [6] should suffice.

The authors would also like to acknowledge that they are indebted to Professor T. Nogura of Ehime University, Japan, for enabling a visit there by D. Robbie; during which time Professor D. Shakhmatov, also of Ehime University, suggested our current definition of a suitable set, which has proved to be a significant improvement over that initially proposed.

2. Suitable Sets in Groups and Semigroups

We first need to see that, in the case of a compact Hausdorff topological group, our definition of suitable set for a topological semigroup given above does correctly generalise the original definition of Hofmann and Morris for a suitable set in a topological group.

Their definition was equivalent to saying that, a suitable set A in a topological group G, is a subset A of G to which the identity, e of G, does not belong, the only possible limit point of A in G is e, and $A \cup \{e\}$ group generates a dense subgroup of G.

Clearly if G is Hausdorff and is regarded as a topological semigroup, and if it has a subset A which is a suitable set in the topological semigroup sense, then that same set A will *a-fortiori* be a suitable set in the topological group sense.

In the other direction, if G is a compact Hausdorf topological group which has a suitable set A in the topological group sense of Hofmann and Morris, then if we consider the subsemigroup of G generated by $A \cup \{e\}$, its closure will be a compact topological subsemigroup of G, and hence will be cancellative as well. It is a folklore theorem by now that such a topological semigroup must be a topological group. So it must contain the inverses of all elements of A and thus contain the subgroup generated by A, and the closure of that subgroup. Thus it must be all of G because A is a suitable set in the topological group sense. Hence A is a suitable set in the topological semigroup sense, for G is regarded as a topological semigroup.

Recall that a subset C of a topological space is called *relatively compact* if $\overline{C}$ is a compact subspace.

Example 2.1. The additive semigroup $\mathbb{H}$ of non-negative reals has countable relatively compact suitable sets. In fact let A be a subset of $\mathbb{H}$ such that 0 is a sole limit point of A and $0 \notin A$. Every $x \in \mathbb{H}$ is within δ of the set $\{n\delta : n \in \mathbb{N}\} \subseteq \langle A \rangle \subseteq \langle A \cup \{0\} \rangle$. Since we can find arbitrarily small δ's in A, we conclude $\overline{\langle A \rangle} = \mathbb{H}$, and thus A is a suitable set of $\mathbb{H}$. It is clear that A may be chosen countable and relatively compact, and indeed contained in any preassigned interval $]0, \epsilon]$, $\epsilon > 0$.

Theorem 2.2. *Closures of continuous homomorphic images of topological semigroups with relatively compact suitable sets have suitable sets.*

Proof. Let S be a topological semigroup with relatively compact suitable set A and $f\colon S \to T$ a morphism of topological semigroups. Set $T = \overline{f(S)}$. We claim that $B \stackrel{\text{def}}{=} f(A) \backslash E(T)$ is a suitable set for T.

Let y be a limit point of $f(A)$ and $f(x_\alpha) \to y$ a net with $x_\alpha \in A$. Since $\overline{A}$ is compact there is a converging subnet $x_{\alpha_j} \to x$. Then by continuity, $f(x) = \lim f(x_{\alpha_j}) = \lim f(x_\alpha) = y$. Since all the limit points of A are idempotents, x must be an idempotent and thus $y = f(x)$ is an idempotent. Note that $f(E(S)) \subseteq E(T)$ and thus $\overline{\langle B \cup E(T) \rangle} =$

$$\begin{aligned}
\overline{\langle (f(A)\backslash E(T)) \cup E(T) \rangle} &\supseteq \overline{\langle f(A) \cup f(E(S)) \rangle} \\
&= \overline{f(\langle A \cup E(S) \rangle)} \quad (f \text{ is a homomorphism}) \\
&\supseteq f(\overline{\langle A \cup E(S) \rangle}) \quad (f \text{ is continuous}) \\
&= f(S).
\end{aligned}$$

Therefore $T = \overline{f(S)} \subseteq \overline{\langle B \cup E(T) \rangle}$. □

Definition 2.3. A one-parameter semigroup in a topological semigroup S is a continuous homomorphism σ from $\mathbb{H}$, the non-negative reals under addition, into S. A compact semigroup with a dense one-parameter semigroup is known as *solenoidal*.

Proposition 2.4. *A solenoidal semigroup S has a suitable set.*

Proof. By definition, S is the closure of a continuous homomorphic image of $\mathbb{H}$, and $\mathbb{H}$ has a relatively compact suitable set by 2.1. Then 2.2 proves the claim. □

Example 2.5. The Real Thread, $I_u = [0, 1]$ under usual multiplication xy and the Nil Thread, $I_n = [1/2, 1]$ under $x \circ y = \max\{xy, 1/2\}$ each have suitable sets. Each Min Thread has a suitable set.

Proof. Each of I_u and I_n is solenoidal. Hence it has a suitable set by 2.4. Any Min Thread, as a compact semilattice, has an empty suitable set. □

In each of the cases I_u and I_n we can ensure that the suitable set has just one limit point at the identity.

Definition 2.6. An O-*semigroup* is a topological monoid S on a compact totally ordered space such that the total order of the space is given by $s \leq t$ iff $s \in St \cap tS$. The minimal element is the semigroup zero, and the maximal element is the semigroup identity. If S is, in addition, connected, then S is called an I-*semigroup*.

Notice that any compact totally ordered space is an O-semigroup with respect to the semilattice multiplication given by $st = \min\{x, y\}$. A topological monoid S is called monothetic if $S = \overline{\{1, x, x^2, \dots\}} = \{1\} \cup \Gamma(x)$ for some $x \in S$ (see [14], p. 15). The element x is called a *generator* of S. A monothetic O-semigroup S is isomorphic to $\mathbb{Z}^+ \cup \{\infty\}$, the additive group of nonnegative integers complemented with a largest element ∞ acting as a semigroup zero, where the total $\mathcal{H}$-order is the reverse of the natural order, or else S is isomorphic to a Rees quotient of this semigroup, in which case it is isomorphic to $\{0, 1, \dots, n\}$ with $x \oplus y = \min\{x + y, n\}$ for some natural number $n > 1$. Clearly, in a monothetic O-semigroup, the set $\{x\}$ with the generator x is a suitable set. A topological monoid on a compact connected totally ordered space with identity and zero as endpoints is an I-semigroup; that is, under these circumstances, the property that $s \le t$ iff $s \in St \cap tS$ follows automatically. The set of real numbers $\{0\} \cup [1, 2]$ with the operation $x \oplus y \stackrel{\text{def}}{=} \min\{x + y, 2\}$ is a totally ordered compact topological monoid with the identity 0 and the zero 2 as only idempotents and with a monotone multiplication, but it fails to be an O-semigroup. This monoid does not have a suitable set because any generating subset has to contain a dense subset of the connected subset $[1, 2[$.

Much of the following material is known; a usful reference for totally ordered topological semigroups is [11].

Proposition 2.7. *An* O*-semigroup S with exactly two idempotents* 0 *and* 1 *is either*

(i) *the two element monoid* $\{0, 1\}$*, or*
(ii) *the Real Thread* I_u*, or*
(iii) *the Nil Thread* I_n*, or*
(iv) *a monothetic* O*-monoid with a generator* $0 < x < 1$*.*

It has a suitable set.

Proof. Assume that $S \ne \{0, 1\}$. If 1 is not isolated, then $S \cong I_u$ or $S \cong I_n$ (see e.g. [14], p. 122, 11(a)). There remains the case that S is not one of type (i), (ii), (iii). Therefore assume now that 1 is isolated and that there are elements $x \in S$ satisfying $x \ne 0$. If $sx = x$ for some $s \in S$, then $s^n x = x$. Accordingly, $ex = x$ for the unique idempotent e in $\Gamma(s)$. Now $e \ne 0$ since $x \ne 0$. Hence $e = 1$ and thus $\Gamma(s) \subseteq \mathcal{H}(1) = \{1\}$, i.e. $s = 1$. Suppose that x is not isolated from below. Then $j \mapsto j : [0, x[\to S$ with the induced order on the index set. Since the total order on S is the $\mathcal{H}$-quasiorder, for each $j \in [0, x[$ there is an $s_j \in S$ such that $j = s_j x$. Now let $(s_{j_\alpha})_{\alpha \in A}$ be any subnet converging to an element s in S. As S is compact, such subnets exist. Then $x = \lim_\alpha j_\alpha = \lim s_{j_\alpha} x = sx$. We just saw that this implies $s = 1$. Since all converging subnets converge to 1 and S is compact, the net itself converges to 1 and since 1 is isolated, this implies the existence of an element $s_0 \in [0, x[$ such that $j \in [s_0, x[$ implies $s_j = 1$. Thus for these j we have $j = s_j x = 1x = x$ contradicting the assumption that x is not isolated from below. Now let $m \stackrel{\text{def}}{=} \max\{s \in S : s < 1\}$. We claim that $S = \{1\} \cup \Gamma(m)$, and a proof of this claim will complete the proof of the lemma since we do know that all four possible types have suitable sets.

By way of contradiction suppose that $C \stackrel{\text{def}}{=} S \setminus \bigl(\{1\} \cup \Gamma(m)\bigr)$ is not empty. Then $c = \sup C > 0$. Since every element of S is isolated from below, $c = \max C$. Then there is a largest natural number $n = 1, 2, \ldots$ such that $m^n > c$. Then $c > m^{n+1}$, and since the total order of S is the $\mathcal{H}$-order, there is an element $s \in S$ such that $sm^n = c$. Since $c \neq m^n$ we have $s < 1$ and by the definition of m this means $s \leq m$. Then by the monotonicty of multiplication with respect to the $\mathcal{H}$ order we have $c = sm^n \leq mm^n < c$, a contradiction. □

This observation is almost contained in Theorem 2.8, p. 22 of [11], but not quite.

With each O-semigroup S there is associated in a canonical fashion the set $K \subseteq S$ of idempotents isolated from below in the $\mathcal{H}$-order. For each $k \in K$ set $k' = \max\{e \in E(S) : e < k\}$. The following is well known for I-semigroups.

Theorem 2.8. *Let S be an O-semigroup. If $e, f \in E(S)$, then $ef = \min\{e, f\}$. Each connected component of $E(S)$ is a Min Thread, and $S \backslash E(S)$ is a union $\bigcup_{k \in K}]k', k[_S$ of disjoint open intervals. Each closed interval $[k', k]_S$, $k \in K$ is an* O*-semigroup with the two idempotents k' and k. If $x \in [k', k]_S$, but $y \notin [k', k]_S$, then $xy = \min\{\mathrm{x}, \mathrm{y}\}$. In particular, any O-semigroup is abelian.*

Corollary 2.9. *An* O*-semigroup S has a suitable set.*

Proof. We know from 2.8 above that an O-semigroup is the union $E(S) \cup \bigcup_{k\in K}[k',k]_S$ of a compact semilattice and a chain of basic semigroups each isomorphic as a topological semigroup to one of I_u, I_n. For each $k \in K$, by 2.5 we find a relatively compact countable suitable set A_k for $[k',k]_S$ contained in $]k',k]$. Indeed A_k may be chosen so as to have k as its unique limit point. We claim that $A = \bigcup_i A_i$ is a suitable set of S.

We need to show that any limit point of A lies inside $E(S)$. If A is empty we are finished. Let $x \in S\backslash E(S)$, then $x \in\,]k',k[_S$ for some $k \in K$. Since x is not an idempotent, since the only limit point of A_k is k, and since $]k',k[_S$ is a neighborhood of x in S, x is not a limit point of A.

Clearly $E(S) \subseteq \overline{\langle A \cup E(S)\rangle}$. For $x \notin E(S)$, $x \in [k',k]_S$ for some $k \in K$ and $x \in \overline{\langle A_k\rangle} \subseteq \overline{\langle A \cup E(S)\rangle}$. □

3. New Suitable Sets from Old

In Theorem 2.2 we have seen for the first time that the class of topological semigroups with suitable sets has fairly agreeable stability properties. The following results will further illustrate this assertion.

Theorem 3.1. *If a compact topological semigroup T is* factorisable *into a set of compact topological subsemigroups $\{X_1, X_2, \dots, X_n\}$, in the sense that each element $t \in T$ is expressible in the form $t = x_1x_2\cdots x_n$ where $x_i \in X_i$, $i = 1,2,\dots,n$; and if each X_i has a suitable set, then so does T.*

Proof. Let S_{X_i} be a suitable set for X_i, and let $E(X_i)$ be the set of idempotents of X_i. We claim that $S = S_{X_1} \cup S_{X_2} \cup \cdots \cup S_{X_n}$ is a suitable set for T.

As T and all of the X_i are compact:

$$\begin{aligned} T &= X_1X_2\cdots X_n \\ &= \overline{\langle S_{X_1} \cup E(X_1)\rangle}\;\overline{\langle S_{X_2} \cup E(X_2)\rangle}\cdots\overline{\langle S_{X_n} \cup E(X_n)\rangle} \\ &= \overline{\langle S_{X_1} \cup E(X_1)\rangle\langle S_{X_2} \cup E(X_2)\rangle\cdots\langle S_{X_n} \cup E(X_n)\rangle} \\ &\subseteq \overline{\langle (S_{X_1} \cup S_{X_2} \cup \cdots \cup S_{X_n}) \cup (E(X_1) \cup E(X_2) \cup \cdots \cup E(X_n))\rangle} \\ &\subseteq \overline{\langle S \cup E(T)\rangle} \subseteq T. \end{aligned}$$

Thus T $= \overline{\langle S \cup E(T)\rangle}$.

Now clearly S has no idempotents since no S_{Xi} does. Any limit point of S would have to be a limit point of at least one of the finitely many S_{X_i}. Hence it would have to be an idempotent in X_i and so be in $E(T)$. And so S is indeed a suitable set for T. □

Example 3.2. The 'triangle semigroup' has a countable suitable set.

Proof. The following set:

$$T = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x, y \leq 1 \text{ and } \mathrm{x} + \mathrm{y} \leq 1\},$$

is a triangle in the plane with vertices at $(0, 0), (1, 0)$ and $(0, 1)$. It is a compact connected semigroup when taken with the relative topology from the plane and the following operation:

$$(x_1, y_1) \circ (x_2, y_2) = (x_1x_2, x_1y_2 + y_1).$$

It is well known and usually called the 'triangle semigroup'. The hypotenuse and the base of this semigroup are each compact connected subsemigroups of $(T, \circ)$ (in fact irreducible ones) which are easily seen to be homeomorphicly isomorphic to the real thread I_u above. Hence they each have countable suitable sets clustering at $(1, 0)$. It is a simple exercise in elementary algebra and calculus to show that T factorises into the product of these two subsemigroups. Hence by the theorem the 'triangle semigroup' has a countable suitable set. □

We note also that this same result also follows as a corollary from 6.1 below where it is derived from Lie semigroup theory. The "triangle semigroup" is a special case of a class of semigroups introduced in [14] on pp. 272 and 273 in Section 16. For some recent systematic investigations see e.g. [1].

Theorem 3.3. *Let $\{S_i\}_{i\in I}$ be a collection of topological monoids with suitable sets $\{A_i\}_{i\in I}$, then $S = \prod_{i\in I} S_i$ has a suitable set.*

Proof. Let p_i be the i-th projection map and 1_i the identity in S_i. Let $A'_i = A_i \times \prod_{j\neq i} 1_j$. We claim that $A = \bigcup_{i\in I} A'_i$ is a suitable set for S.

Let x be a limit point of A. Then for each i, $p_i(x)$ is a limit point of $p_i(A) = A_i \cup \{1_i\}$. Hence $p_i(x)$ is either 1_i or a limit point of A_i. In any case $p_i(x)$ will be an idempotent of S_i.

x is then an idempotent of S as each coordinate of x is an idempotent. Consider a basic open set U containing $x \in S$. Then $U = U_1 \times U_2 \times \cdots \times U_n \times \prod_{j \in I \setminus \{1,\ldots,n\}} S_j$. Each U_i is an open set in S_i. Let $E_i = E(S_i) \times \prod_{j \neq i} 1_j$. Since $\langle A_i \cup E(S_i) \rangle$ is dense in S_i, $\langle A'_i \cup E_i \rangle$ is dense in $S_i \times \prod_{j \neq i} 1_j$. Thus for each i we can find a_i such that $a_i \in U_i \cap \langle A_i \cup E(S_i) \rangle$ Let $\alpha_i = a_i \times \prod_{j \neq i} 1_j$. Then $\alpha_i \in \langle A'_i \cup E_i \rangle$. Now $\alpha_1 \alpha_2 \ldots \alpha_n = a_1 \times \cdots \times a_n \times \prod_{j \in I \setminus \{1,\ldots,n\}} 1_j \in U$. On the other hand, since $E_i \subseteq E(S)$ for each i, $\alpha_1 \alpha_2 \ldots \alpha_n \in \langle A'_1 \cup E_1 \rangle \langle A'_2 \cup E_2 \rangle \ldots \langle A'_n \cup E_n \rangle \subseteq \langle A \cup E(S) \rangle$. Thus $x \in \overline{\langle A \cup E(S) \rangle}$ and $A \cup E(S)$ topologically generates S. $\square$

Definition 3.4. A *cylindrical* semigroup S is a continuous homomorphic image of $T \times G$, where T is solenoidal and G is a compact group.

Realistic examples of cylindrical semigroups which are embeddable in three-space are shown in Hofmann and Mostert [14], pages 242, 243 and 244. These illustrate that *every point* of the minimal ideal $M(S)$ in a cylindrical semigroup S is approximated by points of $S \setminus M(S)$. See also line diagrams in Hofmann [10], page 9.

Corollary 3.5. *A cylindrical semigroup has a suitable set.*

Proof. A solenoidal semigroup T has a suitable set by 2.4 and a compact group G has a suitable set by [13]. Then by 3.3 so does $T \times G$. Then, finally, as a cylindrical semigroup is a continuous homomorphic image of some monoid $T \times G$, by 2.2 it must have a suitable set. $\square$

4. Totally $\mathcal{H}$-ordered Semigroups

We now generalize the total order that we already used for totally ordered monoids. In a monoid S, the $\mathcal{H}$ quasiorder $\leq_{\mathcal{H}}$ is defined by $x \leq_S y$ iff $x \in yS$ and $x \in Sy$ (see e.g. [14], p.29).

Definition 4.1. A compact monoid will be called *totally* $\mathcal{H}$*-ordered* if, in the $\mathcal{H}$ quasiorder $\leq_{\mathcal{H}}$, any two elements are comparable.

In an $\mathcal{H}$-totally ordered compact monoid, each $\mathcal{L}$- class and each $\mathcal{R}$-class is an $\mathcal{H}$-class and thus $\mathcal{D} = \mathcal{L} = \mathcal{R} = \mathcal{H}$ and $\mathcal{H}$ is a congruence. (See e.g. [14], p. 33.) Accordingly, for each $x \in S$, $xS = Sx = SxS$ and S is normal in itself (see [14], p. 17, 2.1).

Clearly, the quotient semigroup $S/\mathcal{H}$ is an O-semigroup. The quotient morphism is bijective on the sets of idempotents. For easy reference we record these remarks:

Lemma 4.2. *If S is a totally $\mathcal{H}$-ordered monoid, then Green's $\mathcal{H}$ relation is a congruence, and the quotient morphism $\eta\colon S \to S/\mathcal{H}$ is a surjective morphism of $\mathcal{H}$-totally ordered monoids onto a* O-*semigroup such that it induces an isomorphism $\eta|E(S)\colon E(S) \to E(S/\mathcal{H})$ of compact semilattices.*

Theorem 4.3. *Let S be a compact totally $\mathcal{H}$-ordered monoid with exactly two idempotents $e > f$. If $\mathcal{H}(e)$ is not open, then S is the union of a cylindrical submonoid Z and $\mathcal{H}(f) = M(S)$.*

If $\mathcal{H}(e)$ is open, then either $S = \mathcal{H}(e) \cup \mathcal{H}(f)$, or for any $g \in S$ such that $\eta(g)$ is a generator according to 2.7 (iv)*, we have $S = \mathcal{H}(e) \cup \mathcal{H}(e)\Gamma(g) \cup \mathcal{H}(f)$.*

Proof. If $\mathcal{H}(e)$ is not open, then the assertion follows directly from [12], p. 150, Corollary 3.5.

Assume now that $\mathcal{H}(e)$ is open and $S/\mathcal{H}$ has more than two points, and let $g \in S$ be such that $\eta(g)$ is a generator. Now for each natural number n such that $\eta(g)^n \neq \mathcal{H}(f)$, the $\mathcal{H}$ class $\mathcal{H}(x^n) = \eta^{-1}(\eta(x)^n)$ is of the form $\mathcal{H}(1)x$ with $x = g^n$ by [14], p. 39, 4.20. In our case, $\Gamma(g) = \{g, g^2, \dots\} \cup M(\Gamma(g))$ with $M(\Gamma(g)) \subseteq \mathcal{H}(f)$, and our assertion follows. □

Corollary 4.4. *A compact totally $\mathcal{H}$-ordered monoid with exactly two idempotents has a suitable set.*

Proof. If $S = Z \cup \mathcal{H}(f)$ with a cylindrical submonoid Z, then Z has a suitable set A_Z by 3.5 and $\mathcal{H}(f)$ has a suitable set A_f by [13]. Then $A_Z \cup A_f$ is a suitable set of S. If S is the union of two groups, it has a suitable set because the compact groups $\mathcal{H}(1)$ and $\mathcal{H}(e)$ have suitable sets A_1 and A_e by [13]. If $S = \mathcal{H}(1) \cup \mathcal{H}(1)\Gamma(x) \cup \mathcal{H}(e)$, then $A_1 \cup \{x\} \cup A_e$ is a suitable set. □

If G is a compact group and α an automorphism one readily forms a semidirect product $G \rtimes_\alpha \mathbb{Z}$ with multiplication $(g, m)(h, n) = (g\alpha^m(h), m + n)$, and the one point compactification of $G \rtimes_\alpha \mathbb{Z}^+ \cup \{\infty\}$ with ∞ acting as zero is an O-semigroup which in general is not a direct product. (Take G abelian with a nontrivial automorphism, e.g. $G = \mathbb{R}^2/\mathbb{Z}^2$ with a nontrivial automorphism represented by an element of $\mathrm{SL}(2, \mathbb{Z})$. A direct product would have to be abelian, but there are ample choices of α such that the semidirect product fails to be abelian.) This example shows, that in the disconnected situation the theory is not completely parallel to the case of cylindrical semigroups; but it is close enough for a treatment of suitable sets so that both cases can be treated largely simultaneously. We also observe that the cylindrical case of Theorem 4.3 involves nontrivial material in the vicinity of the so called "Centralizer Conjecture" (see [12]).

Let S be a totally $\mathcal{H}$-odered monoid. Let us again denote $K \subseteq E(S)$ the set of idempotents isolated from below, and for $k \in K$, set $k' = \max\{e \in E(S) : e < k\}$.

Theorem 4.5. *Let S be a totally $\mathcal{H}$-ordered monoid and assume that the identity e of any nonsingleton subgroup $\mathcal{H}(e)$ is isolated in $E(S)$. Then S has a suitable set.*

Proof. For each $k \in K$ the semigroup $S_k = \eta^{-1}[\eta(k'), \eta(k)]$ is a semigroup of the type classified in 4.3. By 4.4, S_k has a suitable set $A_k = A'_k \cup A_{k'}$ where A'_k is contained in $\eta^{-1}[p_k, \eta(k)]$ for some $p_k \in]\eta(k'), \eta(k)[$ and $A_{k'}$ is a suitable set of $\mathcal{H}(k')$. Set $A = \bigcup_{k\in K} A_k$. Clearly $A \cap E(S) = \emptyset$; and if $e \in E(S)$, then by hypothesis $\mathcal{H}(e)$ is a subgroup of an $\mathcal{H}$-totally ordered semigroup with exactly two idempotents or is singleton (or both), and thus and $\overline{\langle A \cup E(S)\rangle} \supseteq \overline{\langle A \cup \bigcup_{e\in E(S)} \mathcal{H}(e)\rangle} = S$ Now let $x \in \overline{A} \setminus A$. We want to show that x is an idempotent. If $\eta(x)$ is a non-isolated idempotent then by assumption x is an idempotent. If $\eta(x)$ is an isolated idempotent $\eta(k)$ then there exist idempotents k' and f such that $k' < k < f$ and there is no other idempotent in the interval $[\eta(k'), \eta(f)]$; the cases that $\eta(k)$ is an endpoint of $E(S/\mathcal{H})$ is similar. Since $x \in \mathcal{H}(k)$ and $A \cap \eta^{-1}([p_k, \eta(k)]) = A'_k$, the point x must be a limit point of A'_k. This means x is an idempotent in S_k (and thus in S). If $\eta(x)$ is not an idempotent then $\eta(x) \in [\eta(k'), \eta(k)]$ for some I-semigroup $[\eta(k'), \eta(k)]$. Then $x \in \eta^{-1}(]\eta(k'), \eta(k)[)$. From $A \cap \eta^{-1}]\eta(k'), \eta(k)[) \subseteq A_k$, we conclude that x is a limit point of A_k and thus an idempotent. □

Corollary 4.6. *Let S be a totally $\mathcal{H}$-ordered monoid with finitely many idempotents. Then S has a suitable set.*

Proof. If $E(S)$ is finite, then all elements of $E(S)$ are isolated, and the preceding corollary 4.5 applies. □

It remains somewhat inconclusive for which semilattices of idempotents $E(S)$ a totally $\mathcal{H}$-ordered monoid S has a suitable set even in the case that S is a Clifford semigroup, i.e. a union of groups. But we shall consider in the next sequence a class of totally $\mathcal{H}$-ordered semigroup for which we offer more conclusive information.

5. Irreducible Semigroups

We defined the concept of an irreducible semigroup in the introduction. An irreducible semigroup S is an abelian totally $\mathcal{H}$-ordered compact connected monoid (see [14], p. 156, Theorem 156). So everthing said in the preceding section applies, in particular, to irreducible semigroups. However, in the case of irreducible semigroups we can say more. Again let $K \subseteq E(S)$ be the set of idempotents isolated from below in the $\mathcal{H}$-order, and for each $k \in K$ set $k' = \max\{e \in E(S) : e < k\}$. Select a one-parameter subsemigroup $f_k : \mathbb{H} \to S$ with $f_k(0) = k$ and $k' \in M(\overline{f_k(\mathbb{H})})$. Set $C = E(S) \cup \bigcup_{k \in K} f_k(\mathbb{H})$.

Lemma 5.1. $S = \overline{\langle C \rangle}$.

Proof. C is a connected subset of S containing the identity 1 and meeting the minimal ideal $M(S)$. Then the closed subsemigroup generated by C is a connected submonoid meeting the minimal ideal. It then agrees with S by irreducibility of S. □

For each $k \in K$ we select a positive number r_k such that $k' <_{\mathcal{H}} f_k(r_k)$. Set $D = E(S) \cup F$ where $F = \bigcup_{k \in K} f_k([0, r_k])$.

Lemma 5.2. *D is a closed subset of S and the $\mathcal{H}$-class homomorphism $\eta : S \to S/\mathcal{H}$ induces a homeomorphism $\eta|_D : D \to \eta(D)$.*

Proof. Let $\{a_j\}_{j \in J}$ be a net in F converging to $s \in S$. We must show that $s \in D$. If $\{a_j\}$ is cofinaly idempotent, then $s \in E(S)$ and we are done. So assume now that $a_j \in f_{k(j)}([0, r_{k(j)}])$. We first claim that $e = \lim_j k(j)$ exists in $E(S)$. To see this we note that in the I-semigroup $S/\mathcal{H}$ we have $\eta(s) = \lim_j \eta(a_j)$ and $\eta(k(j)') < \eta(a_j) < \eta(k(j))$. In the I-semigroup $S/\mathcal{H}$ each cluster point c of $\eta(k(j))$ is an idempotent in $[\eta(s), \eta(1)]$. Suppose that there are two cluster points $c_1 < c_2$. In the I-semigroup $S/\mathcal{H}$ let x and y be elements such that $c_1 < x < y < c_2$. Then $\eta(s) \leq c_1$ implies that eventually $\eta(a_j) < x$; also for a cofinal subset I of J we have $y < \eta(k(i))$ for $i \in I$. Thus $\eta(a_i) < x < y < \eta(k(i))$ for large enough i in I implies that there are no idempotents in $[x, y]$, and thus we conclude that there cannot be any idempotents in (c_1, c_2). Now $\eta(s) \leq c_1$ implies that eventually $\eta(a_j) < c_2$ and thus $c_2 \leq \eta(k(j))$ eventually, contradicting then assumptions that c_1 is a cluster point of $\eta(k(j))$. Thus there is only one cluster point of $\eta(k(j))$ and since $s \to \eta(s)$ induces an isomorphism on the idempotents, the claim is proved and $e = \lim_j k(j)$ exists in $E(S)$.

Now assume that $s \notin F$. Then we claim $\eta(s) = \eta(e)$. Indeed, $\eta(s) \leq \eta(k)$ and if $\eta(s) < \eta(k)$, then the argument showing the existence of $\lim_j k(j)$ actually shows that there cannot be any idempotents in $(\eta(s), \eta(k))$, and thus $k \in K$ and $\eta(a_j) \in (\eta(k'), \eta(k))$ eventually, and by the choice of r_k and $\lim_j a_j \notin F$ it follows that $s = k$. Thus $\eta(s) = \eta(e)$ as asserted. This last step shows that D is closed.

The restriction and corestriction $\eta|_D : D \to \eta(D)$ is a continuous surjective function. It is injective on $E(S)$ and on each $f_k([0, r_k])$ by the choice of r_k. Since $\eta(E(S)) \cap \eta(F) = \emptyset$ it follows that $\eta|_D$ is injective and thus, by the compactness of D, is a homeomorphism onto its image. □

Let r_k be as above and pick any decreasing sequence of positive real numbers $(r_{kn})_{n\in\mathbb{N}}$ in $[0, r_k]$ converging to 0. Set $A = \{f_k(r_{kn}) : k \in K,\, n \in \mathbb{N}\}$.

Lemma 5.3. *Under the present circumstances A is discrete in $S \setminus E(S)$, and $S = \overline{\langle A \cup E(S)\rangle}$.*

Proof. The first part follows from the fact that $\{r_{kn} : n \in \mathbb{N}\}$ is discrete in $\mathbb{H} \setminus \{0\}$ for each $k \in K$ and $[0, r_k]$ is mapped homeomorphically into S under f_k.

For the second part since $\lim_n r_{kn} = 0$ in $\mathbb{H}$ for each $k \in K$, we have $\mathbb{H} = \overline{\langle\{r_{kn} : n \in \mathbb{N}\}\rangle}$ for each $k \in K$. Accordingly, $f_k(\mathbb{H}) \subseteq \overline{\langle\{r_{kn} : n \in \mathbb{N}\}\rangle}$. The assertion then follows from 6.8. □

Therefore A is a suitable set if and only if $\overline{A} \setminus A \subseteq E(S)$. Let us look at this issue in some detail. For this purpose, let $(a_j)_{j\in J}$ be a net in A and assume that $s = \lim_{j\in J} a_j$ exists and is not contained in A. Projecting into the I-semigroup $S/\mathcal{H}$ we see that

$\eta(s)$ is idempotent and so $\mathcal{H}(s)$ is a group with an idempotent. The question is: When is $s \in E(S)$, that is, $s = e$?

Case 1. If $k(j)$ is eventually constant, equal to $e \in K$, then

$$s = \lim_j f_{k(i)}(r_{k(j)n(j)}) = f_e(0) = e \in E(S).$$

Case 2. A cofinal subset I of J consists of elements such that $k(i) < e$ for $i \in I$. Then $\eta(a_i) < \eta(k(i)) < \eta(k) = \eta(s)$ and $\lim_{i\in I} \eta(a_i) = \lim_{i\in I} \eta(k(i)) = \eta(s)$

Since $\eta|_D$ is a homeomorphism and a_j, $k(j) \in D$ for all $j \in J$ we conclude that $\lim_{i\in I} a_i = \lim_{i\in I} k(i) = e \in E(S)$.

Case 3. $k(j)$ is not eventually constant and eventually all $k(j) \geq e$ with a cofinal subset $I \subseteq J$ such that $k(i) > e$ for all $i \in I$.

This case is critical.

Definition 5.4. We say that an idempotent $e \in E(S)$ of an irreducible semigroup S is *critical* if it is the greatest lower bound, but not the minimum of a subset $X \subseteq K$ of idempotents which in $E(S)$ are isolated from below.

The simplest semilattice exhibiting a critical element is the set $\mathbb{M} \overset{\text{def}}{=} \{0\} \cup \{1/n : n = 1, 2, \dots\}$ with the min operation; here 0 is a critical idempotent.

With the first two cases settled, we can now formulate a positive result.

Theorem 5.5. *Let S be an irreducible compact semigroup such that for every critical idempotent $e \in E(S)$ the group $\mathcal{H}(e)$ is singleton. Then S has a suitable set.*

Proof. We have to show that $s = \lim_{j\in J} a_j$ is an idempotent. If Cases 1 and 2 apply, this has been shown to be true. Now assume Case 3. We have $\eta(s) = \lim_{i\in I} \eta(a_i)$. The idempotent $\eta(s)$ is critical in $E(S/\mathcal{H})$ and thus the idempotent $e \in S$ with $\eta(e) = \eta(s)$ is critical. Then $\mathcal{H}(e) = \{e\}$. Now all clusterpoints of $(a_i)_{i\in I}$ are in $\eta^{-1}(\eta(s)) = \{e\}$. Thus $e = s$ and $s \in E(S)$. ☐

Note that the hypothesis of the theorem is fulfilled in particular if $E(S)$ has no critical idempotents. This holds for any well-ordered set.

Corollary 5.6. *Assume that S is an irreducible compact semigroup and that the space of connected components of $E(S)$ is well-ordered. This is satisfied if $E(S)$ has only finitely many components. Then S has a suitable set.*

Now we turn to the critical case and construct an example of an irreducible semigroup S without a suitable subset and such that $E(S)$ is order isomorphic to $\mathbb{M}$.

The construction of this example is not straightforward even though $S/\mathcal{H}$ is a very simple $I-$semigroup.

We consider an additive version of an $I-$semigroup on the real unit interval $[0,1]$ with the sequence $1, 1/2, 1/3, \cdots, 0$ representing the idempotents.

In fact we take $T \stackrel{\text{def}}{=} (\mathbb{N} \times \mathbb{H}) \cup \{\infty\}$ with the lexicographic order on $\mathbb{N} \times \mathbb{H}$ and ∞ being the largest element. This is an $I-$semigroup with respect to the multiplication

$$(m,r) + (n,s) = \begin{cases} (n, r+s) & \text{if } m = n, \\ (m,r) & \text{if } m > n, \\ (n,s) & \text{if } m < n, \end{cases}$$

where ∞ is acting as a zero.

Theorem 5.7. *There is an irreducible semigroup S whose $\mathcal{H}-$decomposition semigroup is homeomorphically isomorphic to T, and which does not have a suitable subset.*

Proof. We let B be the character group of the discrete reals $\mathbb{R}_d$ and let $\beta : \mathbb{R} \to B$ be the dual of the identity morphism $\mathbb{R}_d \to \mathbb{R}$ (the Bohr compactification of $\mathbb{R}$). Thus $\mathbf{b} \stackrel{\text{def}}{=} \beta(1) \in B = \widehat{\mathbb{R}_d}$, as a character of $\mathbb{R}_d$, is the quotient character $\epsilon : \mathbb{R}_d \to T$.

For an arbitraty set I, the character group of $(\mathbb{R}^I)_d$ is the coproduct $B^{((I))}$ of I copies of B in the category of compact abelian groups with the coprojection $s_i : B \to B^{((I))}$ being the dual of the projection $p_i : B^I \to B$. If I is finite, then $B^{((I))}$ is the algebraic coproduct $B^{(I)} \cong B^I$ with the product topology.

Now define S as a subsemigroup of of $T \times B^{((I))}$ as follows: Define a function $f : \mathbb{N} \times \mathbb{H} \to T \times B^{((I))}$ by $f(m,r) = ((m,r), s_n(\beta(r)))$ and set $S = \overline{f(\mathbb{N} \times \mathbb{H})}$. Notice that $\overline{\beta(\mathbb{H})} = B\mathcal{H}((m,0),0) = \{(m,0)\} \times B^{(m-1)} \subseteq T \times B$ for $m = 2, \cdots$ and that the group of units of S is singleton. The minimal ideal $M(S) = \mathcal{H}((\infty, 0))$ is $\{\infty\} \times B^{((\mathbb{N}))}$.

The subsemigroup S is irreducible (proof by induction) and the $\mathcal{H}-$class homomorphism is equivalent to the restriction pr_1 to S of the projection $T \times B^{((\mathbb{N}))} \to T$.

(Notice that S is an irreducible semigroup in which the maximal subgroups are inductively built up to the coproducts $B^{(n)}$ of n copies of the Bohr compactification of $\mathbb{R}$.)

Let $e \stackrel{\text{def}}{=} (\infty, 0) \in T \times B^{((\mathbb{N}))}$, an idempotent of the minimal ideal $M(S) = \{\infty\} \times B^{((\mathbb{N}))}$ of S. The Clifford-Miller endomorphism $s \mapsto se$ is the restriction pr_2 to S of the projection $T \times B^{((\mathbb{N}))} \to B^{((\mathbb{N}))}$.

Now let $A \subseteq S$ be a subset satisfying that A is discrete in $S \backslash E(S)$ and that $S = \overline{\langle A \cup E(S) \rangle}$. We shall show that $\overline{A} \backslash A$ contains non-idempotent elements. Now $\mathrm{pr}_1(A)$ is a suitable subset of T and therefore is of the form $\{(m, r_{mn}) : m, n \in \mathbb{N}\}$ with $r_{mn} \in \mathbb{H} \backslash \{0\}$ converging to 0 for $n \to \infty$.

There is no real loss in assuming that ${\mathrm{pr}_1}^{-1}((m, r_{m1}))$ contains only one element $a_{m1} = ((m, r_{m1}), c_{m1} + f(r_{m1}))$ with some element $c_{m1} \in B^{(m-1)}$, where

$B^{(0)} = \{0\}$. Then

$$\mathcal{X}_m \stackrel{\text{def}}{=} \mathrm{pr}_2(a_{m1}) = c_{m1} + s_m(\beta(r_{m1})) \in B^{(\mathbb{N})} \subseteq B^{((\mathbb{N}))}.$$

We recall that $\mathcal{X}_m : (\mathbb{R}^{\mathbb{N}}) \to \mathbb{R} \to \mathbb{T}$ is a character of $(\mathbb{R}^{\mathbb{N}})_d$ and we notice that it is of the form $(s_j)_{j \in \mathbb{N}} \mapsto \alpha_{m1} s_1 + \cdots + \alpha_{mm-1} s_{m-1} + r_{m1} s_m \bmod \mathbb{Z}$ with $\alpha_{jm} \in \mathbb{R}$.

The matrix

$$\begin{pmatrix} r_{11} & 0 & 0 & 0 & \cdots \\ a_{21} & r_{21} & 0 & 0 & \cdots \\ a_{31} & a_{32} & r_{31} & 0 & \cdots \\ a_{41} & a_{42} & a_{43} & r_{41} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

implements an automorphism α of the $\mathbb{R}-$vector space $(\mathbb{R}^{\mathbb{N}})_d$ given by

$(s_j)_{j \in \mathbb{N}} \mapsto ({s'}_j)_{j \in \mathbb{N}}$ where

$${s'}_j = \alpha_{j1} s_1 + \cdots + \alpha_{jj-1} s_{j-1} + r_{j1} s_j.$$

Then $\mathcal{X}_m = \epsilon \circ \mathrm{pr}_m \alpha = \mathbf{b} \circ \mathrm{pr}_m$. By duality it follows that

$$\mathrm{pr}_2(a_{m1}) = \hat{\alpha}(s_m(\mathbf{b})),$$

where $\hat{\alpha}$ is an automorphism of the compact abelian group $B^{((\mathbb{N}))}$. Therefore

$$\hat{\alpha}^{-1}(\mathrm{pr}_2(a_{m1})) = s_m(\mathbf{b}).$$

The universal property of the product $(\mathbb{R}^{\mathbb{N}})$ gives us a unique morphism
$\delta : \mathbb{R} \to (\mathbb{R}^{\mathbb{N}})_d$, called the *diagonal morphism* such that $\delta(r) = (r)_{r\in\mathbb{N}}$. Its adjoint morphism $\hat{\delta} : B^{((\mathbb{N}} \to B$ is the *codiagonal morphism* which is uniquely characterised by the fact that $\hat{\delta} \circ s_m : B \to B$ is the identity morphism for each $m \in \mathbb{N}$. Thus

$$\hat{\delta}\hat{\alpha}^{-1}(\mathrm{pr}_2(a_{m1})) = \hat{\delta}(s_m(\mathbf{b})) = \mathbf{b}$$

for all m and thus

$$(\forall m \in \mathbb{N})\ a_{m1} \in [\hat{\delta}\hat{\alpha}^{-1}\mathrm{pr}_2]^{-1}(\mathbf{b}).$$

Since $\mathbf{b} \neq 0$, the closed subset $[\hat{\delta}\hat{\alpha}^{-1}\mathrm{pr}_2]^{-1}(\mathbf{b})$ does not contain any idempotents and thus the sequence $(a_{m1})_{m\in\mathbb{N}}$ does not cluster to any idempotent.

□

These last two theorems taken together seem to be the best obtainable broad structural results concerning the existence of suitable sets in irreducible semigroups. Of course one can choose a nonidempotent limit point $s \in M(S)$ of $(a_m)_{m\in\mathbb{N}}$ and find a character $\mathcal{X} : M(S) \to \mathbb{T}$ which does not vanish on s. Let $S/\ker\mathcal{X}$ denote the quotient of S modulo the congruence whose cosets are singletons outside $M(S)$ and are cosets of $\ker\mathcal{X}$ on $M(S)$. Then $S/\ker\mathcal{X}$ is an irreducible semigroup without a suitable set and such that its minimal ideal is the circle group.

6. Subsemigroups of Lie Groups

Theorem 6.1. (a) *Let G be a connected finite dimensional Lie group and* $\exp : \mathfrak{L}(G) \to G$ *its exponential function. Let C be a convex cone in $\mathfrak{L}(G)$, that is, an additive convex subset containing* 0*. If $S = \overline{\langle \exp(C) \rangle}$ is the smallest closed subsemigroup of G containing* $\exp(C)$*, then S has a countable relatively compact suitable set.*

(b) *Any topological semigroup containing a dense continuous homomorphic image of S has a suitable set.*

Proof. (a) It suffices to exhibit a subset X of $C\backslash\{0\}$ with two properties:

(i) $\exp(X)$ is discrete in $G\backslash\{1\}$ and the closure $\overline{\exp(X)}$ is contained in $\exp(X)\cup\{1\}$.

(ii) The union $U=\bigcup_{x\in X}\mathbb{N}_0\cdot x, \mathbb{N}_0=\{0,1,2,\dots\}$, is dense in C.

(iii) X is countable and $\exp X$ is relatively compact.

(Indeed then $\exp(C)\subseteq\overline{\exp(X)}\subseteq\overline{\langle\exp(X)\rangle}$ whence $S\subseteq\overline{\langle\exp(X)\rangle}$.)

We set $V=C-C$. Then V is a vector subspace of $\mathfrak{L}(G)$. Then the interior I of C **in the vector space** V is dense in V. Give $\mathfrak{L}(G)$ a norm such that the ball of radius 2 is mapped homeomorphically onto an open identity neighbourhood W of G. Let D be the boundary of the unit ball of V. Then there is a countable dense subset N of $D\cap I$, say, $N=\{d_1,d_2,\dots\}$, then we let X be the set of the elements

$$\begin{array}{ccccc} d_1, & d_2, & d_3, & d_4, \cdots \\ 2^{-1}d_1, & 2^{-1}d_2, & 2^{-1}d_3, & 2^{-1}d_4, \cdots \\ 2^{-2}d_1, & 2^{-2}d_2, & 2^{-2}d_3, & 2^{-2}d_4, \cdots \\ 2^{-3}d_1, & 2^{-3}d_2, & 2^{-3}d_3, & 2^{-3}d_4, \cdots \\ \vdots & \vdots & \vdots & \vdots \quad \ddots \end{array}$$

Then X is a countable set which is discrete in $\mathfrak{L}(G)\backslash\{0\}$ so that $\exp(X)\subseteq W$ is discrete in $G\backslash\{1\}$. For each $\mathbb{H}\cdot d_n$, there is an N_n such that for all $m>N_n$, the elements $2^{-m}d_n$ are in X so that the set $\bigcup_{m\in N_n}\mathbb{N}_0 2^{-m}d_n\in U$. Thus $\mathbb{H}d_n\subseteq\overline{U}$ and since $\overline{C}=\mathbb{H}\cdot\overline{D\cap I}=\overline{\mathbb{H}\cdot N}\subseteq\overline{U}$. This is what we had to show.

(b) This is a consequence of (a) and Theorem 2.2. □

As noted in Section 3, this theorem gives an alternative proof that the 'triangle semigroup' of 3.2 has a suitable set.

For material on Lie semigroups see articles in Semigroup Forum **61** (2000), where there is a survey by K.H. Hofmann [10], including a bibliography up to year 2000, and an article by B. Brechner [1]; or see one or more of the following books listed there: [8], [9] or [15].

7. More on $S^2 = S$

It is reasonable that we should concentrate on semigroups satisfying $S^2 = S$, because we have

Theorem 7.1. *If S is a compact connected semigroup in which $S^2 \neq S$, then S cannot have a suitable set.*

Proof. Assume by way of contradiction that there is a suitable set A, then the semigroup generated by $A \cup E(S)$ is dense in S. Now $\bigcup_{n=2}^{\infty}(A \cup E(S))^n \subseteq S^2$, so

$\overline{\bigcup_{n=2}^{\infty}(A \cup E(S))^n} \subseteq S^2$ since S^2 is compact hence closed. Now as $S = \overline{\bigcup_{n=1}^{\infty}(A \cup E(S))^n} = \overline{\bigcup_{n=2}^{\infty}(A \cup E(S))^n} \cup \overline{(A \cup E(S))}$, then $\overline{A \cup E(S)} \supseteq (S \backslash S^2)$. Also $\overline{A \cup E(S)} = \overline{A} \cup \overline{E(S)} = \overline{A} \cup E(S)$ since $\overline{E(S)} = E(S)$. Again $\overline{A} \cup E(S) = A \cup E(S)$ since any limit points of A are in $E(S)$. So $(A \cup E(S)) \supseteq (S \backslash S^2)$. However, $E(S) \subseteq E(S)^2 \subseteq S^2$, so $A \supseteq (S \backslash S^2)$. Now as A is relatively discrete, since it has no idempotents, there exists an open neighbourhood $U(a)$ of some chosen point a of $A \cap (S \backslash S^2)$ such that $U(a) \cap A$ is precisely $\{a\}$. Then we have that $V(a) = U(a) \cap (S \backslash S^2) = \{a\}$ is an open set. However, $\{a\}$ is also a closed set since S is Hausdorff. This contradicts that S is connected, since S has more than one point as $S \neq S^2$. So S cannot have a suitable set. □

Theorem 7.2. *If a compact semigroup S is* simple*, i.e. $M(S) = S$, then S has a suitable set.*

Proof. It follows immediately from the Wallace-Rees-Suschkewich Theorem that $S = M(S) = E(S)GE(S)$, where G is (any) one of the compact maximal topological groups which are subsemigroups of $M(S)$, and whose disjoint union makes up the minimal ideal. Then by the result of Hofmann and Morris G has a suitable set A.

Let $x \in S$. Then $x = egf$ where $e, f \in E(S)$ and $g \in G$. If $U(x)$ is any open subset of S to which x belongs then, by continuity of multiplication, there exists an open neighbourhood $U(g)$ such that $U(x) \supseteq eU(g)f$.

Now $U(g) \cap \langle A \cup 1_G \rangle \neq \emptyset$. So $eU(g)f \cap e\langle A \cup 1_G \rangle f \neq \emptyset$. But $e\langle A \cup 1_G \rangle f \subseteq \langle A \cup E(S) \rangle$ giving that $U(x) \cap \langle A \cup E(S) \rangle \neq \emptyset$ and so $x \in \overline{\langle A \cup E(S) \rangle}$. Since G is a closed subset of S any limit points of A in S must belong to G. However, the only possible limit point of A in G is 1_G which belongs to $E(S)$. Hence A is a suitable set for S, as claimed. □

Corollary 7.3. *A compact semigroup S has a suitable set if the Rees quotient semigroup $S/(M(S))$ has a suitable set whose closure does not contain $0 = M(S)$.*

Proof. Immediate. □

8. Finitely Generated Compact Semigroups

In any discussion of suitable sets A in a compact semigroup S we face a topological generating set $A \cup E(S)$. In the context of topological generating sets it is occasionally useful to observe that certain compact semigroups cannot have a finite topological generating set.

Definition 8.1. A compact semigroup S is *finitely generated* if it contains a finite subset F such that $S = \overline{\langle F \rangle}$.

Some compact semigroups cannot be finitely generated for reasons of sheer size, because a finitely generated compact semigroup is necessarily separable. Compact connected groups whose weight does not exceed the cardinality of the continum are known to be generated by suitable generating sets of no more than two elements.

Continuous homomorphic images of finitely generated compact semigroups are obviously finitely generated. Example application: Let $X \times Y$ be a rectangular semigroup, that is, X and Y are compact spaces and multiplication is given by $(x, y)(x', y') = (x, y')$. If $F \subseteq X \times Y$, then $\langle F \rangle = \mathrm{pr}_X(F) \times \mathrm{pr}_Y(F)$. Thus a finitely generated rectangular semigroup is finite. On a compact simple semigroup S the $\mathcal{H}$-relation is a congruence and $S/\mathcal{H}$ is rectangular. Hence if S is finitely generated, so is $S/\mathcal{H}$. An immediate corollary is

Remark 8.2. *A finitely generated compact connected simple topological semigroup S is a group.*

Definition 8.3. An ideal J of a semigroup S is called a *prime ideal* if $S \setminus J$ is a subsemigroup.

By a result of Fawcett, Koch and Numakura [7], a maximal ideal J of a compact semigroup is a prime ideal whenever $S \setminus J$ is a union of groups. In particular if a compact connected nonsimple semigroup S is a disjoint union of groups, or if S has an identity 1, then this condition is satisfied.

Proposition 8.4. *Assume that S is a compact connected topological semigroup which is not simple (i.e. $M(S) \neq S$), but satisfies $S^2 = S$ and has a maximal ideal which is a prime ideal. Then S is not finitely generated.*

Proof. Suppose by way of contradiction that $S = \overline{\langle B \rangle}$ for a finite sub set $B \subseteq S$. Let $B_J = B \cap J$. The set $\overline{\langle B \cap (S \backslash J) \rangle}$ is contained in $S \backslash J$, because $S \backslash J$ is closed. Hence B_J cannot be empty. Now consider the ideal of S contained in J generated by B_J, specifically: $I_{B_J} = SB_JS \cup SB_J \cup B_JS \cup B_J$. As B_J is finite I_{B_J} is compact hence closed. So no element of $S \backslash J$ can be found in the closure of the subsemigroup of S generated by B_J. Hence $B \cap (S \backslash J) \neq \emptyset$.

Now the topological closure of the semigroup generated by B must be contained in $(S \backslash J) \cup I_B$. So $S = (S \backslash J) \cup I_B$. As $S \backslash J$ and I_B are disjoint this contradicts the fact that S is supposed connected. This contradiction proves the proposition. □

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J. He, S.M. Miller and D.A. Robbie: Department of Mathematics and Statistics, University of Melbourne, Parkville, Victoria, Australia, 3010.

D. Robbie is the corresponding author.

E-mail address: j.he@pgrad.unimelb.edu.au,
smm@ms.unimelb.edu.au,
darobbie@unimelb.edu.au.

K.H. Hofmann: Department of Mathematics, Tulane University, LA, USA and Fachbereich Mathematik, Technische Universität, Schlossgartenstr. 7, Darmstadt, 64289, Germany.

E-mail address: hofmann@mathematik.tu-darmstadt.de.