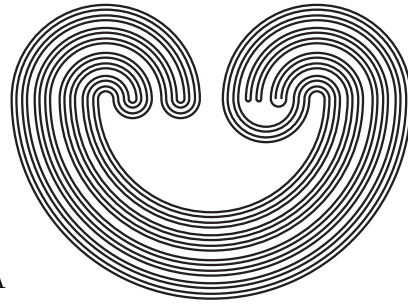


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WEAKLY EBERLEIN COMPACT SPACES

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ABSTRACT. Call a space X weakly splittable if, for each $f \in \mathbb{R}^X$, there exists a σ -compact $F \subset C_p(X)$ such that $f \in \overline{F}$ (the bar denotes the closure in \mathbb{R}^X). A weakly splittable compact space is called weakly Eberlein compact. We prove that weakly Eberlein compact spaces have almost the same properties as Eberlein compact spaces. We show that any weakly Eberlein compact space of cardinality $\leq \mathfrak{c}$ is Eberlein compact. We prove that a compact space X is weakly Eberlein compact if and only if X is splittable over the class of Eberlein compact spaces and that every countably compact weakly splittable space has the Preiss–Simon property.

0. INTRODUCTION

The first one to study weakly compact subspaces of Banach spaces was Eberlein [Eb]. His results showed that these compact spaces are very important and have numerous applications in many areas of mathematics. That is why they were called Eberlein compact spaces. In fact, a compact space X is Eberlein compact if and only if $C_p(X)$ has a σ -compact dense subspace and it is a non-trivial theorem that these two definitions are equivalent. The basic results of the theory of Eberlein compact spaces have many

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applications in functional analysis, topological algebra and topology. The class of Eberlein compact spaces is nice from a categorical point of view because it is closed under continuous images, countable products and closed subspaces; besides, this class contains all metrizable compact spaces. In 1968 Amir and Lindenstrauss proved that a compact space is Eberlein compact if and only if it can be embedded into a Σ_* -product of real lines [AL]. In 1974 an internal characterization in terms of T_0 -separating σ -point-finite families of cozero sets was given by Rosenthal [Ro]. Applying this characterization, Benyamini, Rudin and Wage [BRW] proved in 1977 that any continuous image of an Eberlein compact space is also Eberlein compact. Gul'ko [Gu] proved independently the invariance of the class of Eberlein compact spaces under continuous maps. In 1982 van Mill has constructed an example of a topologically homogeneous non-metrizable Eberlein compact space [vM].

Thus the inner harmony of the class of Eberlein compact spaces as well as their numerous applications show that any new information about this class is of importance.

In this paper we introduce a generalization of the class of Eberlein compact spaces calling a compact space X weakly Eberlein, if for each $f \in \mathbb{R}^X$, there exists a σ -compact $A \subset C_p(X)$ such that $f \in \overline{A}$ (the closure is taken in \mathbb{R}^X). It is easy to see that every Eberlein compact space is also weakly Eberlein compact. A very interesting problem, which is still open is whether any weakly Eberlein compact space is Eberlein space.

Another important class of topological spaces is the class of splittable spaces. These spaces were introduced by Tkachuk in 1986 [T1]. Tkachuk defined a space as splittable if it is Tychonoff and, for each $f \in \mathbb{R}^X$, there exists a countable $N \subset C_p(X)$ such that $f \in \overline{N}$ (the closure is taken in \mathbb{R}^X). Later in 1988 Arhangel'skii and Shakhmatov proved that a Tychonoff X space is splittable if and only if for any $A \subset X$, there exists a continuous map from X into \mathbb{R}^ω such that $f^{-1}(f(A)) = A$. Tkachuk proved in 1986 that pseudocharacter of a splittable space is countable. In [AS] Shakhmatov and Arhangel'skii showed, among other things, that a pseudocompact splittable space is metrizable.

We call a Tychonoff space X weakly splittable if, for each function $f \in \mathbb{R}^X$, there exists a σ -compact subspace $F \subset C_p(X)$ such that $f \in \overline{F}$ (the closure is taken in \mathbb{R}^X). We establish that if X is a weakly Eberlein compact space then it has the Fréchet–Urysohn property and the space $C_p(X)$ is Lindelöf. A weakly Eberlein compact space is Eberlein compact when $|X| \leq \mathfrak{c}$. Another analogy with Eberlein compact spaces is that any weakly Eberlein compact space is metrizable whenever $c(X) \leq \omega$. We will also prove that the Souslin number of a weakly Eberlein compact space coincides with its weight. We give internal and external characterizations of weakly Eberlein compact spaces. The first one is given in terms of T_0 -separating σ -point-finite families of cozero sets, and the second one in terms of splitting sets of maps over the class of Eberlein compact spaces.

The main result of this paper is that a countably compact space is weakly splittable if and only if it splits over the class of Eberlein–Grothendieck spaces. Thus a compact space is weakly Eberlein compact if and only if it is splittable over the class of Eberlein compact spaces.

1. NOTATION AND TERMINOLOGY

All spaces under consideration are assumed to be Tychonoff. The space \mathbb{R} is the set of real numbers with its natural topology, $\mathbb{Q} \subset \mathbb{R}$ is the subspace of rational numbers, $I = [0, 1] \subset \mathbb{R}$ and $\mathbb{D} = \{0, 1\} \subset \mathbb{R}$. For any spaces X and Y let $C_p(X, Y)$ be the space of continuous maps from X to Y endowed with the topology of pointwise convergence. When $Y = \mathbb{R}$ we write $C_p(X)$ instead of $C_p(X, \mathbb{R})$. Let Y be a subspace of a space X ; by $\pi = \pi_Y : C_p(X) \rightarrow C_p(Y)$ we denote the restriction map, i.e. $\pi(f) = f|_Y$ for all $f \in C_p(X)$. Every continuous map $\varphi : X \rightarrow Y$ determines the dual map $\varphi^* : C_p(Y) \rightarrow C_p(X)$ by the rule $\varphi^*(f) = f \circ \varphi$ for any $f \in C_p(Y)$. Given a subspace $Y \subset C_p(X)$ (or $Y \subset \mathbb{R}^X$), the closure of Y in \mathbb{R}^X is denoted by \overline{Y} , and the closure of $Z \subset C_p(X)$ in $C_p(X)$ is denoted by $cl(Z)$.

By \mathbb{R} or \mathbb{R}_α we denote the real line with the natural topology. The subspace of the product space $\prod\{\mathbb{R}_\alpha : \alpha \in A\}$ formed by those $x = (x_\alpha : \alpha \in A)$ for which the set $\{\alpha \in A : |x_\alpha| \geq \epsilon\}$ is finite for all $\epsilon > 0$, is denoted by $\Sigma_*\{\mathbb{R}_\alpha : \alpha \in A\}$ or $\Sigma_*(A)$ and is called the Σ_* -product of $|A|$ real lines.

Given a set $F \subset C_p(X)$, the canonical evaluation map $\Psi : X \rightarrow C_p(F)$ is defined by $\Psi(x)(f) = f(x)$ for all $f \in F$. The set F separates the points of X if, for any distinct $x, y \in X$, there is an $f \in F$ such that $f(x) \neq f(y)$. A family γ of subsets of topological space X is called T_0 -separating if, for any distinct $x, y \in X$, there is $U \in \gamma$ such that $|U \cap \{x, y\}| = 1$.

Given a space X its Souslin number $c(X)$ is the supremum of cardinalities of families of pairwise disjoint open nonempty subsets of X . The i -weight $iw(X)$ of the space X is the minimal weight of all spaces onto which X can be condensed. The tightness $t(X)$ of the space X is the smallest cardinal such that for each set $A \subset X$ and any point $x \in \overline{A}$ there is a set $B \subset A$ for which $|B| \leq t(X)$ and $x \in \overline{B}$. The extent $e(X)$ of X is the supremum of cardinalities of discrete closed subspaces of X . A sequence $\{A_n : n < \omega\}$ of subsets of a space X converges to a point $x \in X$ if, for any neighborhood U of x , there is an $m < \omega$ such that $A_n \subset U$ for all $n \geq m$.

2. PROPERTIES OF WEAKLY EBERLEIN COMPACT SPACES

We will prove that weakly Eberlein compact spaces have almost all properties that the Eberlein compact spaces have. The following theorem is well known (see [Ar2]).

Theorem 2.1. *For a compact space X , the following conditions are equivalent:*

- (i) X is Eberlein compact;
- (ii) there is a compact space K such that X embeds in $C_p(K)$;
- (iii) there exists σ -compact subspace $G \subset C_p(X)$ which separates the points of X ;
- (iv) there is a σ -compact space F such that X embeds in $C_p(F)$;
- (v) X can be homeomorphically mapped into $\Sigma_*(A)$ for some A .

Proposition 2.2. *If X is a weakly splittable space and $Y \subset X$ then Y is weakly splittable.*

Proof. Let $\pi = \pi_Y : C_p(X) \rightarrow C_p(Y)$ be the restriction map. Take any $f \in \mathbb{R}^Y$; there exists $g \in \mathbb{R}^X$ such that $g|_Y = f$. Take a σ -compact subspace $G \subset C_p(X)$ for which $g \in \overline{G}$. The subspace $\pi(G) \subset C_p(Y)$ is σ -compact and, by continuity of π , we have $f = g|_Y \in \pi(\overline{G}) \subset \overline{\pi(G)}$. Therefore Y is weakly splittable space. \square

Corollary 2.3. *If X is a weakly Eberlein compact space and $Y \subset X$ is closed, then Y is weakly Eberlein compact.*

Proposition 2.4. *If X is a weakly splittable countably compact space and $|X| \leq \mathfrak{c}$ then X is Eberlein compact.*

Proof. Let f be an injective function from X to \mathbb{R} . Take a σ -compact space $F \subset C_p(X)$ such that $f \in \overline{F}$. It is easy to see that F separates the points of X . The canonical evaluation map $\Psi : X \rightarrow C_p(F)$ sends X to a countably compact space $Y = \Psi(X) \subset C_p(F)$. The theorem of Grothendieck implies that Y is an Eberlein compact space [Ar2, Theorem III.4.20]. This implies that Y is Fréchet–Urysohn. Since Ψ is a condensation from the countably compact space X to a Fréchet–Urysohn space Y , the map Ψ is a homeomorphism. Hence X is Eberlein compact. \square

Corollary 2.5. *Let X be a weakly Eberlein compact space. If $|X| \leq \mathfrak{c}$ then X is Eberlein compact.*

Lemma 2.6. *If X is weakly Eberlein compact, then the character of the space X is less than or equal to the Souslin number of X .*

Proof. Let $f \in \mathbb{R}^X$ be the characteristic function of the set $\{x\}$ for a given $x \in X$. There exists a subspace $F = \bigcup\{F_i : i < \omega\} \subset C_p(X)$ such that F_i is compact for all $i \in \omega$ and $f \in \overline{F}$. We have $d(F_i) \leq w(F_i) \leq c(X)$ for each $i < \omega$ [Ar2, Theorem III.5.9]. Thus $d(F) \leq c(X)$ so there is $D \subset C_p(X)$ such that $f \in \overline{D}$ and $|D| \leq c(X)$. The cardinality of the family

$$\beta = \{g^{-1}((g(x) - 1/n, g(x) + 1/n)) : g \in D, n < \omega\} \subset \tau(X)$$

does not exceed $c(X)$ and $\bigcap \beta = \{x\}$. Thus x is a $G_{c(X)}$ -point. For each compact space its character and pseudocharacter coincide so $\chi(X) \leq c(X)$. \square

Theorem 2.7. *If X is weakly Eberlein compact then $c(X) = w(X)$.*

Proof. Let $\tau = c(X)$ be the Souslin number of the space X . The previous lemma shows that $|X| \leq 2^{\chi(X)} \leq 2^\tau$, because the inequality $|X| \leq 2^{\chi(X)}$ holds for any compact X [Ar1]. The theorem of Hewitt–Marczewski–Pondiczery implies that $d(\mathbb{R}^X) \leq \tau$. Let $D = \{f_i : i < \tau\} \subset \mathbb{R}^X$ be a dense subspace of \mathbb{R}^X ; for each $f_i \in D$ there is a σ -compact $F_i \subset C_p(X)$ such that $f_i \in \overline{F_i}$. In addition,

every F_i is a countable union of compact subspaces of weight $\leq \tau$, hence for any $i < \tau$ we can take $G_i \subset F_i$ for which $|G_i| \leq \tau$ and $f_i \in \overline{G_i}$. Consequently the subset

$$G = \bigcup \{G_i : i < \tau\} \subset C_p(X)$$

is a dense subspace of $C_p(X)$ and $|G| \leq \tau$. Noble proved in [No] that for any space X we have the equality $iw(X) = d(C_p(X))$. Therefore we conclude that

$$w(X) = iw(X) = d(C_p(X)) \leq |G| \leq \tau = c(X),$$

i.e. $c(X) = w(X)$. □

Corollary 2.8. *Let X be a weakly Eberlein compact space. If the Souslin number of X is countable then X is metrizable.*

Proof. From Theorem 2.7 it follows that weight of the space X is countable, i.e. X is metrizable. □

Corollary 2.9. *Every separable weakly Eberlein compact space X is metrizable.*

Proof. The Souslin number of any separable space is countable. □

A space X is called ω -monolithic if the closure of any countable set in X has a countable network. A space is called monolithic if for every $Y \subset X$ we have $d(Y) = nw(Y)$. A topological space X is called a Fréchet–Urysohn space if for every subset $A \subset X$ and every $x \in \overline{A}$ there is a sequence $\{x_i : i < \omega\} \subset A$ which converges to x . Any Eberlein compact space is an ω -monolithic and Fréchet–Urysohn space. We are going to prove that weakly Eberlein compact spaces also have these properties.

Corollary 2.10. *Every weakly Eberlein compact space is monolithic.*

Proof. Suppose that X is a weakly Eberlein compact space and take any $Y \subset X$. We are going to prove that $nw(Y) = d(Y)$; there is no loss of generality to consider that $\overline{Y} = X$. From Theorem 2.7 it follows that $c(X) = d(X) = w(X)$, and therefore

$$nw(Y) \leq nw(X) = w(X) = d(X) \leq d(Y) \leq nw(Y)$$

so the space X is monolithic. □

Recall that a transfinite sequence $\{x_\alpha : \alpha < \kappa\} \subset X$ is called free sequence of length κ in the space X if for any $\beta < \kappa$, we have $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \beta \leq \alpha\}} = \emptyset$.

Proposition 2.11. *If X is a weakly Eberlein compact space, then the tightness $t(X)$ of the space X is countable.*

Proof. The tightness $t(X)$ of a compact space X is equal to the supremum of all lengths of free sequences of X . Assume that $t(X) > \omega$. Then there is a free sequence $F = \{x_i : i < \omega_1\} \subset X$. For every $\alpha < \omega_1$ we consider the set $X_\alpha = \overline{\{x_\beta : \beta < \alpha\}} \subset X$. The subset $\{x_\beta : \beta < \alpha\}$ is countable; this means that $d(X_\alpha) \leq \omega$. Corollary 2.9 implies that X_α is a compact metrizable space and hence $|X_\alpha| \leq \mathfrak{c}$.

Let $Y = \bigcup\{X_\alpha : \alpha < \omega_1\}$. We are going to prove that Y is a countably compact space. Since $|X_\alpha| \leq \mathfrak{c}$, we have $|Y| \leq \mathfrak{c}$; besides $\alpha < \beta < \omega_1$ implies that $X_\alpha \subset X_\beta$ and $X_\alpha \neq X_\beta$, because $F = \{x_\alpha : \alpha < \omega_1\}$ is a free sequence.

Given a sequence $\{y_n : n < \omega\} \subset Y$, for every $n < \omega$ there is an $\alpha_n < \omega_1$ such that $y_n \in X_{\alpha_n}$. It is easy to see that there exists a $\beta < \omega_1$ with $\beta \geq \alpha_n$ for all $n < \omega$, i.e. $X_{\alpha_n} \subset X_\beta$. This implies that $\{y_n : n < \omega\}$ is contained in the compact metrizable space X_β , and therefore the sequence $\{y_n : n < \omega\}$ has a limit point in $X_\beta \subset Y$. We have proved that Y is countably compact.

Applying Proposition 2.4 we can see that Y is Eberlein compact and hence $t(Y) \leq \omega$ which is a contradiction with the fact that F is a free sequence of length ω_1 in Y . \square

Proposition 2.12. *Every weakly Eberlein compact space X is Fréchet–Urysohn.*

Proof. Take any $A \subset X$ and any $x \in \overline{A}$. The tightness of X is countable, so there is a countable $B \subset A$ for which $x \in \overline{B}$. The ω -monolithity of X implies that \overline{B} is a compact space with countable network weight, hence B is metrizable and therefore there is a sequence in $B \subset A$ which converges to x . \square

Corollary 2.13. *If X is weakly Eberlein compact and $Y \subset X$, $|Y| \leq \mathfrak{c}$, then \overline{Y} is Eberlein compact.*

Proof. The space X is Fréchet–Urysohn, so $|\overline{Y}| \leq \mathfrak{c}^\omega \leq \mathfrak{c}$. Corollary 2.5 implies that \overline{Y} is an Eberlein compact space. \square

Corollary 2.14. *If X is a weakly Eberlein compact space and $f : X \rightarrow Y$ is a continuous onto map such that $w(Y) \leq \mathfrak{c}$, then Y is Eberlein compact.*

Theorem 2.15. *Let X be a weakly Eberlein compact space. Then $C_p(X)$ is Lindelöf.*

Proof. A Baturov's theorem implies that $e(C_p(X)) = l(C_p(X))$ [Ba]. Assume that there is a discrete closed subspace $D \subset C_p(X)$ of cardinality ω_1 .

Let $\Psi : X \rightarrow C_p(D) = \mathbb{R}^{\omega_1}$ be the canonical evaluation map and $Y = \Psi(X)$. It is obvious that $d(Y) \leq w(Y) \leq w(\mathbb{R}^{\omega_1}) \leq \omega_1$, so $\Psi(\overline{W}) = \overline{\Psi(W)} = \overline{Z} = Y$ is an Eberlein compact space by Corollary 2.14.

Since X is compact, Ψ is a quotient map, so the dual map $\Psi^* : C_p(Y) \rightarrow C_p(X)$ is a homeomorphism between $C_p(Y)$ and some closed subspace E of $C_p(X)$. It is easy to see that $D \subset E \subset C_p(X)$. This implies that $e(E) = e(C_p(Y)) \geq \omega_1$, which is a contradiction because Y is an Eberlein compact space. \square

A space X is called a Preiss–Simon space if for each closed subset Y and each point $y \in Y$ there is a sequence $\{U_n : n < \omega\}$ of nonempty open subsets of Y which converges to y . Any Eberlein compact space is a Preiss–Simon space [PS].

Proposition 2.16. *Every weakly splittable countably compact space X is a Preiss–Simon space.*

Proof. Since every closed subspace of X is also a weakly splittable countably compact space, it suffices to prove that, for any $x \in X$, there is a sequence $\{U_n : n \in \omega\}$ of non-empty open subsets of $X \setminus \{x\}$ which converges to x . Let $f = \chi_x$ be the characteristic function of the set $\{x\}$. Since X is weakly splittable, there is a σ -compact subspace $F \subset C_p(X)$ such that $f \in \overline{F}$. Let $\Psi : X \rightarrow C_p(F)$ be the canonical evaluation map. Since X is countably compact and F is σ -compact, the space $E = \Psi(X) \subset C_p(F)$ is a countably compact Eberlein–Grothendieck space. Thus the space E is Eberlein compact, and hence Preiss–Simon space.

It is easy to see that $\Psi^{-1}\Psi(x) = \{x\}$. Let $y = \Psi(x)$. There exists a sequence $\{V_n : n < \omega\} \subset \tau(Y)$ which converges to y . Let $U_n = \Psi^{-1}(V_n)$ for all $n \in \omega$; we claim that the sequence $\{U_n : n < \omega\}$ converges to x . Let W be an open neighborhood of

the point x . Since Ψ is a closed map, there is a set $V \in \tau(Y, y)$ for which $\Psi^{-1}(V) \subset W$. There is $m \in \omega$ such that $V_n \subset V$ for any $n \geq m$. Thus $U_n = \Psi^{-1}(V_n) \subset \Psi^{-1}(V) \subset W$ for all $n \geq m$ and hence $\{U_n : n < \omega\}$ converges to the point x . \square

Corollary 2.17. *Every weakly Eberlein compact space has the Preiss–Simon property.*

Corollary 2.18. *Any pseudocompact subspace of a weakly Eberlein compact space is compact.*

Proof. It is evident that every pseudocompact subspace of a compact space with the Preiss–Simon property, is compact. \square

3. TWO CHARACTERIZATIONS OF WEAKLY EBERLEIN COMPACT SPACES

Let X be a space and let A be its subspace. A family γ of subsets of the space X is called T_0 -separating the points of A from the points of A^c , if for any two points $x, y \in X$ such that $x \in A$ and $y \in A^c$ there is a $V \in \gamma$ for which $|V \cap \{x, y\}| = 1$. A space X is splittable over the class of Eberlein compact spaces if for any $A \subset X$ there is an Eberlein compact space Y and a continuous onto map $f : X \rightarrow Y$ such that $f^{-1}f(A) = A$. Recall that a compact space X is Eberlein compact if and only if it contains a T_0 -separating σ -point-finite family of cozero sets [Ro].

Lemma 3.1. *For any $f \in \mathbb{R}^X$ there is a countable set $N \subset (\mathbb{D}^X)_{\mathbb{Q}} = \{q_1 a_1 + \cdots + q_n a_n : n < \omega, a_i \in \mathbb{D}^X \text{ and } q_i \in \mathbb{Q}\}$ such that $f \in \overline{N}$.*

Proof. Let n be a natural number. Given an integer k we define $B_k^n = [\frac{k}{n}, \frac{(k+1)}{n}] \subset \mathbb{R}$ and $A_k^n = f^{-1}(B_k^n)$. The family

$$\mathcal{U} = \{A_k^n : n \in \mathbb{N}, k \in \mathbb{Z}\}$$

is countable and hence the set

$$N = \{q_1 \chi_{A_1} + \cdots + q_k \chi_{A_k} : k \in \mathbb{N}, A_i \in \mathcal{U}, q_i \in \mathbb{Q}\}$$

is also countable. It is easy to see that $f \in \overline{N}$. \square

A Tychonoff space X is called Eberlein–Grothendieck if it is homeomorphic to a subspace of the space $C_p(K)$ for some compact space K . Each compact Eberlein–Grothendieck space is Eberlein compact; it is well known that for any Eberlein–Grothendieck space X there exists a σ -compact dense subspace of $C_p(X)$ [Ar2].

Proposition 3.2. *A Tychonoff space X is weakly splittable if and only if for any $A \subset X$ there is an Eberlein–Grothendieck space Y and a continuous map $f : X \rightarrow Y$ such that $f^{-1}f(A) = A$. In other words a Tychonoff space X is weakly splittable if and only if it is splittable over the class of Eberlein–Grothendieck spaces.*

Proof. Necessity. Suppose that X is a weakly splittable space and $A \subset X$. Let χ_A be the function from X to $\{0, 1\}$ defined by $\chi_A(x) = 1$, if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Since X is weakly splittable, there is a σ -compact subspace $F \subset C_p(X)$ such that $\chi_A \in \overline{F}$. Let $\Psi : X \rightarrow C_p(F)$ the canonical evaluation map. Since F is σ -compact, the map Ψ sends X to the Eberlein–Grothendieck space $Y = \Psi(X) \subset C_p(F)$. It is easy to see that $\Psi^{-1}\Psi(A) = A$.

Sufficiency. By Lemma 3.1 it is sufficient to prove that for any $f \in \mathbb{D}^X$ there is σ -compact $F \subset C_p(X)$ such that $f \in \overline{F}$. Let $A = f^{-1}(1)$. There is an Eberlein–Grothendieck space Y and a continuous map $\varphi : X \rightarrow Y$ for which $\varphi^{-1}\varphi(A) = A$. The space $C_p(Y)$ contains a σ -compact dense subspace F . The dual map $\varphi^* : C_p(Y) \rightarrow C_p(X)$ sends the subspace F to a σ -compact space $Z = \varphi^*(F) \subset C_p(X)$. We will prove that $f \in \overline{Z}$. Take a standard basic open neighborhood $U = W(f; a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_n; \epsilon)$ of the function f where $a_i \in A$ and $b_i \in A^c$. It is clear that

$$\varphi(\{a_1, \dots, a_k\}) \cap \varphi(\{b_1, \dots, b_n\}) = \emptyset.$$

Let $a'_i = \varphi(a_i)$ and $b'_i = \varphi(b_i)$. Since F is dense in $C_p(Y)$, there is a $g \in F$ such that $|g(a'_i) - 1| < \epsilon$ for all $i \leq k$ and $|g(b'_i)| < \epsilon$ for any $i \leq n$; i.e. $|g(\varphi(a_i)) - 1| < \epsilon$ for all $i \leq k$ and $|g(\varphi(b_i))| < \epsilon$ for any $i \leq n$. Since $h = g \circ \varphi \in Z$ and $h \in U$ we have proved that $Z \cap U \neq \emptyset$. \square

Corollary 3.3. *A compact space X is weakly Eberlein compact if and only if for any $A \subset X$ there is an Eberlein compact space Y and a continuous map $f : X \rightarrow Y$ such that $f^{-1}f(A) = A$. In other words a compact space X is weakly Eberlein compact if and only if it is splittable over the class of Eberlein compact spaces.*

If a space X is splittable over the class of weakly Eberlein compact spaces, then it is splittable over the class of Eberlein compact spaces. Hence we have proved the following:

Corollary 3.4. *A compact space X is weakly Eberlein compact if and only if it is splittable over the class of weakly Eberlein compact spaces.*

Proposition 3.5. *A compact space X is weakly Eberlein compact if and only if for any $A \subset X$ there exists a σ -point-finite family of cozero sets which separates the points of A from the points of $X \setminus A$.*

Proof. Let X be a weakly Eberlein compact space. Take an $A \subset X$. The previous theorem implies that there is an Eberlein compact space Y and a continuous map $f : X \rightarrow Y$ such that $f^{-1}f(A) = A$. The space Y has a T_0 -separating σ -point-finite family γ of cozero sets; it is evident that the family $\beta = \{f^{-1}(U) : U \in \gamma\}$ is σ -point-finite and consists of cozero sets of X . Take $x_1, x_2 \in X$ such that $x_1 \in A$ and $x_2 \in A^c$. We have $f(x_1) = y_1 \neq y_2 = f(x_2)$. There is $V \in \gamma$ such that $|V \cap \{y_1, y_2\}| = 1$, so $|f^{-1}(V) \cap \{x_1, x_2\}| = 1$. We have proved that β indeed T_0 -separates the points of A from the points of A^c , which proves the necessity.

To establish sufficiency, take any $A \subset X$. We will prove that there is an Eberlein compact space Y and a continuous map $f : X \rightarrow Y$ such that $f^{-1}f(A) = A$. For each $n < \omega$ there is a family $\gamma_n = \{U_{\alpha,n} : \alpha \in B_n\}$ of cozero sets in X such that $\gamma = \bigcup \{\gamma_n : n < \omega\}$ is T_0 -separating the points of A from the points of A^c , and any γ_n is point-finite.

Given an $n < \omega$ and an $\alpha \in B_n$, we can take a function $f_{\alpha,n} \in C_p(X)$ such that $|f_{\alpha,n}(x)| \leq 1/n$ for all $x \in U_{\alpha,n}$, and $f_{\alpha,n}^{-1}(0) = X \setminus U_{\alpha,n}$. For $B = \{(\alpha, n) : \alpha \in B_n \text{ and } n < \omega\}$ we define the map $f : X \rightarrow \mathbb{R}^B$ by $f(x) = y$ where $y((\alpha, n)) = f_{\alpha,n}(x)$ for all $(\alpha, n) \in B$. Since the maps $f_{\alpha,n}$ are continuous, f is also continuous. It is easy to check that $f(X) \subset \Sigma_*(B)$; Theorem 2.1 implies that $Y = f(X)$ is an Eberlein compact space.

Let z be a point of A^c . Given a point $x \in A$ there is an $U_{\alpha,n} \in \gamma$ such that $|U_{\alpha,n} \cap \{x, z\}| = 1$. If $x \in U_{\alpha,n}$, then $f_{\alpha,n}(x) \neq 0$ and $f_{\alpha,n}(z) = 0$; this implies that $f(x) \neq f(z)$. If $z \in U_{\alpha,n}$, then $f_{\alpha,n}(z) \neq 0$ and $f_{\alpha,n}(x) = 0$; this implies that $f(x) \neq f(z)$. Thus we can conclude that $f(z) \neq f(x)$ and therefore $f^{-1}f(A) = A$. Now, Proposition 3.3 implies that X is weakly Eberlein compact. \square

4. OPEN PROBLEMS

There are still many interesting open questions left as shown by the list below.

Problem 4.1. Is any continuous image of a weakly Eberlein compact space also weakly Eberlein compact?

Problem 4.2. Is any countable (or finite) product of weakly Eberlein compact spaces also weakly Eberlein compact?

Problem 4.3. Does weak Eberlein compactness imply Eberlein compactness?

Problem 4.4. Let X be a weakly splittable pseudocompact space. Must X be compact? How about a countably compact X ?

Problem 4.5. Let X be a weakly Eberlein compact Corson compact space. Must X be Eberlein compact?

Problem 4.6. Let X be a weakly Eberlein compact such that $|X| = \mathfrak{c}^+$. Must X be an Eberlein compact space?

Problem 4.7. Let X be a weakly Eberlein compact space. Must $C_p(X)$ be a Lindelöf Σ -space?

Problem 4.8. Let X be a scattered weakly Eberlein compact space. Must X be an Eberlein compact space?

Problem 4.9. Let a compact space X be splittable over the class of Corson compact spaces. Must X be a Corson compact space?

Problem 4.10. Let a compact space X be splittable over the class of Gul'ko compact spaces. Must X be a Gul'ko compact space?

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